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EXISTENCE AND UNIQUENESS OF THE SOLUTION FOR THE COAGULATION-FRAGMENTATION EQUATION OF WATER DROPS IN FALL*

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Abstract

We consider the integro-differential equation, which describes the fall and the coagulation-fragmentation process of the droplets. By constructing the approximate solutions, which are constituted by families of piece-wise analytic functions, and verifying their convergence, we prove the existence and the uniqueness of the local solution.

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1 Introduction

As it is commonly known by meteorologists (see eg [12]), the water droplets in the atmosphere fall with different velocities, (mainly determined by the mass of each droplet) and contemporarily undergo the coagulation and fragmentation process. There are several works in the mathematical description of these process we cite here a few. The coagulation process was given by Smoluchowski [14] and Müller [11], the equation of the coagulation-fragmentation process has been studied by Melzak [9]. When the equation of droplets which move and undergo the coagulation process, in [7], Galkin proved the existence and the uniqueness of the solution (see also [4], [8]). Also, in 2001 Dubovskii [3], demonstrated the existence and the uniqueness of the global solution of the displacement and coagulation-fragmentation equation of the droplets. To construct the solution, Dubovskii similarly to Galkin used an essential way "the maximum principale" to control the norm L^∞ of the solution.

In this work, we consider the equation of droplets which fall in the air and undergo the coagulation-fragmentation process as in Dubovskii's work [3]. But to construct the solution, instead of following the time $t \geq 0$, we follow the trajectories of droplets and their position $z \leq 0$, which permit us to remove a condition posed in [3] (it's about the condition (33) in [3]) on the velocity of droplets $u(m)$ which can lead to the relation $\frac{du(m)}{dm} \geq cm^\alpha$, $\alpha > 0$ (see in [3] the formula (38) and it's comments). Indeed, it seems that this condition can be difficult to achieve in the case of droplets in the atmosphere. More precisely, denoting by $\sigma(m,t,z)$ the density of liquid water contained in the droplets of mass m at time t and in position z, we consider the equation with the entry condition $\sigma(m,t,0) = \overline{\sigma}_0(m,t)$ and prove the existence and the uniqueness of the local solution (i.e in a domain $-L < z \leq 0$). To do this, using the Melzak's method [9], we construct approximate solutions, consisting of analytic functions in $s = -z$ in each interval $[\frac{\nu}{N}, \frac{\nu+1}{N}]$, $\nu = 0, 1, 2, \cdots$; $N \in \mathbb{N}\backslash\{0\}$, and prove their convergence to the solution of the equation.

The density $\sigma(m,t,z)$ of water liquid is a density with respect to the unit volume of the air containing possible droplets. The equation can be written with respect to the number (in the purely statistical sense) $\tilde{n}(m,t,z)$ of droplets that Dubovskii and Galkin use in their works. We see clearly that the density $\sigma(m,t,z)$ and the number $\tilde{n}(m,t,z)$ are connected by the relation $\tilde{n}(m,t,z) = \frac{\sigma(m,t,z)}{m}$.

We will use the density $\sigma(m,t,z)$ to be conform with the symbolism of

[2], [10] and the known literature of general modeling of weather phenomena ([1], [5], [6], [13]).

2 Position of the problem

We suppose that the drops undergo the coagulation and the fragmentation process and in the same time move in the air by the gravitational force while undergoing also the friction effect with surrounding air. In this situation, we can formulate the coagulation-fragmentation process as Melzak's equation ([9]) and the displacement of drops by a velocity given by the friction coefficient between the drops and the air, as the meteorologists commonly use it (see for example [12]). These considerations lead us to the equation (see[1], [2], [10] , [13])

$$\partial_t \sigma(m,t,z) + \partial_z(\sigma(m,t,z)u(m)) = \tag{1}$$

$$= \frac{m}{2}\int_0^m \beta(m-m',m')\sigma(m',t,z)\sigma(m-m',t,z)dm' +$$

$$-m\int_0^\infty \beta(m,m')\sigma(m,t,z)\sigma(m',t,z)dm' - \frac{m}{2}\sigma(m,t,z)\int_0^m \vartheta(m-m',m')dm' +$$

$$+m\int_0^\infty \vartheta(m,m')\sigma(m+m',t,z)dm',$$

where $\beta(m_1,m_2)$ represents the probability of meeting between two drops of mass m_1, m_2 respectively whereas $\vartheta(m_1,m_2)$ is the probability of fragmentation of a droplet of mass $m = m_1 + m_2$ into one of mass m_1 and another one of mass m_2. In addition, $u(m)$ indicate the velocity of drops with mass m. The equation (1) will be considered for $(m,t,z) \in \mathbb{R}_+ \times \mathbb{R} \times [-L,0]$ with $L > 0$ or possibly in $\mathbb{R}_+ \times \mathbb{R} \times]-\infty,0]$ and with the entry condition

$$\sigma(m,t,0) = \overline{\sigma}_0(m,t). \tag{2}$$

The functions $\beta(m_1,m_2)$ and $\vartheta(m_1,m_2)$, according to their physical nature, are supposed

$$\beta(\cdot,\cdot) \in C(\mathbb{R}_+ \times \mathbb{R}_+), \quad \beta(m_1,m_2) \geq 0 \ \ \forall (m_1,m_2) \in \mathbb{R}_+ \times \mathbb{R}_+, \tag{3}$$

$$\vartheta(\cdot,\cdot) \in C(\mathbb{R}_+ \times \mathbb{R}_+), \quad \vartheta(m_1,m_2) \geq 0 \ \ \forall (m_1,m_2) \in \mathbb{R}_+ \times \mathbb{R}_+, \tag{4}$$

$$\beta(m_1,m_2) = \beta(m_2,m_1), \quad \vartheta(m_1,m_2) = \vartheta(m_2,m_1)$$

and we admit that $u(m)$ is given by

$$u(m) = -\frac{g}{\alpha(m)}, \tag{5}$$

where g is a positive constant representing the gravitational acceleration and $\alpha(m)$ is the friction coefficient between drops and air. The relation (5) corresponds, in a good approximation, to the real velocity of drops in the atmosphere (see for example [1], [13]).

For the convenience of presentation, we will use the notation

$$w(m) := -u(m), \tag{6}$$

so that $w(m) > 0$ for all $m > 0$. For $w(m)$ we suppose that:

$$w(\cdot) \in C(\mathbb{R}_+), \qquad 0 < w(m_1) \leq w(m_2) \quad \text{si} \;\; 0 < m_1 \leq m_2; \tag{7}$$

the growth of the function $w(m)$ corresponds to the phenomena observed in nature (see for example [12]).

Moreover, we suppose that there exists a positive constant $C_0 < \infty$ such that:

$$\sup_{m \in \mathbb{R}_+, m' \in [0,m]} \frac{m}{w(m)} \beta(m - m', m') \leq C_0, \tag{8}$$

$$\sup_{m, m' \in \mathbb{R}_+} \frac{m}{w(m)} \beta(m, m') \leq C_0, \tag{9}$$

$$\sup_{m \in \mathbb{R}_+} \frac{m}{w(m)} \int_0^m \vartheta(m - m', m') dm' \leq C_0, \tag{10}$$

$$\sup_{m \in \mathbb{R}_+} \int_0^m \frac{m'}{w(m')} \vartheta(m - m', m') dm' \leq C_0, \tag{11}$$

$$\sup_{m, m' \in \mathbb{R}_+} \frac{m}{w(m)} \vartheta(m, m') \leq C_0. \tag{12}$$

It is clear that, if $\frac{m}{w(m)}$ is an increasing function of m, then the conditions (8) and (10) imply (9) and (11). The conditions on the function $\overline{\sigma}_0(m,t)$ will be specified in the following paragraphs (see (23), (71)-(72)).

3 Preliminaries - characteristics and description on them

To solve the equation (1) with conditions (2), firstly we define the family of characteristics $\chi_{m,\tilde{t}}$ by the equations system

$$\begin{cases} \frac{dz(s)}{ds} = -1, \\ \frac{dt(s)}{ds} = \frac{1}{w(m)}, \end{cases} \tag{13}$$

with the initial conditions

$$z(0) = 0, \qquad t(0) = \tilde{t}. \tag{14}$$

The characteristics $\chi_{m,\tilde{t}}$ as defined have, in the space $\mathbb{R}\times\,]-\infty,0]$, the expression:

$$\chi_{m,\tilde{t}} = \{(t,z) \in \mathbb{R}\times\,]-\infty,0] \,|\, t = \tilde{t} + \frac{s}{w(m)},\ z = -s,\ s \in [0,\infty[\,\}.$$

In the following, we will use the coordinates $(m,\tilde{t},s) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ and $\sigma(m,\tilde{t},s)$ instead of $\sigma(m,t,z) \in \mathbb{R}_+ \times \mathbb{R}\times\,]-\infty,0]$ and $\sigma(m,t,z)$ when $t = \tilde{t} + \frac{s}{w(m)}$ and $z = -s$.

Now we introduce, for each fixed $s \geq 0$, the curves family given by:

$$\gamma_{qs} = \{(m,\tilde{t}) \in \mathbb{R}_+ \times \mathbb{R} \,|\, \tilde{t} = q - \frac{s}{w(m)}\,\}, \qquad q \in \mathbb{R}. \tag{15}$$

The curve γ_{qs} is none other than the set of points $(m,\tilde{t})$ (on the half-plane $\{z = -s\}$) such as the characteristics $\chi_{m,\tilde{t}}$ passes by the point $t = q, z = -s$ on the plan (t,z).

In a similar way to [10] and [2] we define a measure $\mu_\gamma = \mu_{\gamma_{qs}}$ on the curves γ_{qs} by $P_{\mathbb{R}_+}$ the projection of γ_{qs} on $\mathbb{R}_+(\ni m)$, i.e. by the relations:

i) $A' \subset \gamma_{qs}$ is measurable if and only if $P_{\mathbb{R}_+}A'$ is measurable according to Lebesgue on $\mathbb{R}_+$,

ii) $\mu_\gamma(A') = \mu_{L,\mathbb{R}_+}(P_{\mathbb{R}_+}A')$, where $\mu_{L,\mathbb{R}_+}(\cdot)$ is the Lebesgue's measure on $\mathbb{R}_+$.

The measure $\mu_{\gamma_{qs}}(\cdot)$ enjoys a suitable properties for the calculus of integrals on the curves γ_{qs} (for more details, see [10]).

In particular, we recall that, if φ and ψ are two functions belonging to $L^1(\gamma_{qs},\mu_{\gamma_{qs}})$, then we have $\varphi * \psi \in L^1(\gamma_{qs},\mu_{\gamma_{qs}})$ and

$$\|\varphi * \psi\|_{L^1(\gamma_{qs},\mu_{\gamma_{qs}})} \leq \|\varphi\|_{L^1(\gamma_{qs},\mu_{\gamma_{qs}})}\|\psi\|_{L^1(\gamma_{qs},\mu_{\gamma_{qs}})}, \tag{16}$$

where

$$(\varphi * \psi)(m) = \int_{\gamma_{qs}} \varphi(m - m')\psi(m')\mu_{\gamma_{qs}}(dm').$$

Let $\varphi(\cdot,\cdot)$ be a measurable function defined on $\mathbb{R}_+ \times \mathbb{R}$. We put

$$\{\varphi\}_{qs}(m) = \varphi(m, q - \frac{s}{w(m)}), \tag{17}$$

which represents the values of $\varphi(m,\tilde{t})$ on the curve γ_{qs} expressed according to m. Moreover, $\gamma_{qs(m,\tilde{t})}$ designate the curve γ_{qs} with $q = \tilde{t} + \frac{s}{w(m)}$. It is clear that, the curve $\gamma_{qs(m,\tilde{t})}$ passes by the point $(m, \tilde{t}, s)$ and that

$$\gamma_{qs(m,\tilde{t})} = \{(m', \tilde{t}') \in \mathbb{R}_+ \times \mathbb{R} \,|\, \tilde{t}' = \tilde{t} + \frac{s}{w(m)} - \frac{s}{w(m')}\}. \tag{18}$$

Let $\gamma^{[0,m]}_{qs(m,\tilde{t})}$ be defined as:

$$\gamma^{[0,m]}_{qs(m,\tilde{t})} = \gamma_{qs(m,\tilde{t})} \cap ([0,m] \times \mathbb{R}).$$

Now, we define the operators $K_{\gamma_{qs}}[\varphi,\psi]$ and $L_{\gamma_{qs}}[\varphi]$ as follows:

$$\begin{aligned} K_{\gamma_{qs}}[\varphi,\psi](m,\tilde{t}) = \frac{1}{2}\int_{\gamma^{[0,m]}_{qs(m,\tilde{t})}} \beta(m-m',m')\{\varphi\}_{qs}(m-m')\{\psi\}_{qs}(m')\mu_{\gamma}(dm') + \\ -\frac{1}{2}\varphi(m,\tilde{t})\int_{\gamma_{qs(m,\tilde{t})}} \beta(m,m')\{\psi\}_{qs}(m')\mu_{\gamma}(dm') + \\ -\frac{1}{2}\psi(m,\tilde{t})\int_{\gamma_{qs(m,\tilde{t})}} \beta(m,m')\{\varphi\}_{qs}(m')\mu_{\gamma}(dm'), \end{aligned} \tag{19}$$

$$L_{\gamma_{qs}}[\varphi](m,\tilde{t}) = -\frac{1}{2}\varphi(m,\tilde{t})\int_{\gamma^{[0,m]}_{qs(m,\tilde{t})}} \vartheta(m-m',m')\mu_{\gamma}(dm') + \tag{20}$$

$$+\int_{\gamma_{qs(m,\tilde{t})}} \vartheta(m,m')\{\varphi\}_{qs}(m+m')\mu_{\gamma}(dm'),$$

provided that all the integrals in the right sides are well defined. From these relations, it results that $K_{\gamma_{qs}}[\varphi,\psi]$ is a symmetric, bilinear operator and $L_{\gamma_{qs}}[\varphi]$ is a linear operator. If $\varphi(m,\tilde{t})$ and $\psi(m,\tilde{t})$ are continuous, $K_{\gamma_{qs}}[\varphi,\psi](m,\tilde{t})$ and $L_{\gamma_{qs}}[\varphi](m,\tilde{t})$ are too .

The operators $K_{\gamma_{qs}}[\cdot,\cdot]$ and $L_{\gamma_{qs}}[\cdot]$ being defined, we can transform the equation (1) to

$$\frac{\partial}{\partial s}\sigma(m,\tilde{t},s)=\frac{m}{w(m)}\big(K_{\gamma_{qs}}[\sigma(\cdot,\cdot,s),\sigma(\cdot,\cdot,s)](m,\tilde{t})+L_{\gamma_{qs}}[\sigma(\cdot,\cdot,s)](m,\tilde{t})\big), \tag{21}$$

in the coordinates $(m,\tilde{t},s)$ defined above. The equation (21) will be considered with the condition

$$\sigma(m,\tilde{t},0)=\overline{\sigma}_0(m,\tilde{t}), \tag{22}$$

which is the transcription of the condition (2) in the coordinates $(m,\tilde{t},s)$. We suppose that $\overline{\sigma}_0(m,\tilde{t})$ is continuous in $(m,\tilde{t})\in\mathbb{R}_+\times\mathbb{R}$ and that

$$0\leq\overline{\sigma}_0(m,\tilde{t}),\qquad \sup_{(m,\tilde{t})\in\mathbb{R}_+\times\mathbb{R}}\overline{\sigma}_0(m,\tilde{t})<\infty,\qquad \sup_{\tilde{t}\in\mathbb{R}}\int_0^\infty\overline{\sigma}_0(m,\tilde{t})dm<\infty. \tag{23}$$

In the case where $\overline{\sigma}_0(m,\tilde{t})$ depends on $\tilde{t}$, we need to construct a sequence of approximate solutions. Indeed, for each $N\in\mathbb{N}\backslash\{0\}$, we introduce the partition of $\mathbb{R}_+$ into $[\frac{\nu}{N},\frac{\nu+1}{N}[$, $\nu=0,1,2,\cdots$, and we consider the approximate equation

$$\frac{\partial}{\partial s}\sigma(m,\tilde{t},s)=\frac{m}{w(m)}\big(K_{\gamma_{q\,\overline{s}_\nu}}[\sigma(\cdot,\cdot,s),\sigma(\cdot,\cdot,s)](m,\tilde{t})+L_{\gamma_{q\,\overline{s}_\nu}}[\sigma(\cdot,\cdot,s)](m,\tilde{t})\big) \tag{24}$$

for

$$\overline{s}_\nu=\frac{\nu}{N}\leq s<\frac{\nu+1}{N},\qquad \nu=0,1,2,\cdots.$$

Remark 1. *In* $[\frac{\nu}{N},\frac{\nu+1}{N}[$ *the curves family* $\{\gamma_{q\,\overline{s}_\nu}\}_{q\in\mathbb{R}}$ *is fixed and does not depend on* s*. By solving* (24) *for* $0\leq s<\frac{1}{N}$ *with the condition* (22) *and using, if possible,* $\sigma(m,\tilde{t},\frac{1}{N})$ *as entry condition of the equation* (24) *for* $\frac{1}{N}\leq s<\frac{2}{N}$*, we will be solving it in* $[\frac{1}{N},\frac{2}{N}[$*; by repeating this procedure for* $\nu=0,1,2,\cdots$*, we construct the approximate solution* $\sigma(m,\tilde{t},s)=\sigma^{[N]}(m,\tilde{t},s)$*.*

Before examining the equation (21) or (24), we recall the inequalities concerning the operators $K_{\gamma_{qs}}[\cdot,\cdot]$ and $L_{\gamma_{qs}}[\cdot]$.

Lemma 1. *For all* $s\geq 0$*, we have*

$$\sup_{(m,\tilde{t})\in\mathbb{R}_+\times\mathbb{R}}\frac{m}{w(m)}|K_{\gamma_{qs}}[\varphi,\psi](m,\tilde{t})|\leq \tag{25}$$

$$\leq\frac{3C_0}{4}\Big[\sup_{(m,\tilde{t})\in\mathbb{R}_+\times\mathbb{R}}|\varphi(m,\tilde{t})|\int_{\gamma_{qs(m,\tilde{t})}}|\{\psi\}_{qs}(m)|\mu_\gamma(dm)+$$

$$+ \sup_{(m,\tilde{t})\in\mathbb{R}_+\times\mathbb{R}} |\psi(m,\tilde{t})| \int_{\gamma_{qs(m,\tilde{t})}} |\{\varphi\}_{qs}(m)|\mu_\gamma(dm)\Big],$$

$$\sup_{q\in\mathbb{R}} \int_{\gamma_{qs}} \frac{m}{w(m)} |\{K_{\gamma_{qs}}[\varphi,\psi]\}_{qs}(m)|\mu_\gamma(dm) \leq \tag{26}$$

$$\leq \frac{3C_0}{2} \sup_{q\in\mathbb{R}} \int_{\gamma_{qs}} |\{\varphi\}_{qs}(m)|\mu_\gamma(dm) \int_{\gamma_{qs}} |\{\psi\}_{qs}(m)|\mu_\gamma(dm),$$

$$\sup_{(m,\tilde{t})\in\mathbb{R}_+\times\mathbb{R}} \frac{m}{w(m)} |L_{\gamma_{qs}}[\varphi](m,\tilde{t})| \leq \tag{27}$$

$$\leq C_0\Big[\frac{1}{2} \sup_{(m,\tilde{t})\in\mathbb{R}_+\times\mathbb{R}} |\varphi(m,\tilde{t})| + \sup_{q\in\mathbb{R}} \int_{\gamma_{qs}} |\{\varphi\}_{qs}(m)|\mu_\gamma(dm)\Big],$$

$$\sup_{q\in\mathbb{R}} \int_{\gamma_{qs}} \frac{m}{w(m)} |\{L_{\gamma_{qs}}[\varphi]\}_{qs}(m)|\mu_\gamma(dm) \leq \tag{28}$$

$$\leq \frac{3C_0}{2} \sup_{q\in\mathbb{R}} \int_{\gamma_{qs}} |\{\varphi\}_{qs}(m)|\mu_\gamma(dm).$$

Proof. The inequalities (25) and (27) result immediately from the definition (19) and (20) of operators $K_{\gamma_{qs}}[\cdot,\cdot]$ and $L_{\gamma_{qs}}[\cdot]$ and the conditions (8)-(10), (12). On the other hand, the inequality (26) results from relations (19), (8), (9) and the property of the convolution (16).

Last, let's use the change of variables $m'' = m + m'$. Hence, for any fixed arbitrary curve γ_{qs}, we have:

$$\int_{\gamma_{qs}} \frac{m}{w(m)} \int_{\gamma_{qs}} \vartheta(m,m')\{\varphi\}_{qs}(m+m')\mu_\gamma(dm')\mu_\gamma(dm) = \tag{29}$$

$$= \int_{\gamma_{qs}} \int_{\gamma_{qs}^{[0,m'']}} \frac{m''-m'}{w(m''-m')} \vartheta(m''-m',m')\mu_\gamma(dm')\{\varphi\}_{qs}(m'')\mu_\gamma(dm'').$$

Thus, taking into account the conditions (11), (12), we deduce from the definition (20) of the operator $L_{\gamma_{qs}}[\cdot]$ the inequality (28). □

4 Local solution of the approximate equation

In this paragraph and in the following one, we consider the equation (24)

$$\frac{\partial}{\partial s}\sigma(m,\tilde{t},s) = \frac{m}{w(m)}\big(K_{\gamma_{q\bar{s}_\nu}}[\sigma(\cdot,\cdot,s),\sigma(\cdot,\cdot,s)](m,\tilde{t}) + L_{\gamma_{q\bar{s}_\nu}}[\sigma(\cdot,\cdot,s)](m,\tilde{t})\big)$$

for $s \geq \overline{s}_\nu = \frac{\nu}{N}$ with the condition

$$\sigma(m, \tilde{t}, \frac{\nu}{N}) = \overline{\sigma}_\nu(m, \tilde{t}),$$

by considering $\overline{\sigma}_\nu(m, \tilde{t})$ as a given function.

As the curves $\gamma_{q\,\overline{s}_\nu}$ depend only on q, we use the simplified notation for this problem

$$\gamma_q = \gamma_{q\,\overline{s}_\nu}, \qquad \gamma_{q(m,\tilde{t})} = \gamma_{q\,\overline{s}_\nu(m,\tilde{t})}, \qquad \{\varphi\}_q = \{\varphi\}_{q\,\overline{s}_\nu}. \tag{30}$$

It would be enough to consider the equation in the interval $[\frac{\nu}{N}, \frac{\nu+1}{N}[$, but it will be more convenient to consider it in the interval $[\frac{\nu}{N}, \infty[$. Still to simplify the writing, we use the change of variables $s' = s - \frac{\nu}{N}$, to get $[0, \infty[$ and we write s instead of s'. by these writing conventions, we can write the problem in the form

$$\frac{\partial}{\partial s}\sigma(m, \tilde{t}, s) = \frac{m}{w(m)}\big(K_{\gamma_q}[\sigma(\cdot,\cdot,s), \sigma(\cdot,\cdot,s)](m, \tilde{t}) + L_{\gamma_q}[\sigma(\cdot,\cdot,s)](m, \tilde{t})\big), \tag{31}$$

$$\sigma(m, \tilde{t}, 0) = \overline{\sigma}_\nu(m, \tilde{t}). \tag{32}$$

Consider the integrate form of the latter equation:

$$\begin{aligned}\sigma(m, \tilde{t}, s) = \overline{\sigma}_\nu(m, \tilde{t}) \quad &+ \quad \int_0^s \frac{m}{w(m)}\big(K_{\gamma_q}[\sigma(\cdot,\cdot,s'), \sigma(\cdot,\cdot,s')](m, \tilde{t}) \\ &+ \quad L_{\gamma_q}[\sigma(\cdot,\cdot,s')](m, \tilde{t})\big) ds'.\end{aligned}$$

We suppose that for each $(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}$ the function $\sigma(m, \tilde{t}, s)$ is analytic in s, i.e. there exist the functions $a_k(m, \tilde{t})$, $k \in \mathbb{N}$, such as

$$\sigma(m, \tilde{t}, s) = \sum_{k=0}^{\infty} a_k(m, \tilde{t}) s^k. \tag{33}$$

Thus,

$$\frac{\partial}{\partial s}\sigma(m, \tilde{t}, s) = \sum_{k=0}^{\infty}(k+1) a_{k+1}(m, \tilde{t}) s^k.$$

We recall the definitions (19), (20), and by equalizing the terms having the same power of s, we deduce from the equality (31) that

$$a_{k+1}(m, \tilde{t}) = \frac{m}{w(m)} \frac{1}{k+1}\Big(\sum_{i+j=k} K_{\gamma_q}[a_i, a_j](m, \tilde{t}) + L_{\gamma_q}[a_k](m, \tilde{t})\Big) \tag{34}$$

for $k = 0, 1, 2, \cdots$.

Lemma 2. *We suppose that $\beta(\cdot,\cdot)$, $\vartheta(\cdot,\cdot)$ and $w(\cdot)$ satisfy the conditions mentioned in paragraph* 2, *and that $\overline{\sigma}_\nu(m,\tilde{t})$ is continuous in $(m,\tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}$ and satisfy the conditions*

$$\sup_{q\in\mathbb{R}} \int_{\gamma_q} \{\overline{\sigma}_\nu\}_q(m)\mu_\gamma(dm) \equiv A_0 < \infty, \tag{35}$$

$$\sup_{(m,\tilde{t})\in\mathbb{R}_+\times\mathbb{R}} \overline{\sigma}_\nu(m,\tilde{t}) \equiv B_0 < \infty. \tag{36}$$

Then, there exists a positive constant $C_0 < \infty$ such as the power-series of the second member of (33) *converges in the interval* $[0, \frac{1}{M}[$, *where*

$$M = C_0\Big(\frac{3}{2}(A_0+1) + \frac{A_0}{B_0}\Big). \tag{37}$$

Proof. We put

$$A_k = \sup_{q\in\mathbb{R}} \int_{\gamma_q} |\{a_k\}_q(m)|\mu_\gamma(dm), \qquad B_k = \sup_{(m,\tilde{t})\in\mathbb{R}_+\times\mathbb{R}} |a_k(m,\tilde{t})|. \tag{38}$$

We recall that, according to (32), the values of A_0 and B_0 given by (35) and (36) coincide with those given by (38).

By (26), (28) and (34) we have

$$\int_{\gamma_q} |\{a_{k+1}\}_q(m)|\mu_\gamma(dm) \leq$$

$$\leq \frac{1}{k+1}\int_{\gamma_q} \frac{m}{w(m)}\Big(\sum_{i+j=k} |\{K_{\gamma_q}[a_i,a_j]\}_q(m)| + |\{L_{\gamma_q}[a_k]\}_q(m)|\Big)\mu_\gamma(dm) \leq$$

$$\leq \frac{1}{k+1}\frac{3C_0}{2}\Big(\sum_{i+j=k}\int_{\gamma_q} |\{a_i\}_q(m)|\mu_\gamma(dm)\int_{\gamma_q} |\{a_j\}_q(m)|\mu_\gamma(dm) +$$

$$+ \int_{\gamma_q} |\{a_k\}_q(m)|\mu_\gamma(dm)\Big).$$

We deduce that

$$A_{k+1} \leq \frac{1}{k+1}\frac{3C_0}{2}\Big(\sum_{i+j=k} A_iA_j + A_k\Big). \tag{39}$$

On the other hand, according to (25), (27), (34), we obtain

$$B_{k+1} \leq \frac{C_0}{k+1}\Big(\frac{3}{2}\sum_{i+j=k} A_iB_j + \frac{1}{2}B_k + A_k\Big). \tag{40}$$

Now, we will prove by induction that the inequalities

$$A_k \leq A_0 M^k, \qquad B_k \leq B_0 M^k \qquad \forall \;\; k \in \mathbb{N}, \tag{41}$$

hold, where M is defined in (37).

For $k = 0$ the inequalities (41) hold. Moreover, we suppose that they are verified for every $k \leq n$, and substitue the estimates of A_k and B_k in (39) and (40) respectively, we get:

$$A_{k+1} \leq \frac{1}{k+1}\frac{3C_0}{2}A_0 M^k((k+1)A_0 + 1),$$

$$B_{k+1} \leq \frac{C_0}{k+1}B_0 M^k(\frac{3}{2}(k+1)A_0 + \frac{1}{2} + \frac{A_0}{B_0}),$$

which means that

$$A_{n+1} \leq A_0 M^{n+1}, \qquad B_{n+1} \leq B_0 M^{n+1}.$$

We conclude that the relation (41) is satisfied for every k.

The proved inequalities (41) imply that

$$\sum_{k=0}^{\infty} |a_k(m,\tilde{t})| s^k \leq \sum_{k=0}^{\infty} B_0 M^k s^k \qquad \forall \;\; (m,\tilde{t}) \in \mathbb{R}_+ \times \mathbb{R},$$

which means that, if $Ms < 1$, then the formal power-series of the second member of (33) converges absolutely. $\square$

Lemma 3. *Let $\sigma(m,\tilde{t},s)$ be the solution of the problem* (31)-(32) *constructed in lemma 2. Then for $0 \leq s < \frac{1}{M}$ we have:*

$$|\sigma(m,\tilde{t},s)| \leq \frac{B_0}{1-Ms}, \quad \sup_{q\in\mathbb{R}} \int_{\gamma_q} |\{\sigma(\cdot,\cdot,s)\}_q(m)| \mu_\gamma(dm) \leq \frac{A_0}{1-Ms},$$

$$\Big|\frac{\partial\sigma(m,\tilde{t},s)}{\partial s}\Big| \leq \frac{B_0 M}{(1-Ms)^2}, \quad \sup_{q\in\mathbb{R}} \int_{\gamma_q} \Big|\{\frac{\partial\sigma(\cdot,\cdot,s)}{\partial s}\}_q(m)\Big| \mu_\gamma(dm) \leq \frac{A_0 M}{(1-Ms)^2},$$

$$\Big|\frac{\partial^2\sigma(m,\tilde{t},s)}{\partial s^2}\Big| \leq \frac{2B_0 M^2}{(1-Ms)^3}, \quad \sup_{q\in\mathbb{R}} \int_{\gamma_q} \Big|\{\frac{\partial^2\sigma(\cdot,\cdot,s)}{\partial s^2}\}_q(m)\Big| \mu_\gamma(dm) \leq \frac{2A_0 M^2}{(1-Ms)^3}.$$

Proof. These inequalities result from (33), (38), (41) and elementary calculus. $\square$

Lemma 4. *Let $\sigma(m,\tilde{t},s)$ be the solution of the problem (31)-(32) constructed in lemma 2. If*

$$\overline{\sigma}_\nu(m,\tilde{t}) \geq 0 \qquad \forall (m,\tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}, \tag{42}$$

then we have

$$\sigma(m,\tilde{t},s) \geq 0 \qquad \textit{for} \;\; 0 \leq s < \frac{1}{M}.$$

Proof. The lemma is proved in a similar way to Lemma 2 of [9]. Indeed, we choose a number $\tau \in]0, \frac{1}{M}[$; in the following (see (52)) we will impose a further restriction on τ. We will construct an approximation $G_n(m,\tilde{t},s)$ ($n \in \mathbb{N}$) of $\sigma(m,\tilde{t},s)$ in the interval $0 \leq s < \tau$, putting

$$G_n(m,\tilde{t},s) = g_{k\,n}(m,\tilde{t}) \qquad \text{for} \;\; \frac{k\tau}{n} \leq s < \frac{(k+1)\tau}{n}, \quad k = 0, 1, \cdots, n-1, \tag{43}$$

$$g_{0\,n}(m,\tilde{t}) = \sigma(m,\tilde{t},0) = \overline{\sigma}_\nu(m,\tilde{t}), \tag{44}$$

$$g_{k+1\,n}(m,\tilde{t}) = g_{k\,n}(m,\tilde{t}) + \frac{\tau}{n}\frac{m}{w(m)}\big(K_{\gamma_q}[g_{k\,n}, g_{k\,n}](m,\tilde{t}) + L_{\gamma_q}[g_{k\,n}](m,\tilde{t})\big). \tag{45}$$

We put

$$T_{k\,n} = \sup_{q \in \mathbb{R}} \int_{\gamma_q} |\{g_{k\,n}\}_q(m)| \mu_\gamma(dm), \qquad L_{k\,n} = \sup_{(m,\tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}} |g_{k\,n}(m,\tilde{t})|, \tag{46}$$

from (35)-(36) we have

$$T_{0\,n} = A_0, \qquad L_{0\,n} = B_0.$$

On the other hand, according to (45) and the inequalities (25)-(28) we have

$$T_{k+1\,n} \leq \big(1 + \frac{\tau}{n}\frac{3C_0}{2}\big)T_{k\,n} + \frac{\tau}{n}\frac{3C_0}{2}T_{k\,n}^2,$$

$$L_{k+1\,n} \leq \big(1 + \frac{\tau}{n}\frac{C_0}{2}\big)L_{k\,n} + \frac{\tau}{n}\frac{3C_0}{2}L_{k\,n}T_{k\,n} + \frac{\tau}{n}C_0 T_{k\,n}.$$

In particular, if we put

$$\Lambda_{k\,n} = \max(T_{k\,n}, L_{k\,n}), \tag{47}$$

we get

$$\Lambda_{0\,n} = \max(A_0, B_0), \tag{48}$$

$$\Lambda_{k+1\,n} \leq \big(1 + \frac{\tau}{n}\frac{3C_0}{2}\big)\Lambda_{k\,n} + \frac{\tau}{n}\frac{3C_0}{2}\Lambda_{k\,n}^2. \tag{49}$$

Let

$$a = 1 + \frac{\tau}{n}\frac{3C_0}{2}, \qquad \lambda_{k\,n} = \frac{1}{a}\frac{\tau}{n}\frac{3C_0}{2}\Lambda_{k\,n}, \tag{50}$$

so, we have

$$\lambda_{k+1\,n} \le a\lambda_{k\,n}(1+\lambda_{k\,n}),$$

or, if we define the function $h(x) = ax(1+x)$,

$$\lambda_{k+1\,n} \le h(\lambda_{k\,n}),$$

and, in the following,

$$\lambda_{k\,n} \le h^{(k)}(\lambda_{0\,n}) \le h^{(n)}(\lambda_{0\,n}).$$

Now, it is not difficult to see, by induction on $k = 1, 2, \dots$ that

$$0 < h^{(k)}(x) \le \frac{a^k x}{1 - \frac{a^k-1}{a-1}x}, \qquad k = 1, 2, \dots$$

provided that $\frac{a^k-1}{a-1}x < 1$. So we have

$$\lambda_{k\,n} \le \frac{a^n\lambda_{0\,n}}{1 - \frac{a^n-1}{a-1}\lambda_{0\,n}}, \qquad k = 0, 1, \cdots, n,$$

provided that $\frac{a^n-1}{a-1}\lambda_{0\,n} < 1$. As

$$a^n = \big(1 + \frac{\tau}{n}\frac{3C_0}{2}\big)^n \le e^{\frac{3\tau C_0}{2}},$$

returning to the expression of $\Lambda_{k\,n}$ (see (50)) and taking into account (47)-(48) and from the expression of a (see (50)), we have

$$\max(T_{k\,n}, L_{k\,n}) \le \frac{e^{\frac{3\tau C_0}{2}}\max(A_0, B_0)}{1 - (e^{\frac{3\tau C_0}{2}} - 1)\max(A_0, B_0)}, \tag{51}$$

provided that

$$\tau < \frac{2}{3C_0}\log\big(1 + \frac{1}{\max(A_0, B_0)}\big), \tag{52}$$

we also note that (52) ensures the condition $\frac{a^n-1}{a-1}\lambda_{0\,n} < 1$.
The inequality (51) (see also (46)) implies that the functions $g_{k\,n}(m, \tilde{t})$ are bounded and integrable on all the curves γ_q. Furthermore, if we recall the formulas (44)-(45) which defined the functions $g_{k\,n}(m, \tilde{t})$, we can see that they are continuous in $(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}$.

On the other hand, recalling the explicit expressions of the operators $K_{\gamma_q}[\cdot,\cdot]$ and $L_{\gamma_q}[\cdot]$ (see (19)-(20)), the definition of functions $g_{k\,n}(m,\tilde{t})$ (see (44)-(45)) and the conditions (3)-(4), imply that, if $g_{k\,n}(m,\tilde{t}) \geq 0$, $\forall(m,\tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}$, then

$$g_{k+1\,n}(m,\tilde{t}) \geq g_{k\,n}(m,\tilde{t})\Big(1 - \frac{\tau}{n}\frac{m}{w(m)}\Big[\int_{\gamma_{q(m,\tilde{t})}} \beta(m,m')\{g_{k\,n}\}_q(m')\mu_\gamma(dm') + \\ + \frac{1}{2}\int_{\gamma^{[0,m]}_{q(m,\tilde{t})}} \vartheta(m-m',m')\mu_\gamma(dm')\Big]\Big).$$

Taking into account the relations (9), (10), (46) and (51), we see that, if n is sufficiently large, then $g_{k+1\,n}(m,\tilde{t}) \geq 0$, which means that $g_{k\,n}(m,\tilde{t}) \geq 0$, $\forall(m,\tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}$, $\forall k = 0,1,\cdots,n$, in other terms if n is sufficiently large then

$$G_n(m,\tilde{t},s) \geq 0 \qquad \forall(m,\tilde{t},s) \in \mathbb{R}_+ \times \mathbb{R} \times [0,\tau]. \tag{53}$$

Now we examine the difference

$$\sigma(m,\tilde{t},s) - G_n(m,\tilde{t},s)$$

in the interval $[0,\tau]$. For this, we pose

$$\alpha_k = \sup_{q\in\mathbb{R},\, \frac{k\tau}{n}\leq s\leq\frac{(k+1)\tau}{n}} \int_{\gamma_q} |\{\sigma(\cdot,\cdot,s) - G_n(\cdot,\cdot,s)\}_q(m)|\mu_\gamma(dm) = \tag{54}$$

$$= \sup_{q\in\mathbb{R},\, \frac{k\tau}{n}\leq s\leq\frac{(k+1)\tau}{n}} \int_{\gamma_q} |\{\sigma(\cdot,\cdot,s) - g_{k\,n}\}_q(m)|\mu_\gamma(dm),$$

$$\beta_k = \sup_{(m,\tilde{t})\in\mathbb{R}_+\times\mathbb{R},\, \frac{k\tau}{n}\leq s\leq\frac{(k+1)\tau}{n}} |\sigma(m,\tilde{t},s) - G_n(m,\tilde{t},s)| = \tag{55}$$

$$= \sup_{(m,\tilde{t})\in\mathbb{R}_+\times\mathbb{R},\, \frac{k\tau}{n}\leq s\leq\frac{(k+1)\tau}{n}} |\sigma(m,\tilde{t},s) - g_{k\,n}(m,\tilde{t})|.$$

Substituting (45) in the difference $\sigma(m,\tilde{t},s) - g_{k\,n}(m,\tilde{t})$, and by adding $0 = -\sigma(m,\tilde{t},s-\frac{\tau}{n}) + \sigma(m,\tilde{t},s-\frac{\tau}{n})$, we have

$$\sigma(m,\tilde{t},s) - g_{k\,n}(m,\tilde{t}) = \sigma(m,\tilde{t},s) - \sigma(m,\tilde{t},s-\frac{\tau}{n}) + \sigma(m,\tilde{t},s-\frac{\tau}{n}) - g_{k-1\,n}(m,\tilde{t}) + \\ - \frac{\tau}{n}\frac{m}{w(m)}\big(K_{\gamma_q}[g_{k-1\,n},g_{k-1\,n}](m,\tilde{t}) + L_{\gamma_q}[g_{k-1\,n}](m,\tilde{t})\big). \tag{56}$$

As

$$\sigma(m,\tilde{t},s)-\sigma(m,\tilde{t},s-\frac{\tau}{n})=\frac{\tau}{n}\frac{\partial\sigma(m,\tilde{t},s-\frac{\tau}{n})}{\partial s}+\frac{1}{2}\frac{\tau^2}{n^2}\frac{\partial^2\sigma(m,\tilde{t},s-\delta_1)}{\partial s^2}$$

with $0\le\delta_1\le\frac{\tau}{n}$.

By substituting the expression (31) and using the symmetric propriety of $K_{\gamma_q}[\varphi,\psi]$ (see (19)) and the linearity of the operator $L_{\gamma_q}[\varphi]$ (see (20)), by (56) we deduce that

$$|\sigma(m,\tilde{t},s)-g_{k\,n}(m,\tilde{t})|\le|\sigma(m,\tilde{t},s-\frac{\tau}{n})-g_{k-1\,n}(m,\tilde{t})|+ \tag{57}$$

$$+\frac{\tau}{n}\frac{m}{w(m)}|K_{\gamma_q}[\sigma(\cdot,\cdot,s-\frac{\tau}{n})+g_{k-1\,n},\ \sigma(\cdot,\cdot,s-\frac{\tau}{n})-g_{k-1\,n}](m,\tilde{t})|+$$

$$+\frac{\tau}{n}\frac{m}{w(m)}|L_{\gamma_q}[\sigma(\cdot,\cdot,s-\frac{\tau}{n})-g_{k-1\,n}](m,\tilde{t})|+\frac{1}{2}\frac{\tau^2}{n^2}\Big|\frac{\partial^2\sigma(m,\tilde{t},s-\delta_1)}{\partial s^2}\Big|.$$

As $0\le s\le\tau$, according to Lemma 3 and from (51) (see also (46)) the terms

$$\int_{\gamma_q}|\{\sigma(\cdot,\cdot,s-\frac{\tau}{n})+g_{k-1\,n}\}_q(m)|\mu_\gamma(dm),\ \ \frac{1}{2}\int_{\gamma_q}\Big|\Big\{\frac{\partial^2\sigma(\cdot,\cdot,s-\delta_1)}{\partial s^2}\Big\}_q(m)\Big|\mu_\gamma(dm)$$

are uniformly bounded by some constant, that we denote by C_1, and according to (26), (28) (see also (54)), we deduce from (57) that

$$\alpha_k\le\big(1+\frac{\tau}{n}\frac{3C_0}{2}(1+C_1)\big)\alpha_{k-1}+\frac{\tau^2}{n^2}C_1. \tag{58}$$

In a similar way, majoring the terms

$$|\sigma(m,\tilde{t},s-\frac{\tau}{n})+g_{k-1\,n}(m,\tilde{t})|,\qquad \frac{1}{2}\Big|\frac{\partial^2\sigma(m,\tilde{t},s-\delta_1)}{\partial s^2}\Big|$$

through a constant, that we denote by C_2, and taking into account (25), (27) (see also (55)), we have

$$\beta_k\le\big(1+\frac{\tau}{n}C_0(\frac{3C_1}{4}+\frac{1}{2})\big)\beta_{k-1}+\frac{\tau}{n}C_0(\frac{3C_2}{4}+1)\alpha_{k-1}+\frac{\tau^2}{n^2}C_2. \tag{59}$$

If we put

$$\zeta_k=\max(\alpha_k,\beta_k),\qquad C_3=\max(C_1,C_2), \tag{60}$$

then from (58)-(59) we deduce that

$$\zeta_k \leq \big(1+\frac{\tau}{n}\frac{3C_0}{2}(1+C_3)\big)\zeta_{k-1}+\frac{\tau^2}{n^2}C_3. \tag{61}$$

By repeating the application of the inequality (61), we obtain

$$\max_{k=1,\cdots,n-1}\zeta_k \leq \big(1+\frac{\tau}{n}\frac{3C_0}{2}(1+C_3)\big)^{n-1}\zeta_0+\frac{\tau^2}{n^2}C_3\sum_{k=0}^{n-2}\big(1+\frac{\tau}{n}\frac{3C_0}{2}(1+C_3)\big)^k \leq \tag{62}$$

$$\leq e^{\tau\frac{3C_0}{2}(1+C_3)}\max(\alpha_0,\beta_0)+\frac{\tau}{n}C_3\frac{e^{\tau\frac{3C_0}{2}(1+C_3)}-1}{\frac{3C_0}{2}(1+C_3)}.$$

As for α_0 and β_0, from (44), (54), (55) we deduce that

$$\alpha_0 \leq \frac{\tau}{n}\sup_{q\in\mathbb{R},0\leq s\leq\frac{\tau}{n}}\int_{\gamma_q}\big|\big\{\frac{\partial\sigma(\cdot,\cdot,s)}{\partial s}\big\}_q(m)\big|\mu_\gamma(dm),$$

$$\beta_0 \leq \frac{\tau}{n}\sup_{(m,\tilde{t})\in\mathbb{R}_+\times\mathbb{R},0\leq s\leq\frac{\tau}{n}}\big|\frac{\partial\sigma(m,\tilde{t},s)}{\partial s}\big|.$$

Therefore, according to lemma 3 there is a constant C_4 such as

$$\max(\alpha_0,\beta_0)\leq C_4\frac{\tau}{n},$$

that enables us to deduce from (62),

$$\max_{k=0,1,\cdots,n-1}\big[\max(\alpha_k,\beta_k)\big]\leq\frac{\tau}{n}\Big[e^{\tau\frac{3C_0}{2}(1+C_3)}C_4+C_3\frac{e^{\tau\frac{3C_0}{2}(1+C_3)}-1}{\frac{3C_0}{2}(1+C_3)}\Big]. \tag{63}$$

Recalling (55), we see that (63) implies that, for $0\leq s\leq\tau$, $G_n(m,\tilde{t},s)$ converges uniformly to $\sigma(m,\tilde{t},s)$. Therefore, according to (53), we have $\sigma(m,\tilde{t},s)\geq 0\ \forall(m,\tilde{t},s)\in\mathbb{R}_+\times\mathbb{R}\times[0,\tau]$.

The non-negativity of $\sigma(m,\tilde{t},s)$ in $[0,\tau]$ being proved, we construct $[\tau_1,\tau_2]$ (we take $\tau_1=0$, $\tau_2=\tau$) and, by repeating the procedure, to get the successive intervals $[\tau_n,\tau_{n+1}]$, $n=1,2,\cdots$. In a similar way to (52), which gives the restriction of the choice of τ, we can take τ_{n+1} such that

$$\tau_{n+1}-\tau_n<\frac{2}{3C_0}\log\big(1+\frac{1}{\max(A_0^{[n]},B_0^{[n]})}\big),$$

where

$$A_0^{[n]} = \sup_{q\in\mathbb{R}} \int_{\gamma_q} |\{\sigma(\cdot,\cdot,\tau_n)\}_q(m)|\mu_\gamma(dm), \qquad B_0^{[n]} = \sup_{(m,\tilde{t})\in\mathbb{R}_+\times\mathbb{R}} |\sigma(m,\tilde{t},\tau_n)|.$$

The previous Lemma 3 implies that we can construct a sequence of intervals $[\tau_n, \tau_{n+1}]$, $n = 0, 1, \cdots$, such that

$$[0, \frac{1}{M}[\ \subset \bigcup_{n\in\mathbb{N}} [\tau_n, \tau_{n+1}],$$

which completes the proof of Lemma.

To summarize things up, we proved the existence of a solution in the interval $[0, \frac{1}{M}[$, solution which is analytic in s, non-negative, continuous, bounded and integrable on each $\gamma_q = \gamma_{q\,\bar{s}_\nu}$, $q \in \mathbb{R}$.

5 Global solution of the approximate equation

Being established the existence of a local solution, now we will prove that we can extend it on the interval $[0, \infty[$.

Proposition 1. *Under the conditions of the lemma 2 and 4 the problem* (31)-(32) *admits, in the interval* $[0, \infty[$, *a solution* $\sigma(m, \tilde{t}, s)$, *which is analytic in s, continuous, non-negative and integrable on each curve* $\gamma_q = \gamma_{q\,\bar{s}_\nu}$.

Proof. the proposition 1 is proved in a similar way to lemma 3 of [9]. More precisely, the first interval is considered $[0, D_1]$ with $D_1 = \frac{1}{2M}$, $M = C_0\big(\frac{3}{2}(A_0+1)+\frac{A_0}{B_0}\big)$ (see (37)), then successively the intervals $[D_n, D_{n+1}]$ with

$$D_{n+1} - D_n = \frac{1}{C_0(3(A(D_n)+1) + 2\frac{A(D_n)}{B(D_n)})}, \tag{64}$$

where

$$A(s) = \sup_{q\in\mathbb{R}} \int_{\gamma_q} |\{\sigma(\cdot,\cdot,s)\}_q(m)|\mu_\gamma(dm), \qquad B(s) = \sup_{(m,\tilde{t})\in\mathbb{R}_+\times\mathbb{R}} |\sigma(m,\tilde{t},s)|.$$

The lemmas 2, 3 and 4, reformulated with the initial data $\sigma(m, \tilde{t}, D_n)$, give the solution in the interval $[D_n, D_{n+1}]$.

We return to equation (31) and integrate it on γ_q. To examine the term $\int_{\gamma_q} \frac{m}{w(m)}\{(K_{\gamma_q}[\sigma(\cdot,\cdot,s),\sigma(\cdot,\cdot,s)]\}_q(m)\mu_\gamma(dm)$ (recall the expression (19)), we note that

$$\int_{\gamma_q} \frac{1}{2}\frac{m}{w(m)} \int_{\gamma_q^{[0,m]}} \beta(m-m', m')\{\sigma(\cdot,\cdot,s)\}_q(m-m')\{\sigma(\cdot,\cdot,s)\}_q(m')$$

$$\mu_\gamma(dm')\mu_\gamma(dm) =$$

$$=\int_{\gamma_q}\int_{\gamma_q}\frac{1}{2}\frac{m+m'}{w(m+m')}\beta(m,m')\{\sigma(\cdot,\cdot,s)\}_q(m)\{\sigma(\cdot,\cdot,s)\}_q(m')\mu_\gamma(dm)\mu_\gamma(dm').$$

Therefore, according to the symmetry of the function $\beta(m, m')$, the condition (7), and the non-negativity of $\sigma(m, \tilde{t}, s)$, we have

$$\int_{\gamma_q}\frac{m}{w(m)}\big\{K_{\gamma_q}[\sigma(\cdot,\cdot,s),\sigma(\cdot,\cdot,s)]\big\}_q(m)\mu_\gamma(dm){=}\int_{\gamma_q}\int_{\gamma_q}\Big(\frac{m}{w(m+m')}-\frac{m}{w(m)}\Big)\times$$

$$\times\beta(m,m')\{\sigma(\cdot,\cdot,s)\}_q(m)\{\sigma(\cdot,\cdot,s)\}_q(m')\mu_\gamma(dm')\mu_\gamma(dm)\le 0.$$

On one hand, similarly to the proof of (28) (see in particular (29)), and taking account the sign of each term, we deduce from the expression of (20) that

$$\int_{\gamma_q}\frac{m}{w(m)}\big\{L_{\gamma_q}[\sigma(\cdot,\cdot,s)]\big\}_q(m)\mu_\gamma(dm)\le C_0\int_{\gamma_q}\{\sigma(\cdot,\cdot,s)\}_q(m)\mu_\gamma(dm).$$

Using these inequalities, from the integral form of (31), we obtain

$$A(s)=\sup_{q\in\mathbb{R}}\int_{\gamma_q}\{\sigma(\cdot,\cdot,s)\}_q(m)\mu_\gamma(dm)\le$$

$$\le\sup_{q\in\mathbb{R}}\int_{\gamma_q}\{\sigma(\cdot,\cdot,0)\}_q(m)\mu_\gamma(dm)+C_0\int_0^s\sup_{q\in\mathbb{R}}\int_{\gamma_q}\{\sigma(\cdot,\cdot,s')\}_q(m)\mu_\gamma(dm)ds',$$

from where it results that

$$A(s)\le A(0)e^{C_0 s}. \tag{65}$$

On the other hand, according to (19), (20) and from the non-negativity of $\sigma(m, \tilde{t}, s)$, we deduce from (31) that

$$\frac{\partial}{\partial s}\sigma(m,\tilde{t},s)\ge-\sigma(m,\tilde{t},s)\Big[\int_{\gamma_q}\frac{m}{w(m)}\beta(m,m')\{\sigma(\cdot,\cdot,s)\}_q(m')\mu_\gamma(dm')+$$

$$+\frac{1}{2}\int_{\gamma_q^{[0,m]}}\frac{m}{w(m)}\vartheta(m-m',m')\mu_\gamma(dm')\Big],$$

from where, according to conditions (9), (10), we obtain

$$B(s)\ge B(0)-C_0\int_0^s B(s')(A(s')+\frac{1}{2})ds',$$

therefore

$$B(s)\ge B(0)e^{-C_0\int_0^s(A(s')+\frac{1}{2})ds'}. \tag{66}$$

The relations (64)-(66) implies that the sequence $\{D_n\}_{n=0}^{\infty}$ can't converge to a finite value, i.e. it is necessary that $\lim_{n\to\infty} D_n = \infty$. □

Proposition 2. *Under the same hypothesis of the proposition 1, the solution of the problem* (31)-(32) *is unique in the class* Φ *which satisfy the conditions:*
i) $\varphi(m, \tilde{t}, s)$ *is continuous in* $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$,
ii) $\varphi(m, \tilde{t}, s)$ *is integrable on each curve* γ_q $(q \in \mathbb{R})$,
iii) for all $\overline{s}_1 \in [0, \infty[$, *we have* $\sup_{q\in\mathbb{R}, s\in[0,\overline{s}_1]} \int_{\gamma_q} |\{\varphi(\cdot,\cdot,s\}_q(m)|\mu_\gamma(dm) < \infty$.

Proof. Let φ_1 and φ_2 two solutions of the problem (31)-(32) belonging to the class Φ. As $\varphi_1(m, \tilde{t}, 0) - \varphi_2(m, \tilde{t}, 0) = 0$, using the symmetry of the operator $K_{\gamma_q}[\varphi, \psi]$ and the linearity of $L_{\gamma_q}[\varphi]$, we have

$$\varphi_1(m, \tilde{t}, s) - \varphi_2(m, \tilde{t}, s) =$$

$$= \int_0^s \frac{m}{w(m)} (K_{\gamma_q}[\varphi_1(\cdot,\cdot,s') + \varphi_2(\cdot,\cdot,s'), \varphi_1(\cdot,\cdot,s') - \varphi_2(\cdot,\cdot,s')] +$$
$$+ L_{\gamma_q}[\varphi_1(\cdot,\cdot,s) - \varphi_2(\cdot,\cdot,s)])ds'.$$

Therefore, from (26) and (28) we have

$$\int_{\gamma_q} |\varphi_1(m, \tilde{t}, s) - \varphi_2(m, \tilde{t}, s)| \leq \tag{67}$$

$$\leq \frac{3C_0}{2} \int_0^s \sup_{q\in\mathbb{R}} \int_{\gamma_q} |\{\varphi_1(\cdot,\cdot,s') + \varphi_2(\cdot,\cdot,s')\}_q(m)|\mu_\gamma(dm) \times$$
$$\times \sup_{q\in\mathbb{R}} \int_{\gamma_q} |\{\varphi_1(\cdot,\cdot,s') - \varphi_2(\cdot,\cdot,s')\}_q(m)|\mu_\gamma(dm)ds' +$$
$$+ \frac{3C_0}{2} \int_0^s \sup_{q\in\mathbb{R}} \int_{\gamma_q} |\{\varphi_1(\cdot,\cdot,s') - \varphi_2(\cdot,\cdot,s')\}_q(m)|\mu_\gamma(dm)ds'.$$

We choose $\overline{s}_1$ such that $\overline{s}_1 < \infty$. Hence, according to the condition *iii*) we have

$$\sup_{q\in\mathbb{R}, s\in[0,\overline{s}_1]} \int_{\gamma_q} |\{\varphi_1(\cdot,\cdot,s') + \varphi_2(\cdot,\cdot,s')\}_q(m)|\mu_\gamma(dm) \equiv M_1 < \infty. \tag{68}$$

Therefore, if we put

$$g(s) = \sup_{q\in\mathbb{R}} \int_{\gamma_q} |\{\varphi_1(\cdot,\cdot,s) - \varphi_2(\cdot,\cdot,s)\}_q(m)|\mu_\gamma(dm),$$

then it results from (67) that

$$g(s) \leq \frac{3C_0}{2}(M_1 + 1) \int_0^s g(s')ds',$$

which implies that

$$g(s) = 0 \qquad \forall s \in [0, \overline{s}_1].$$

Or, from *iii*), to have the relation (68), we can choose any $\overline{s}_1 < \infty$ (even if M_1 can be different, but always $M_1 < \infty$), so that, by repeating the same reasoning, we can prove $g(s) = 0$ for all $s \in \mathbb{R}_+$, which completes the proof. □

6 Estimates of the approximate solutions

We note that, if $\overline{\sigma}_\nu(m, \tilde{t})$ is continuous in $(m, \tilde{t})$ and satisfies the conditions (35), (36) and (42), then from the propositions 1 and 2, there exists a unique solution $\sigma(m, \tilde{t}, s)$ of the problem (31)-(32) for $\frac{\nu}{N} \leq s < \infty$ (here we return to the initial formulation of the variable s). Let's put $\overline{\sigma}_{\nu+1}(m, \tilde{t}) = \sigma(m, \tilde{t}, \frac{\nu+1}{N})$, it satisfies the conditions (35), (36) and (42), and it is continuous in $(m, \tilde{t})$ so that we can repeat the resolution of the equation for $\frac{\nu+1}{N} \leq s$, with the entry condition $\overline{\sigma}_{\nu+1}(m, \tilde{t}) = \sigma(m, \tilde{t}, \frac{\nu+1}{N})$. Thus, by iterating this procedure on the intervals $[\frac{\nu}{N}, \frac{\nu+1}{N}]$ for $\nu = 0, 1, 2, \cdots$, we construct on $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ the solution of the equation (24) with the entry condition (22); we indicate this solution by $\sigma^{[N]}(m, \tilde{t}, s)$. It is useful to recall that this last solution is bounded, continuous in $(m, \tilde{t}, s)$ and non-negative.

To solve the problem (21)-(22) in the field $\mathbb{R}_+ \times \mathbb{R} \times [0, \overline{s}]$ with $\overline{s} > 0$, we suppose that $w(m)$ satisfies the additional condition

$$0 < \frac{1}{w(m)} \leq \sup_{m \in \mathbb{R}_+} \frac{1}{w(m)} \equiv \overline{C}_w < \infty \tag{69}$$

and that $\overline{\sigma}_0(m, \tilde{t})$ satisfies the condition (23), and the following ones

$$\int_0^\infty \sup_{\tilde{t} \in \mathbb{R}} \overline{\sigma}_0(m, \tilde{t}) dm \equiv \overline{\omega}_0 < \infty, \tag{70}$$

$$\sup_{m \in \mathbb{R}_+, \tilde{t}_1, \tilde{t}_2 \in \mathbb{R}, \tilde{t}_1 \neq \tilde{t}_2} \frac{|\overline{\sigma}_0(m, \tilde{t}_1) - \overline{\sigma}_0(m, \tilde{t}_2)|}{|\tilde{t}_1 - \tilde{t}_2|} \equiv \overline{\lambda}_0 < \infty, \tag{71}$$

$$\int_0^\infty \sup_{\tilde{t}_1, \tilde{t}_2 \in \mathbb{R}, \tilde{t}_1 \neq \tilde{t}_2} \frac{|\overline{\sigma}_0(m, \tilde{t}_1) - \overline{\sigma}_0(m, \tilde{t}_2)|}{|\tilde{t}_1 - \tilde{t}_2|} dm \equiv \overline{J}_0 < \infty. \tag{72}$$

In this paragraph, we are interested by some estimates for the values of $\omega^{[N]}(s)$, $\psi^{[N]}(s)$, $J^{[N]}(s)$ and $\lambda^{[N]}(s)$ defined by:

$$\omega^{[N]}(s) = \int_0^\infty u^{[N]}(m, s) dm, \qquad u^{[N]}(m, s) = \sup_{\tilde{t} \in \mathbb{R}} \sigma^{[N]}(m, \tilde{t}, s), \tag{73}$$

$$\psi^{[N]}(s) = \sup_{(m,\tilde{t})\in\mathbb{R}_+\times\mathbb{R}} \sigma^{[N]}(m,\tilde{t},s) = \sup_{m\in\mathbb{R}_+} u^{[N]}(m,s), \tag{74}$$

$$J^{[N]}(s) = \int_0^\infty j^{[N]}(m,s)dm, \tag{75}$$

$$j^{[N]}(m,s) = \sup_{\tilde{t}_1,\tilde{t}_2\in\mathbb{R},\tilde{t}_1\neq\tilde{t}_2} \frac{|\sigma^{[N]}(m,\tilde{t}_1,s) - \sigma^{[N]}(m,\tilde{t}_2,s)|}{|\tilde{t}_1-\tilde{t}_2|},$$

$$\lambda^{[N]}(s) = \sup_{m\in\mathbb{R}_+,\tilde{t}_1,\tilde{t}_2\in\mathbb{R},\tilde{t}_1\neq\tilde{t}_2} \frac{|\sigma^{[N]}(m,\tilde{t}_1,s) - \sigma^{[N]}(m,\tilde{t}_2,s)|}{|\tilde{t}_1-\tilde{t}_2|} = \sup_{m\in\mathbb{R}_+} j^{[N]}(m,s). \tag{76}$$

Lemma 5. *For all $N \in \mathbb{N}\backslash\{0\}$, we have*

$$\omega^{[N]}(s) \leq \overline{\omega}(s), \qquad \forall s \in [0, S_1[, \tag{77}$$

where

$$\overline{\omega}(s) = \frac{1}{(\frac{1}{\overline{\omega}_0}+\frac{1}{2})e^{-C_0 s} - \frac{1}{2}}, \qquad S_1 = \frac{1}{C_0}\log\big(\frac{2}{\overline{\omega}_0}+1\big). \tag{78}$$

Proof. As $\sigma^{[N]}(m,\tilde{t},s) \geq 0$, using (19), (20), we deduce from (24) that for $\frac{\nu}{N} \equiv \overline{s}_\nu < s < \frac{\nu+1}{N}$ we have

$$\frac{\partial}{\partial s}\sigma^{[N]}(m,\tilde{t},s) \leq$$

$$\leq \frac{1}{2}\frac{m}{w(m)}\int_{\gamma_{q\,\overline{s}_\nu}^{[0,m]}} \beta(m-m',m')\{\sigma^{[N]}(\cdot,\cdot,s)\}_{q\,\overline{s}_\nu}(m-m')\{\sigma^{[N]}(\cdot,\cdot,s)\}_{q\,\overline{s}_\nu}(m')$$

$$\mu_\gamma(dm') + \frac{m}{w(m)}\int_{\gamma_{q\,\overline{s}_\nu}} \vartheta(m,m')\{\sigma^{[N]}(\cdot,\cdot,s)\}_{q\,\overline{s}_\nu}(m+m')\mu_\gamma(dm').$$

We deduce from it that

$$\frac{\partial}{\partial s}\sigma^{[N]}(m,\tilde{t},s) \leq \frac{1}{2}\frac{m}{w(m)}\int_0^m \beta(m-m',m')u^{[N]}(m-m',s)u^{[N]}(m',s)dm' +$$

$$+\frac{m}{w(m)}\int_0^\infty \vartheta(m,m')u^{[N]}(m+m',s)dm', \qquad \forall s \geq 0, s \neq \overline{s}_\nu, \nu \in \mathbb{N},$$

which, joined with the continuity of $\sigma^{[N]}(m,\tilde{t},s)$, leads to

$$\omega^{[N]}(s) \leq \overline{\omega}_0 +$$

$$+\frac{1}{2}\int_0^s\int_0^\infty \frac{m}{w(m)}\int_0^m \beta(m-m',m')u^{[N]}(m-m',s')u^{[N]}(m',s')dm'dmds' +$$

$$+\int_0^s\int_0^\infty \frac{m}{w(m)}\int_0^\infty \vartheta(m,m')u^{[N]}(m+m',s')dm'dmds'.$$

Finally, with conditions (8), (11), (29) and from the convolution property, we deduce that

$$\omega^{[N]}(s) \le \overline{\omega}_0 + \frac{C_0}{2}\int_0^s (\omega^{[N]}(s'))^2 ds' + C_0\int_0^s \omega^{[N]}(s')ds'. \tag{79}$$

On the other hand, we see immediately that the function

$$\overline{\omega}(s) = \frac{1}{(\frac{1}{\overline{\omega}_0}+\frac{1}{2})e^{-C_0 s}-\frac{1}{2}}$$

is the solution of the Cauchy problem

$$\frac{d}{ds}\overline{\omega}(s) = \frac{C_0}{2}(\overline{\omega}(s))^2 + C_0\overline{\omega}(s), \qquad \overline{\omega}(0)=\overline{\omega}_0 \tag{80}$$

and that its maximum interval of existence is $[0, S_1[$ with S_1 given in (78). We get (77) by comparing (79) and (80). □

Lemma 6. *For all $N \in \mathbb{N}\backslash\{0\}$, we have*

$$\psi^{[N]}(s) \le \overline{\psi}(s) \qquad \textit{for} \;\; 0 \le s < S_1, \tag{81}$$

where $\overline{\psi}(s)$ is the solution of the Cauchy problem

$$\frac{d}{ds}\overline{\psi}(s) = \frac{C_0}{2}\big[(3\overline{\omega}(s)+1)\overline{\psi}(s)+2\overline{\omega}(s)\big], \qquad \overline{\psi}(0) = \sup_{(m,\tilde{t})\in\mathbb{R}_+\times\mathbb{R}} \overline{\sigma}_0(m,\tilde{t}). \tag{82}$$

Proof. Applying (25) and (27) to the right side of (24) and by recalling the definitions (73), (74) and (78), we have

$$\psi^{[N]}(s) \le \psi^{[N]}(\frac{\nu}{N}) + \frac{C_0}{2}\int_{\frac{\nu}{N}}^s [(3\overline{\omega}(s')+1)\psi(s')+2\overline{\omega}(s')]ds'$$

for $\frac{\nu}{N} \le s \le \frac{\nu+1}{N}$, $\nu = 0,1,2,\cdots$. This leads, according (23) and by the usual reasoning we obtain (81). □

Lemma 7. *For all $N \in \mathbb{N}\backslash\{0\}$, we have*

$$J^{[N]}(s) \le \overline{J}(s) \qquad \textit{for} \;\; 0 \le s < S_1, \tag{83}$$

where $\overline{J}(s)$ is the solution of the Cauchy problem

$$\frac{d}{ds}\overline{J}(s) = \frac{3C_0}{2}(2\overline{\omega}(s)+1)\overline{J}(s), \qquad \overline{J}(0)=\overline{J}_0. \tag{84}$$

Proof. we consider $\tilde{t}_1, \tilde{t}_2 \in \mathbb{R}$, $\tilde{t}_1 \neq \tilde{t}_2$, $m \in \mathbb{R}_+$, $s \in [\frac{\nu}{N}, \frac{\nu+1}{N}]$. Then, putting

$$\overline{s}_\nu = \frac{\nu}{N}, \qquad q_1 = \tilde{t}_1 + \frac{\overline{s}_\nu}{w(m)}, \qquad q_2 = \tilde{t}_2 + \frac{\overline{s}_\nu}{w(m)},$$

we have

$$|\sigma^{[N]}(m, \tilde{t}_1, s) - \sigma^{[N]}(m, \tilde{t}_2, s)| \leq |\sigma^{[N]}(m, \tilde{t}_1, \frac{\nu}{N}) - \sigma^{[N]}(m, \tilde{t}_2, \frac{\nu}{N})| + \tag{85}$$

$$+ \int_{\frac{\nu}{N}}^{s} \frac{m}{w(m)} |D_K^{[N]}(m, \tilde{t}_1, \tilde{t}_2, s')| ds' + \int_{\frac{\nu}{N}}^{s} \frac{m}{w(m)} |D_L^{[N]}(m, \tilde{t}_1, \tilde{t}_2, s')| ds',$$

where

$$D_K^{[N]}(m, \tilde{t}_1, \tilde{t}_2, s) =$$

$$= K_{\gamma_{q_1 \overline{s}_\nu}}[\sigma^{[N]}(\cdot, \cdot, s), \sigma^{[N]}(\cdot, \cdot, s)](m, \tilde{t}_1) - K_{\gamma_{q_2 \overline{s}_\nu}}[\sigma^{[N]}(\cdot, \cdot, s'), \sigma^{[N]}(\cdot, \cdot, s)](m, \tilde{t}_2),$$

$$D_L^{[N]}(m, \tilde{t}_1, \tilde{t}_2, s) = L_{\gamma_{q_1 \overline{s}_\nu}}[\sigma^{[N]}(\cdot, \cdot, s)](m, \tilde{t}_1) - L_{\gamma_{q_2 \overline{s}_\nu}}[\sigma^{[N]}(\cdot, \cdot, s)](m, \tilde{t}_2).$$

Even if $K_{\gamma_{q_1 \overline{s}_\nu}}[\cdot, \cdot]$ and $K_{\gamma_{q_2 \overline{s}_\nu}}[\cdot, \cdot]$ are defined on two different curves $\gamma_{q_1 \overline{s}_\nu}$ and $\gamma_{q_2 \overline{s}_\nu}$, if we pay attention to the expression of the right hand side of (19), we note that, once definite $\{\sigma^{[N]}(\cdot, \cdot, s)\}_{q_1 \overline{s}_\nu}(m)$ and $\{\sigma^{[N]}(\cdot, \cdot, s)\}_{q_2 \overline{s}_\nu}(m)$ (see (17)), $D_K^{[N]}(m, \tilde{t}_1, \tilde{t}_2, s)$ can be written in the form

$$D_K^{[N]}(m, \tilde{t}_1, \tilde{t}_2, s) = \tag{86}$$

$$= \frac{1}{2} \int_0^m \beta(m - m', m')(\{\sigma^{[N]}\}_{q_1}(m - m') - \{\sigma^{[N]}\}_{q_2}(m - m')) \times$$

$$\times (\{\sigma^{[N]}\}_{q_1}(m') + \{\sigma^{[N]}\}_{q_2}(m')) dm' +$$

$$- \frac{1}{2} (\{\sigma^{[N]}\}_{q_1}(m) - \{\sigma^{[N]}\}_{q_2}(m)) \int_0^\infty \beta(m, m')(\{\sigma^{[N]}\}_{q_1}(m') + \{\sigma^{[N]}\}_{q_2}(m')) dm' +$$

$$- \frac{1}{2} (\{\sigma^{[N]}\}_{q_1}(m) + \{\sigma^{[N]}\}_{q_2}(m)) \int_0^\infty \beta(m, m')(\{\sigma^{[N]}\}_{q_1}(m') - \{\sigma^{[N]}\}_{q_2}(m')) dm',$$

where

$$\{\sigma^{[N]}\}_{q_1}(m) = \{\sigma^{[N]}(\cdot, \cdot, s)\}_{q_1 \overline{s}_\nu}(m), \qquad \{\sigma^{[N]}\}_{q_2}(m) = \{\sigma^{[N]}(\cdot, \cdot, s)\}_{q_2 \overline{s}_\nu}(m).$$

Using (8), (9) and definitions (73), (75), we deduce from (86) that

$$\frac{m}{w(m)} \sup_{\tilde{t}_1, \tilde{t}_2 \in \mathbb{R}, \tilde{t}_1 \neq \tilde{t}_2} \frac{|D_K^{[N]}(m, \tilde{t}_1, \tilde{t}_2, s)|}{|\tilde{t}_1 - \tilde{t}_2|} \leq \tag{87}$$

$$\leq C_0 \int_0^m j^{[N]}(m-m',s)u^{[N]}(m',s)dm' + C_0\omega^{[N]}(s)j^{[N]}(m,s)+$$

$$C_0 u^{[N]}(m,s)J^{[N]}(s).$$

On the other hand, for $D_L^{[N]}(m,\tilde{t}_1,\tilde{t}_2,s)$, from definition (20) we obtain without difficulty

$$\frac{m}{w(m)} \sup_{\tilde{t}_1,\tilde{t}_2\in\mathbb{R},\tilde{t}_1\neq\tilde{t}_2} \frac{|D_L^{[N]}(m,\tilde{t}_1,\tilde{t}_2,s)|}{|\tilde{t}_1-\tilde{t}_2|} \leq \tag{88}$$

$$\leq \frac{C_0}{2} j^{[N]}(m,s) + \frac{m}{w(m)} \int_0^\infty \vartheta(m,m')j^{[N]}(m+m',s)dm'.$$

Using the relation

$$\int_0^\infty \frac{m}{w(m)} \int_0^\infty \vartheta(m,m')j^{[N]}(m+m',s)dm'dm =$$

$$= \int_0^\infty \int_0^{m''} \frac{m''-m'}{w(m''-m')}\vartheta(m''-m',m')j^{[N]}(m'',s)dm'dm''$$

joined with (11), we deduce from the last three estimates and from property of the convolution that

$$J^{[N]}(s) \leq J^{[N]}(\frac{\nu}{N}) + 3C_0\int_{\frac{\nu}{N}}^s J^{[N]}(s')\omega^{[N]}(s')ds' + \frac{3C_0}{2}\int_{\frac{\nu}{N}}^s J^{[N]}(s')ds'.$$

As this inequality has the same form in all intervals $[\frac{\nu}{N},\frac{\nu+1}{N}]$, $\nu = 0, 1, \cdots,$ we obtain

$$J^{[N]}(s) \leq J^{[N]}(0) + 3C_0\int_0^s J^{[N]}(s')\omega^{[N]}(s')ds' + \frac{3C_0}{2}\int_0^s J^{[N]}(s')ds',$$

or, taking into account (77) and from the relation $J^{[N]}(0) = \overline{J}_0$,

$$J^{[N]}(s) \leq \overline{J}_0 + 3C_0\int_0^s J^{[N]}(s')\overline{\omega}(s')ds' + \frac{3C_0}{2}\int_0^s J^{[N]}(s')ds',$$

that implies (83) with (84). □

Lemma 8. *For all $N \in \mathbb{N}\backslash\{0\}$, we have*

$$\lambda^{[N]}(s) \leq \overline{\lambda}(s) \qquad \textit{for} \ \ 0 \leq s < S_1, \tag{89}$$

where $\overline{\lambda}(s)$ is the solution of the Cauchy problem

$$\frac{d}{ds}\overline{\lambda}(s) = C_0(2\overline{\omega}(s)+\frac{1}{2})\overline{\lambda}(s) + C_0(\overline{\psi}(s)+1)\overline{J}(s), \qquad \overline{\lambda}(0) = \overline{\lambda}_0. \tag{90}$$

Proof. Using the relations

$$\sup_{m\in\mathbb{R}_+} u^{[N]}(m,s) = \psi^{[N]}(s) \le \overline{\psi}(s), \qquad \sup_{m\in\mathbb{R}_+} j^{[N]}(m,s) = \lambda^{[N]}(s),$$

$$\omega^{[N]}(s) \le \overline{\omega}(s), \qquad J^{[N]}(s) \le \overline{J}(s),$$

we deduce from (85), (87), (88) and from the property of the convolution that

$$\lambda^{[N]}(s) \le \lambda^{[N]}(\frac{\nu}{N}) + 2C_0 \int_{\frac{\nu}{N}}^{s} \overline{\omega}(s')\lambda^{[N]}(s')ds' +$$

$$+C_0 \int_{\frac{\nu}{N}}^{s} \overline{\psi}(s')\overline{J}(s')ds' + \frac{C_0}{2} \int_{\frac{\nu}{N}}^{s} \lambda^{[N]}(s')ds' + C_0 \int_{\frac{\nu}{N}}^{s} \overline{J}(s')ds'.$$

In a similar way to the proof of the previous lemma, from this inequality we deduce (89) with (90). □

7 Convergence of the approximate solutions

To solve the problem (21)-(22), it is essential to prove the convergence of the approximate solutions $\sigma^{[N]}(m,\tilde{t},s)$. Thus, we will prove the convergence of a subsequence of the approximate solutions in the interval $[0, S_1[$, which will give us the solution of the problem in this interval.

Theorem 1. *We suppose that $\beta(\cdot,\cdot)$, $\vartheta(\cdot,\cdot)$ and $w(\cdot)$, satisfy the conditions mentioned in paragraph 2 and the condition (69) and that $\overline{\sigma}_0(m,\tilde{t})$ is continuous in $(m,\tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}$ and satisfies the conditions (23), (70)-(72). Let S_1 the number given in (78). Then the problem (21)-(22) admits a solution in the interval $[0, S_1[$. Moreover the solution is unique in the class of the functions $\sigma(m,\tilde{t},s)$ which satisfy the conditions*

i) $\sigma(m,\tilde{t},s)$ *is continuous in* $\mathbb{R}_+ \times \mathbb{R} \times [0, S_1[$,

ii) *for all* $s \in [0, S_1[$, $u(m,s) = \sup_{\tilde{t}\in\mathbb{R}} |\sigma(m,\tilde{t},s)|$ *is integrable in* $m \in \mathbb{R}_+$,

iii) *for all* $\overline{s}_1 \in [0, S_1[$, *we have* $\sup_{s\in[0,\overline{s}_1]} \int_0^\infty u(m,s)dm < \infty$.

Proof. We construct the sequence of approximate solutions $\sigma^{[2^n]}$, $n = 1, 2, \cdots$, which are the solutions of the problem (24), (22) with $N = 2^n$. For simplicity, we write σ_n instead of $\sigma^{[2^n]}$. In the interval $[\frac{\nu}{2^n}, \frac{2\nu+1}{2^{n+1}}[$ the approximate solutions σ_n and σ_{n+1} are defined by integral operators on the same curves $\gamma_{q\,\overline{s}_1}$, $\overline{s}_1 = \frac{\nu}{2^n}$, while in $[\frac{2\nu+1}{2^{n+1}}, \frac{\nu+1}{2^n}[$ the approximate solutions σ_n and σ_{n+1} are defined on the different curves $\gamma_{q\,\overline{s}_1}$, $\gamma_{q\,\overline{s}_2}$, $\overline{s}_2 = \frac{2\nu+1}{2^{n+1}}$ respectively.

We put

$$\eta_n(m,s) = \sup_{\tilde{t}\in\mathbb{R}} |\sigma_n(m,\tilde{t},s) - \sigma_{n+1}(m,\tilde{t},s)|, \tag{91}$$

$$\overline{\alpha}_n(s) = \int_0^\infty \eta_n(m,s)dm, \tag{92}$$

$$\overline{\beta}_n(s) = \sup_{(m,\tilde{t})\in\mathbb{R}_+\times\mathbb{R}} |\sigma_n(m,\tilde{t},s) - \sigma_{n+1}(m,\tilde{t},s)| = \sup_{m\in\mathbb{R}_+} \eta_n(m,s). \tag{93}$$

We will also write $u_n(m,s)$, $j_n(m,s)$, $\omega_n(s)$, $\psi_n(s)$, $J_n(s)$, $\lambda_n(s)$ instead of $u^{[2^n]}(m,s)$, $j^{[2^n]}(m,s)$, $\omega^{[2^n]}(s)$, $\psi^{[2^n]}(s)$, $J^{[2^n]}(s)$, $\lambda^{[2^n]}(s)$ (see (73)-(76)).

We recall that, for all $q\in\mathbb{R}$ and $\overline{s}\geq 0$, the definition of the operator $K_{\gamma_q\overline{s}}[\cdot,\cdot]$ gives us

$$K_{\gamma_q\overline{s}}[\sigma_n(\cdot,\cdot,s'),\sigma_n(\cdot,\cdot,s')](m,\tilde{t}) - K_{\gamma_q\overline{s}}[\sigma_{n+1}(\cdot,\cdot,s'),\sigma_{n+1}(\cdot,\cdot,s')](m,\tilde{t}) =$$

$$= K_{\gamma_q\overline{s}}[\sigma_n(\cdot,\cdot,s')+\sigma_{n+1}(\cdot,\cdot,s'),\ \sigma_n(\cdot,\cdot,s')-\sigma_{n+1}(\cdot,\cdot,s')](m,\tilde{t}).$$

Therefore, with the linearity of the operator $L_{\gamma_q\overline{s}}[\varphi]$, we get

$$\sigma_n(m,\tilde{t},s) - \sigma_{n+1}(m,\tilde{t},s) = \sigma_n(m,\tilde{t},\frac{\nu}{2^n}) - \sigma_{n+1}(m,\tilde{t},\frac{\nu}{2^n}) + \tag{94}$$

$$+\frac{m}{w(m)}\int_{\frac{\nu}{2^n}}^{s}\Big[K_{\gamma_q\overline{s}_1}[\sigma_n(\cdot,\cdot,s')+\sigma_{n+1}(\cdot,\cdot,s'),\ \sigma_n(\cdot,\cdot,s')-\sigma_{n+1}(\cdot,\cdot,s')](m,\tilde{t})+$$

$$+L_{\gamma_q\overline{s}_1}[\sigma_n(\cdot,\cdot,s')-\sigma_{n+1}(\cdot,\cdot,s')](m,\tilde{t})\Big]ds'$$

for

$$\overline{s}_1 = \frac{\nu}{2^n} \leq s \leq \frac{2\nu+1}{2^{n+1}}$$

and

$$\sigma_n(m,\tilde{t},s) - \sigma_{n+1}(m,\tilde{t},s) = \sigma_n(m,\tilde{t},\frac{2\nu+1}{2^{n+1}}) - \sigma_{n+1}(m,\tilde{t},\frac{2\nu+1}{2^{n+1}}) + \tag{95}$$

$$+\frac{m}{w(m)}\int_{\frac{2\nu+1}{2^{n+1}}}^{s}\Big[K_{\gamma_q\overline{s}_2}[\sigma_n(\cdot,\cdot,s')+\sigma_{n+1}(\cdot,\cdot,s'),\ \sigma_n(\cdot,\cdot,s')-\sigma_{n+1}(\cdot,\cdot,s')](m,\tilde{t})+$$

$$+L_{\gamma_q\overline{s}_2}[\sigma_n(\cdot,\cdot,s')-\sigma_{n+1}(\cdot,\cdot,s')](m,\tilde{t})\Big]ds' + \Delta_{\overline{s}_1\,\overline{s}_2}(m,\tilde{t})$$

for

$$\frac{2\nu+1}{2^{n+1}} \leq s \leq \frac{\nu+1}{2^n}, \qquad \overline{s}_1 = \frac{\nu}{2^n}, \quad \overline{s}_2 = \frac{2\nu+1}{2^{n+1}},$$

where

$$\Delta_{\overline{s}_1\,\overline{s}_2}(m,\tilde{t}) =$$

$$= \frac{m}{w(m)} \int_{\frac{2\nu+1}{2^{n+1}}}^{s} \Big[K_{\gamma_{q\,\overline{s}_1}}[\sigma_n(\cdot,\cdot,s'),\, \sigma_n(\cdot,\cdot,s')](m,\tilde{t}) - K_{\gamma_{q\,\overline{s}_2}}[\sigma_n(\cdot,\cdot,s'),$$

$$\sigma_n(\cdot,\cdot,s')](m,\tilde{t}) + L_{\gamma_{q\,\overline{s}_1}}[\sigma_n(\cdot,\cdot,s')](m,\tilde{t}) - L_{\gamma_{q\,\overline{s}_2}}[\sigma_n(\cdot,\cdot,s')](m,\tilde{t})\Big] ds'.$$

According to the conditions (8), (9), (10), it results from (19), (20) (see also (77)) that, for all $q \in \mathbb{R}$ and $\overline{s}, s \in [0, S_1[$, we have

$$\sup_{\tilde{t}\in\mathbb{R}} \frac{m}{w(m)} |K_{\gamma_{q\,\overline{s}}}[\sigma_n(\cdot,\cdot,s) + \sigma_{n+1}(\cdot,\cdot,s),\ \sigma_n(\cdot,\cdot,s) - \sigma_{n+1}(\cdot,\cdot,s)](m,\tilde{t})| \leq \tag{96}$$

$$\leq \frac{C_0}{2} \int_0^m (u_n(m-m',s) + u_{n+1}(m-m',s))\eta_n(m',s)dm' +$$

$$+ C_0 \eta_n(m,s)\overline{\omega}(s) + \frac{C_0}{2}(u_n(m,s) + u_{n+1}(m,s))\overline{\alpha}_n(s),$$

$$\sup_{\tilde{t}\in\mathbb{R}} \frac{m}{w(m)} |L_{\gamma_{q\,\overline{s}}}[\sigma_n(\cdot,\cdot,s) - \sigma_{n+1}(\cdot,\cdot,s)](m,\tilde{t})| \leq \tag{97}$$

$$\leq \frac{C_0}{2}\eta_n(m,s) + \frac{m}{w(m)} \int_0^\infty \vartheta(m,m')\eta_n(m+m',s)dm'.$$

On the other hand, from the definitions (17) and (18) the values of $\sigma_n(m', \tilde{t}', s)$ on the curves $\gamma_{q\overline{s}_1(m,\tilde{t})}$ and $\gamma_{q\overline{s}_2(m,\tilde{t})}$ are given by:

$$\{\sigma_n(\cdot,\cdot,s)\}_{q\overline{s}_1(m,\tilde{t})}(m') = \sigma_n(m', \tilde{t} + \frac{\overline{s}_1}{w(m)} - \frac{\overline{s}_1}{w(m')}, s),$$

$$\{\sigma_n(\cdot,\cdot,s)\}_{q\overline{s}_2(m,\tilde{t})}(m') = \sigma_n(m', \tilde{t} + \frac{\overline{s}_2}{w(m)} - \frac{\overline{s}_2}{w(m')}, s).$$

Therefore, taking into account the relation $\overline{s}_2 - \overline{s}_1 = \frac{2\nu+1}{2^{n+1}} - \frac{\nu}{2^n} = \frac{1}{2^{n+1}}$ and the hypothesis (69), we have

$$|\{\sigma_n(\cdot,\cdot,s)\}_{q\overline{s}_1(m,\tilde{t})}(m') - \{\sigma_n(\cdot,\cdot,s)\}_{q\overline{s}_2(m,\tilde{t})}(m')| \leq \tag{98}$$

$$\leq j_n(m',s)\Big|\frac{\overline{s}_1}{w(m)} - \frac{\overline{s}_1}{w(m')} - \Big(\frac{\overline{s}_2}{w(m)} - \frac{\overline{s}_2}{w(m')}\Big)\Big| \leq j_n(m',s)\frac{\overline{C}_w}{2^n}.$$

With the information of (8), (9), and (10), we deduce from (19), (20) and (98), in a similar manner to (86)) that

$$\frac{m}{w(m)}\big|K_{\gamma_{q\,\overline{s}_1}}[\sigma_n(\cdot,\cdot,s'),\, \sigma_n(\cdot,\cdot,s')](m,\tilde{t}) - K_{\gamma_{q\,\overline{s}_2}}[\sigma_n(\cdot,\cdot,s'),\, \sigma_n(\cdot,\cdot,s')](m,\tilde{t})\big| \leq \tag{99}$$

$$\le \frac{\overline{C}_w}{2^n} C_0 \Big[\int_0^m u_n(m-m',s) j_n(m',s)dm' + j_n(m,s)\omega_n(s) + u_n(m,s)J_n(s) \Big],$$

$$\frac{m}{w(m)} \big| L_{\gamma_q \overline{s}_1}[\sigma_n(\cdot,\cdot,s')](m,\tilde{t}) - L_{\gamma_q \overline{s}_2}[\sigma_n(\cdot,\cdot,s')](m,\tilde{t}) \big| \le \tag{100}$$

$$\le \frac{\overline{C}_w}{2^n} \Big[\frac{C_0}{2} j_n(m,s) + \frac{m}{w(m)} \int_0^\infty \vartheta(m,m') j_n(m+m',s)dm' \Big].$$

As

$$\int_0^\infty \int_0^m (u_n(m-m',s) + u_{n+1}(m-m',s))\eta_n(m',s)dm'dm \le 2\overline{\omega}(s)\overline{\alpha}_n(s),$$

$$\int_0^\infty \frac{m}{w(m)} \int_0^\infty \vartheta(m,m')\eta_n(m+m',s)dm'dm \le C_0\overline{\alpha}_n(s),$$

and $\overline{\alpha}_n(0) = \overline{\beta}_n(0) = 0$, and by using (96), (97), (99), (100) (see also (77), (81), (83), (89)), we deduce from (94)-(95) that

$$\overline{\alpha}_n(s) \le \frac{3C_0}{2} \int_0^s (2\overline{\omega}(s')+1)\overline{\alpha}_n(s')ds' + \frac{1}{2^n}\frac{3C_0\overline{C}_w}{2} \int_0^s (2\overline{\omega}(s')+1)\overline{J}(s')ds', \tag{101}$$

$$\overline{\beta}_n(s) \le C_0 \int_0^s ((2\overline{\psi}(s')+1)\overline{\alpha}_n(s') + (\overline{\omega}(s')+\frac{1}{2})\overline{\beta}_n(s'))ds' + \tag{102}$$

$$+ \frac{1}{2^n}\overline{C}_w C_0 \int_0^s \big((2\overline{\psi}(s')+1)\overline{J}(s') + (\overline{\omega}(s')+\frac{1}{2})\overline{\lambda}(s')\big)ds'.$$

It follows that

$$\overline{\alpha}_n(s) \le \overline{y}(s), \qquad \overline{\beta}_n(s) \le \overline{z}(s),$$

where $\overline{y}(s)$ is the solution of the following Cauchy problem

$$\frac{d}{ds}\overline{y}(s) = \frac{3C_0}{2}(2\overline{\omega}(s)+1)\overline{y}(s) + \frac{1}{2^n}\frac{3\overline{C}_w C_0}{2}(2\overline{\omega}(s)+1)\overline{J}(s), \qquad \overline{y}(0)=0,$$

while $\overline{z}(s)$ is the solution of the following Cauchy problem

$$\frac{d}{ds}\overline{z}(s) = C_0(\overline{\omega}(s)+\frac{1}{2})\overline{z}(s) + C_0(2\overline{\psi}(s)+1)\overline{y}(s) +$$

$$+ \frac{1}{2^n}\overline{C}_w C_0 \big((2\overline{\psi}(s)+1)\overline{J}(s) + (\overline{\omega}(s)+\frac{1}{2})\overline{\lambda}(s)\big), \qquad \overline{z}(0)=0.$$

To summarize, if we put

$$\overline{A}(s) = \frac{3C_0\overline{C}_w}{2}\int_0^s (2\overline{\omega}(s')+1)\overline{J}(s')e^{\frac{3C_0}{2}\int_{s'}^s(2\overline{\omega}(s'')+1)ds''}ds', \tag{103}$$

and

$$\overline{B}(s) = C_0\int_0^s \big[(2\overline{\psi}(s')+1)\overline{A}(s')+\overline{C}_w\big((2\overline{\psi}(s')+1)\overline{J}(s')+(\overline{\omega}(s')+\frac{1}{2})\overline{\lambda}(s')\big)\big]\times$$
$$\times e^{\frac{C_0}{2}\int_{s'}^s(2\overline{\omega}(s'')+1)ds''}ds', \tag{104}$$

we find that

$$\overline{\alpha}_n(s) \le \frac{1}{2^n}\overline{A}(s), \qquad \overline{\beta}_n(s) \le \frac{1}{2^n}\overline{B}(s). \tag{105}$$

As $\overline{B}(s)$ defined in (103)-(104) does not depend on n and it's an increasing function well defined on $[0, S_1[$, i.e.:

$$0 \le \overline{B}(s_1) \le \overline{B}(s_2) < \infty \qquad \forall s_1, s_2 \in [0, S_1[\,, \quad s_1 \le s_2,$$

from (93) and (105) we deduce that

$$\forall\varepsilon>0, \forall\overline{s}\in[0,S_1[, \exists\overline{n}\in\mathbb{N} : \overline{n}>\xi \Rightarrow \sup_{(m,\tilde{t},s)\in\mathbb{R}_+\times\mathbb{R}\times[0,\overline{s}]} |\sigma_{n_1}(m,\tilde{t},s)-\sigma_{n_2}(m,\tilde{t},s)|<\varepsilon,$$

$\forall n_1, n_2 \ge \overline{n}$, where $\overline{n} > \frac{1}{\log 2}(\log\overline{B}(\overline{s}) + \log\frac{1}{\varepsilon}) + 1$.

This proves the uniform convergence of $\sigma_n(m,\tilde{t},s)$ in $\mathbb{R}_+\times\mathbb{R}\times[0,\overline{s}]$ as $n\to\infty$. Moreover, this result about the convergence remains valid for all $\overline{s}\in[0,S_1[$.

Let us provisionally designate by $\sigma_\infty(m,\tilde{t},s)$ limit of the sequence $\{\sigma_n(m,\tilde{t},s)\}_{n=0}^\infty$, i.e.

$$\sigma_\infty(m,\tilde{t},s) = \lim_{n\to\infty}\sigma_n(m,\tilde{t},s).$$

As $\sigma_n(m,\tilde{t},s)$ converges uniformly to $\sigma_\infty(m,\tilde{t},s)$ in $\mathbb{R}_+\times\mathbb{R}\times[0,\overline{s}]$ for any $\overline{s}\in\,]0,S_1[$, it clear that $\sigma_\infty(m,\tilde{t},s)$ is also continuous and non-negative; moreover, from the first inequality of (105) we deduce that $\sup_{\tilde{t}\in\mathbb{R}}\sigma_\infty(m,\tilde{t},s)$ is integrable on $\mathbb{R}_+(\ni m)$ for all $s\in[0,S_1[$.

Let $\overline{s}\in\,]0,S_1[$. We put

$$\Delta_\infty = \sup_{(m,\tilde{t},s)\in\mathbb{R}_+\times\mathbb{R}\times[0,\overline{s}]}\big|\sigma_\infty(m,\tilde{t},s)-\overline{\sigma}_0(m,\tilde{t})-I(m,\tilde{t},s)\big|, \tag{106}$$

where

$$I(m,\tilde{t},s) = \int_0^s \frac{m}{w(m)}\big(K_{\gamma_{qs}}[\sigma_\infty(\cdot,\cdot,s'),\sigma_\infty(\cdot,\cdot,s')](m,\tilde{t}) + L_{\gamma_{qs}}[\sigma_\infty(\cdot,\cdot,s')](m,\tilde{t})\big)ds'.$$

As

$$\sigma_n(m,\tilde{t},s) = \overline{\sigma}_0(m,\tilde{t}) + \int_0^s \frac{m}{w(m)}\big(K_{\gamma_{q\tilde{s}_\nu(n,s)}}[\sigma_n(\cdot,\cdot,s'),\sigma_n(\cdot,\cdot,s')](m,\tilde{t})+ L_{\gamma_{q\tilde{s}_\nu(n,s)}}[\sigma_n(\cdot,\cdot,s')](m,\tilde{t})\big)ds'$$

with

$$\tilde{s}_\nu(n,s) = \frac{\nu}{2^n} \qquad \text{for} \quad \frac{\nu}{2^n} \le s < \frac{\nu+1}{2^n},$$

we have

$$\sigma_\infty(m,\tilde{t},s) - \overline{\sigma}_0(m,\tilde{t}) - I(m,\tilde{t},s) = \tag{107}$$

$$= \sigma_\infty(m,\tilde{t},s) - \sigma_n(m,\tilde{t},s) - I_n^{[1]}(m,\tilde{t},s) - I_n^{[2]}(m,\tilde{t},s),$$

$$I_n^{[1]}(m,\tilde{t},s) = \int_0^s \frac{m}{w(m)}\Big(K_{\gamma_{qs}}[\sigma_\infty(\cdot,\cdot,s'),\sigma_\infty(\cdot,\cdot,s')](m,\tilde{t})+ +L_{\gamma_{qs}}[\sigma_\infty(\cdot,\cdot,s')](m,\tilde{t}) - K_{\gamma_{qs}}[\sigma_n(\cdot,\cdot,s'),\sigma_n(\cdot,\cdot,s')](m,\tilde{t})+ -L_{\gamma_{qs}}[\sigma_n(\cdot,\cdot,s')](m,\tilde{t})\Big)ds',$$

$$I_n^{[2]}(m,\tilde{t},s) = \int_0^s \frac{m}{w(m)}\Big(K_{\gamma_{qs}}[\sigma_n(\cdot,\cdot,s'),\sigma_n(\cdot,\cdot,s')](m,\tilde{t})+ +L_{\gamma_{qs}}[\sigma_n(\cdot,\cdot,s')](m,\tilde{t}) - K_{\gamma_{q\tilde{s}_\nu(n,s)}}[\sigma_n(\cdot,\cdot,s'),\sigma_n(\cdot,\cdot,s')](m,\tilde{t})+ -L_{\gamma_{q\tilde{s}_\nu(n,s)}}[\sigma_n(\cdot,\cdot,s')](m,\tilde{t})\Big)ds'.$$

On one hand, the uniform convergence of $\sigma_n(m,\tilde{t},s)$ to $\sigma_\infty(m,\tilde{t},s)$ implies that

$$\lim_{n\to\infty}(|\sigma_\infty(m,\tilde{t},s) - \sigma_n(m,\tilde{t},s)| + |I_n^{[1]}(m,\tilde{t},s)|) = 0.$$

On the other hand, recalling the reasoning used to obtain (99)-(100), there is no difficulty to find that

$$\forall\varepsilon > 0, \exists\overline{n}_\varepsilon \in \mathbb{N} : n \ge \overline{n}_\varepsilon \Rightarrow \sup_{(m,\tilde{t},s)\in\mathbb{R}_+\times\mathbb{R}\times[0,\overline{s}]} |I_n^{[2]}(m,\tilde{t},s)| < \varepsilon.$$

We deduce that

$$\Delta_\infty = 0$$

or

$$\sigma_\infty(m,\tilde{t},s) = \overline{\sigma}_0(m,\tilde{t}) + \tag{108}$$

$$+\int_0^s \frac{m}{w(m)}\big(K_{\gamma_{qs}}[\sigma_\infty(\cdot,\cdot,s'),\sigma_\infty(\cdot,\cdot,s')](m,\tilde{t}) + L_{\gamma_{qs}}[\sigma_\infty(\cdot,\cdot,s')](m,\tilde{t})\big)ds'.$$

According to the continuity of $\sigma_\infty(m,\tilde{t},s)$, the derivative with respect to s of the right hand side of (108) is well defined, which allows us to pass from (108) to the differential version (21), i.e. $\sigma_\infty(m,\tilde{t},s)$ is a solution of the problem (21)-(22).

To demonstrate the uniqueness, we consider two solutions σ and φ of the problem (21)-(22) belonging to the class of functions defined in the statement of the theorem. As $\sigma(m,\tilde{t},0)-\varphi(m,\tilde{t},0)=0$ and

$$K_{\gamma_{qs}}[\sigma(\cdot,\cdot,s),\sigma(\cdot,\cdot,s)](m,\tilde{t}) - K_{\gamma_{qs}}[\varphi(\cdot,\cdot,s),\varphi(\cdot,\cdot,s)](m,\tilde{t}) =$$

$$= K_{\gamma_{qs}}[\sigma(\cdot,\cdot,s)+\varphi(\cdot,\cdot,s),\sigma(\cdot,\cdot,s)-\varphi(\cdot,\cdot,s)](m,\tilde{t}),$$

integrating (21) with respect to s with $s\in]0,S_1[$, we have

$$\sigma(m,\tilde{t},s)-\varphi(m,\tilde{t},s) =$$

$$= \int_0^s \frac{m}{w(m)}\big(K_{\gamma_{qs'}}[\sigma(\cdot,\cdot,s')+\varphi(\cdot,\cdot,s'),\sigma(\cdot,\cdot,s')-\varphi(\cdot,\cdot,s')](m,\tilde{t}) +$$

$$+L_{\gamma_{qs'}}[\sigma(\cdot,\cdot,s')-\varphi(\cdot,\cdot,s')](m,\tilde{t})\big)ds'.$$

Therefore, putting

$$\eta(m,s) = \sup_{\tilde{t}\in\mathbb{R}} |\sigma(m,\tilde{t},s)-\varphi(m,\tilde{t},s)|, \quad u_2^*(m,s) = \sup_{\tilde{t}\in\mathbb{R}} |\sigma(m,\tilde{t},s)+\varphi(m,\tilde{t},s)|,$$

and in a similar way to (96)-(97), we obtain

$$\eta(m,s) \le \int_0^s \Big[\frac{C_0}{2}\int_0^m u_2^*(m-m',s')\eta(m',s')dm' +$$

$$\frac{C_0}{2}\eta(m,s')\int_0^\infty u_2^*(m',s')dm' + \frac{C_0}{2}u_2^*(m,s')\int_0^\infty \eta(m',s')dm' + \frac{C_0}{2}\eta(m,s') +$$

$$+\frac{m}{w(m)}\int_0^\infty \vartheta(m,m')\eta(m+m',s')dm'\Big]ds'.$$

Consequently, if we put

$$\overline{g}(s) = \int_0^\infty \eta(m,s)dm,$$

in a similar way to (101), we have

$$\overline{g}(s) \leq \frac{3C_0}{2} \int_0^s \Big(1 + \int_0^\infty u_2^*(m, s')dm\Big)\overline{g}(s')ds'.$$

We deduce from the condition *iii*) that

$$\overline{g}(s) = 0 \qquad \forall s \in [0, S_1],$$

that proves the uniqueness of the solution. □

Remark 2. *If the entry condition does not depend on time $\tilde{t}$ (i.e $\overline{\sigma}_0(m, \tilde{t}) = \overline{\sigma}_0(m)$), we can directly construct the solution, which will be an analytic function in $s = -z$; or rather, the equation with the homogeneous entry rewritten on the trajectories will be a formal variant of equation studied by Melzak in [9]. In addition, the result can be deduced almost immediately from the proposition 1 and 2.*

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