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EXPONENTIAL DICHOTOMY CONCEPTS FOR EVOLUTION OPERATORS IN THE HALF-LINE*

Mihai-Gabriel Babuţia† Mihail Megan‡

Abstract

The paper considers three concepts of nonuniform exponential dichotomy and their correspondents for the case of uniform exponential dichotomy on the half-line in the general framework of evolution operators in Banach spaces. Two of these concepts can be considered for evolution operators that are not invertible on the unstable manifold yielding more general behaviors. Using two particular classes of evolution operators defined on the Banach space of bounded real-valued sequences, we give some illustrative examples which clarify the relations between these concepts.

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1 Introduction

The notion of exponential dichotomy introduced by Perron in [24] plays a central role in the qualitative theory of dynamical systems, which has

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†mbabutia@math.uvt.ro, Department of Mathematics, West University of Timişoara, 4 V. Pârvan Blvd., 300223 Timişoara, Romania
‡megan@math.uvt.ro, Academy of Romanian Scientists, 54 Independenţei St., 050094 Bucharest, Romania

an impressive development. The exponential dichotomy property for linear dynamical systems has gained prominence since the appearance of two fundamental monographs of J. L. Massera and J. J. Schäffer [15], J. L. Daleckii and M. G. Krein [12]. These were followed by the important books of C. Chicone and Y. Latushkin [11] and L. Barreira and C. Valls [5].

The most important dichotomy concept used in the qualitative theory of ordinary differential equations is the uniform exponential dichotomy (see [13], [14], [9], [16], [18], [27], [29], [32], [31], [33]). In the nonautonomous setting, the concept of uniform exponential dichotomy is too restrictive and it is important to look for more general behaviors, for example the nonuniform case, where a consistent contribution is due to L. Barreira and C. Valls ([6], [7], [8]). Their study is motivated by ergodic theory and nonuniform hyperbolic theory (we refer the reader to the monograph of L. Barreira and Ya. Pesin [4] for details and further information). Furthermore, an important property of this asymptotic behavior (both in the uniform and nonuniform case) is the roughness of the dichotomy which can be seen from the papers [27], [34] and [35]. Another direction for the study of nonuniform behaviors is due to the members of the Research Center in Differential Equations from West University of Timişoara, Romania, who study a more general type of nonuniform exponential dichotomy which does not impose an upper bound on the dichotomy projections (see [16], [20], [19], [17],[25], [26], [3], [22], [21], [28], [30]).

We prove that in the particular case when the nonuniformity is of exponential type and the dichotomy projections are exponentially bounded, the three dichotomy concepts presented in this paper are equivalent (Theorem 3).

In this paper we consider three concepts of nonuniform exponential dichotomy (exponential dichotomy, strong exponential dichotomy, weak exponential dichotomy) and their correspondents for the case of uniform exponential dichotomy for evolution operators on the half-line. Thus we obtain a systematic classification of exponential dichotomy concepts with the connections between them. Using two general classes of evolution operators, we clarify the relations between these concepts. In contrast with the concept of exponential dichotomy, two concepts of strong exponential dichotomy and weak exponential dichotomy (see Proposition 1 and Open Problem 2) can be defined for evolution operators which are not invertible on the unstable manifolds, but, in contrast with the invertible case, more general behaviors are obtained.

We remark that in this paper we assume the existence of a family of projections P which is compatible with a given evolution operator U. At a fist view the existence of such a family P is a strong hypothesis. The impediment can be eliminated using the notion of admissibility, by associating to an evolution operator $U : \Delta \to \mathcal{B}(X)$ the integral equation

$$f(t) = U(t,s)f(s) + \int_s^t U(t,\tau)v(\tau)\ d\tau, \quad (t,s) \in \Delta$$

where f and v belong to some Banach function spaces. Under the hypothesis of admissibility, the existence of the family of projections and the dichotomy property is deduced (for details, see for example [8], [19], [30], [32], [23] [31], [18]).

2 Dichotomic pairs

Let X be a real or complex Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X. The norms on X and $\mathcal{B}(X)$ will be denoted by $\|\cdot\|$. Denote by I the identity operator on X.
We will also use the following notations:

$$\Delta = \{(t,s) \in \mathbb{R}_+^2\ :\ t \geq s\} \quad \text{and} \quad T = \Delta \times X.$$

Definition 1. *A map $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is called a* **family of projections** *on X if*

$$P(t)^2 = P(t), \quad \textit{for every } t \geq 0.$$

In particular

- *if there are $M \geq 1$ and $\gamma \geq 0$ such that*

  $$\|P(t)\| \leq Me^{\gamma t}, \quad \textit{for all } t \geq 0$$

  then we say that $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is **exponentially bounded***;*

- *if there is $M \geq 1$ such that*

  $$\|P(t)\| \leq M, \quad \textit{for all } t \geq 0$$

  then we say that P is **bounded***.*

Remark 1. *If $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is a family of projections on X then*

$$Q : \mathbb{R}_+ \to \mathcal{B}(X) \quad \textit{defined by} \quad Q(t) = I - P(t)$$

is also a family of projections on X, which is called the **complementary family of projections** *of P.*

Definition 2. *A map $U : \Delta \to \mathcal{B}(X)$ is called an* ***evolution operator*** *on X if*

(e_1) $U(t,t) = I$ for every $t \geq 0$;

(e_2) $U(t,s)U(s,t_0) = U(t,t_0)$ for all $(t,s),(s,t_0) \in \Delta$.

Definition 3. *A family of projections $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is said to be* **invariant** *for the evolution operator $U : \Delta \to \mathcal{B}(X)$ if*

$$U(t,s)P(s) = P(t)U(t,s)$$

for all $(t,s) \in \Delta$.

Definition 4. *A family of projections $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is said to be* **compatible** *with the evolution operator $U : \Delta \to \mathcal{B}(X)$ if*

(c_1) P is invariant for U;

(c_2) for every $(t,s) \in \Delta$ the restriction of $U(t,s)$ on $Ker\ P(s)$ is an isomorphism from $Ker\ P(s)$ to $Ker\ P(t)$

If P is compatible with U then the pair (U,P) is called a **dichotomic pair***.*

Remark 2. *If (U,P) is a dichotomic pair then for all $(t,s) \in \Delta$ one has that*

$$U(t,s)Q(s) = Q(t)U(t,s).$$

Remark 3. *If (U,P) is a dichotomic pair and for all $(t,s) \in \Delta$ the linear operator $U(t,s)$ is invertible (for example, if the evolution operator arises from linear ODEs) then (U,Q) is also a dichotomic pair, where Q is the complementary family of P.*

Remark 4. *If (U,P) is a dichotomic pair then there exists $V : \Delta \to \mathcal{B}(X)$ such that $V(t,s)$ is an isomorphism from $Ker\ P(t)$ to $Ker\ P(s)$ and*

(v_1) $U(t,s)V(t,s)Q(t) = Q(t)$;

(v_2) $V(t,s)U(t,s)Q(s) = Q(s)$;

(v_3) $V(t,s)Q(t) = Q(s)V(t,s)Q(t)$

for all $(t,s) \in \Delta$. The map V is called the **skew-evolution operator** *associated to the pair (U,P).*

3 Exponential dichotomy

In this section we present the exponential dichotomy concepts considered (both in the uniform and the nonuniform case), for example, in [1], [15], [18], [19], [23], [28], [29], [30].

In what follows, let (U, P) be a dichotomic pair and let V be the skew-evolution operator associated to the pair (U, P).

Definition 5. *We say that the pair (U, P) is* **exponentially dichotomic** *(e.d) if there are $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that*

$$(ed_1) \quad e^{\alpha(t-s)} \|U(t,s)P(s)x\| \leq Ne^{\beta s} \|P(s)x\|$$

$$(ed_2) \quad e^{\alpha(t-s)} \|Q(s)x\| \leq Ne^{\beta t} \|U(t,s)Q(s)x\|$$

for all $(t, s, x) \in T$, where Q is the complementary family of P.
In the particular case when $\beta = 0$ we say that (U, P) is **uniformly exponentially dichotomic** *(u.e.d).*

Remark 5. *As particular cases of the above defined concept, we obtain the following concepts:*

(i) if $P(t) = I$ for all $t \geq 0$, then we obtain the **exponential stability property***;*

(ii) if $P(t) = I$ for all $t \geq 0$ and $\beta = 0$, then we obtain the **uniform exponential stability property***.*

Remark 6. *If (U, P) is u.e.d then it is e.d. The converse is not generally true, as shown in Example 1, (vii).*

The above concept allows us to define the exponential dichotomy property for evolution operators in the general case in which the invertibility on the kernels of the projections is not assumed i.e. P is only invariant for U. Next, we present another result concerning the nonuniform exponential dichotomy which, as it can be seen from the two conditions of the theorem, can also be asserted in the general (noninvertible) case.

Theorem 1. *The dichotomic pair (U, P) is exponentially dichotomic with $\beta \in [0, \alpha)$ (where α and β are given by Definition 5) if and only if there exists $N \geq 1$ such that*

$$(ed_1') \quad e^{\alpha(t-s)} \|U(t,s)P(s)x\| \leq Ne^{\beta s} \|P(s)x\|$$

$$(ed_2') \quad e^{\alpha(t-s)} \|Q(s)x\| \leq Ne^{\beta s} \|U(t,s)Q(s)x\|$$

for all $(t, s, x) \in T$.

Proof. It is sufficient to show that $(ed_2) \Leftrightarrow (ed_2')$.
For $(ed_2) \Rightarrow (ed_2')$ we have that

$$e^{(\alpha-\beta)(t-s)}\|Q(s)x\| \leq Ne^{\beta t}e^{-\beta(t-s)}\|U(t,s)Q(s)x\| = Ne^{\beta s}\|U(t,s)Q(s)x\|$$

and for $(ed_2') \Rightarrow (ed_2)$ we observe that

$$e^{\alpha(t-s)}\|Q(s)x\| \leq Ne^{\beta s}\|U(t,s)Q(s)x\| = Ne^{\beta t}\|U(t,s)Q(s)x\|$$

for all $(t,s,x) \in T$. □

As an immediate consequence we obtain

Corollary 1. *If the dichotomic pair (U,P) is exponentially dichotomic with $\beta \in [0,\alpha)$ then*

$$\lim_{t\to\infty} U(t,s)P(s)x = 0 \quad \textit{for every} \quad (s,x) \in \mathbb{R}_+ \times X \quad \textit{and}$$

$$\lim_{t\to\infty} \|U(t,s)Q(s)x\| = \infty \quad \textit{for every} \quad (s,x) \in \mathbb{R}_+ \times X \textit{ with } Q(s)x \neq 0.$$

Remark 7. *The condition $\beta \in [0,\alpha)$ is essential for the validity of the previous corollary, phenomenon illustrated in Example 1, (x).*

A characterization of the concept of exponential dichotomy is given by

Theorem 2. *Let (U,P) be a dichotomic pair. Then (U,P) is exponentially dichotomic if and only if there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that*

$$(ed_1'') \quad e^{\alpha(t-s)}\|U(t,s)P(s)x\| \leq Ne^{\beta s}\|P(s)x\|$$

$$(ed_2'') \quad e^{\alpha(t-s)}\|V(t,s)Q(t)x\| \leq Ne^{\beta t}\|Q(t)x\|$$

for all $(t,s,x) \in T$.

Proof. We only have to prove the equivalence between the instability properties (i.e. $(ed_2) \Leftrightarrow (ed_2'')$). To prove that $(ed_2'') \Rightarrow (ed_2)$, we observe that

$$e^{\alpha(t-s)}\|Q(s)x\| \stackrel{(v_2)}{=} e^{\alpha(t-s)}\|V(t,s)U(t,s)Q(s)x\| =$$
$$= e^{\alpha(t-s)}\|V(t,s)Q(t)U(t,s)Q(s)x\| \leq Ne^{\beta t}\|U(t,s)Q(s)x\|$$

for all $(t,s,x) \in T$.
Similarly, by (v_3), (ed_2) and (v_1) it results that

$$e^{\alpha(t-s)}\|V(t,s)Q(t)x\| \stackrel{(v_3)}{=} e^{\alpha(t-s)}\|Q(s)V(t,s)Q(t)x\| \leq$$
$$\leq Ne^{\beta t}\|U(t,s)Q(s)V(t,s)Q(t)x\| =$$
$$= Ne^{\beta t}\|Q(t)U(t,s)V(t,s)Q(t)x\| \stackrel{(v_1)}{=} Ne^{\beta t}\|Q(t)x\|$$

for all $(t,s,x) \in T$. □

As a particular case we obtain

Corollary 2. *Let (U, P) be a dichotomic pair. Then (U, P) is uniformly exponentially dichotomic if and only if there are $N \geq 1$ and $\alpha > 0$ such that*

$$(ued'_1) \quad e^{\alpha(t-s)}\|U(t,s)P(s)x\| \leq N\|P(s)x\|$$

$$(ued'_2) \quad e^{\alpha(t-s)}\|V(t,s)Q(t)x\| \leq N\|Q(t)x\|$$

for all $(t, s, x) \in T$.

4 Weak exponential dichotomy

Let (U, P) be a dichotomic pair, Q the complementary family of P and V the skew-evolution operator associated to the pair (U, P). We introduce the following dichotomy concept:

Definition 6. *We say that the pair (U, P) is* **weakly exponentially dichotomic** *(w.e.d) if there are $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that*

$$(wed_1) \quad e^{\alpha(t-s)}\|U(t,s)P(s)\| \leq Ne^{\beta s}\|P(s)\|$$

$$(wed_2) \quad e^{\alpha(t-s)}\|V(t,s)Q(t)\| \leq Ne^{\beta t}\|Q(t)\|$$

for all $(t, s) \in \Delta$.
In the particular case when $\beta = 0$ we say that (U, P) is **uniformly weakly exponentially dichotomic** *(u.w.e.d).*

Remark 8. *It is obvious that u.w.e.d $\Rightarrow$ w.e.d. The converse implication is not generally valid (for details, see Example 1 (vii)).*

Remark 9. *The following implications hold:*

$$e.d \Rightarrow w.e.d \quad and \quad u.e.d \Rightarrow u.w.e.d$$

Open Problems.

1) We ask wether the reciprocal implications from Remark 9 hold.

2) For example, in [1], a "weak exponential dichotomy" concept was introduced in the uniform case, in the general framework of evolution operators, in which the assumption of invertibility of the given evolution operator on the kernels of the projections was dropped. Having in mind such "weak" behavior in our nonuniform case, we propose for solving or disproving the following implication:

$$(U,P) \text{ is w.e.d } \Rightarrow \begin{cases} \exists\ N \geq 1,\ \alpha > 0,\ \beta \geq 0 \text{ such that } \forall (t,s) \in \Delta \\ (wed_1') \quad e^{\alpha(t-s)}\|U(t,s)P(s)\| \leq Ne^{\beta s}\|P(s)\|; \\ (wed_2') \quad e^{\alpha(t-s)}\|Q(s)\| \leq Ne^{\beta t}\|U(t,s)Q(s)\|. \end{cases}$$

In what concerns Open Problem 2, we posses a partial result, given by the following assertion.

Remark 10. *The converse of the implication from Open Problem 2 is not generally valid (see Example 2).*

5 Strong exponential dichotomy

In this section we consider another exponential dichotomy concept used in the papers of L. Barreira and C. Valls ([6], [7], [8]). Connections with the previous dichotomy concepts are given. It is shown that in the particular case when the family of projections is exponentially bounded then the exponential dichotomy concepts presented in this paper are equivalent.
Let (U, P) be a dichotomic pair and let Q be the complementary family of P. Let V be the skew-evolution operator associated to the pair (U, P).

Definition 7. *We say that the pair (U, P) is* **strongly exponentially dichotomic** *(s.e.d) if there are $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that*

$$(sed_1) \quad e^{\alpha(t-s)}\|U(t,s)P(s)x\| \leq Ne^{\beta s}\|x\|$$

$$(sed_2) \quad e^{\alpha(t-s)}\|V(t,s)Q(t)x\| \leq Ne^{\beta t}\|x\|$$

for all $(t, s, x) \in T$.

If the conditions (sed_1) and (sed_2) hold for $\beta = 0$ then we say that (U, P) is **uniformly strongly exponentially dichotomic** (u.s.e.d).

Remark 11. *It is obvious that u.s.e.d $\Rightarrow$ s.e.d. The converse implication is not generally true (see Example 1 (ix)).*

Remark 12. *If (U, P) is s.e.d then from (sed_1), for $t = s$, we obtain that*

$$\|P(s)\| \leq Ne^{\beta s} \quad \textit{for all} \quad s \geq 0$$

i.e. P is exponentially bounded. In particular, if (U, P) is u.s.e.d then P is bounded.

Remark 13. *If (U,P) is s.e.d then by substituting x by $P(s)x$ in (sed_1) respectively by $Q(s)x$ in (sed_2) we obtain the implication s.e.d $\Rightarrow$ e.d. In particular, u.s.e.d $\Rightarrow$ u.e.d. The converse implications are not generally valid (see Example 1 (viii)).*

Remark 14. *Having in mind the wide usage of the e.d concept and the s.e.d concept, it is reasonable to consider a dichotomy concept which has the estimations in the operator norm (see Remark 15) as in the s.e.d concept, but in the meantime, as in the case of the e.d concept, not to assume any restriction on the family of projections (see Remark 12).*

Remark 15. *From Definition 7 it results that (U,P) is s.e.d if and only if there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that*

$$(sed_1') \quad e^{\alpha(t-s)}\|U(t,s)P(s)\| \leq Ne^{\beta s}$$

$$(sed_2') \quad e^{\alpha(t-s)}\|V(t,s)Q(t)\| \leq Ne^{\beta t}$$

for all $(t,s) \in \Delta$.
In particular, for $\beta = 0$ we have that (U,P) is u.s.e.d if and only if there are $N \geq 1$ and $\alpha > 0$ with the following properties:

$$(used_1') \quad e^{\alpha(t-s)}\|U(t,s)P(s)\| \leq N$$

$$(used_2') \quad e^{\alpha(t-s)}\|V(t,s)Q(t)\| \leq N$$

for all $(t,s) \in \Delta$.

A difference between the result of Theorem 1 and its correspondent for the s.e.d property is given by

Proposition 1. *If the pair (U,P) is s.e.d then there exists $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that*

$$(sed_1'') \quad e^{\alpha(t-s)}\|U(t,s)P(s)\| \leq Ne^{\beta s}$$

$$(sed_2'') \quad e^{\alpha(t-s)} \leq Ne^{\beta t}\|U(t,s)Q(s)\|$$

for all $(t,s) \in \Delta$.

Proof. It is sufficient to prove that $(sed_2') \Rightarrow (sed_2'')$. Indeed, from (sed_2') , (v_2) and (c_1) we obtain that

$$\begin{aligned} e^{\alpha(t-s)} \leq e^{\alpha(t-s)}\|Q(s)\| &= e^{\alpha(t-s)}\|V(t,s)Q(t)U(t,s)Q(s)\| \\ &\leq Ne^{\beta t}\|U(t,s)Q(s)\| \end{aligned}$$

for all $(t,s) \in \Delta$. □

Remark 16. *The converse of the above proposition is not generally valid (see Example 2).*

Remark 17. *Having in mind the above proposition and remark, we can observe that if we consider the s.e.d property in the general case of invariant families of projections (without the invertibility on the unstable direction of the evolution operator), we obtain a more general behavior. Such behaviors were also pointed out in [1] (in the uniform case) and [2] (in the discrete case).*

The main result of this section is

Theorem 3. *Let (U, P) be a dichotomic pair with the property that P is exponentially bounded. Then the following properties are equivalent:*

(i) (U, P) *is strongly exponentially dichotomic;*

(ii) (U, P) *is exponentially dichotomic;*

(iii) (U, P) *is weakly exponentially dichotomic.*

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ follow from Remarks 13 and 9. For $(iii) \Rightarrow (i)$ assume that (U, P) is w.e.d. Then there exist $M \geq 1$ and $\gamma \geq 0$ such that for all $t \geq 0$,

$$\|P(t)\| \leq Me^{\gamma t}.$$

Then, for all $(t, s) \in \Delta$, from (wed_1) and (wed_2) it follows that

$$e^{\alpha(t-s)}\|U(t, s)P(s)\| \leq Ne^{\beta s}\|P(s)\| \leq 2MNe^{(\beta+\gamma)s}$$

and

$$e^{\alpha(t-s)}\|V(t, s)Q(t)\| \leq Ne^{\beta t}\|Q(t)\| \leq 2MNe^{(\beta+\gamma)t}$$

which, by Remark 15, shows that (U, P) is s.e.d. □

As a particular case, we have

Corollary 3. *Let (U, P) be a dichotomic pair with the property that P is a bounded family of projections. Then the following assertions are equivalent:*

(i) (U, P) *is u.s.e.d;*

(ii) (U, P) *is u.e.d;*

(iii) (U, P) *is u.w.e.d.*

Remark 18. *By Remarks 6, 8, 9, 13 and 16, we obtain the connections between the dichotomy concepts studied in this paper. These are illustrated in the following diagram:*

$$\begin{array}{ccccccc}
\boxed{u.s.e.d} & \overset{\not\Leftarrow}{\Longrightarrow} & \boxed{u.e.d} & \Longrightarrow & \boxed{u.w.e.d} & \overset{\not\Rightarrow}{\Longleftarrow} & \boxed{u.s.e.d} \\
\Downarrow\not\Uparrow & & \Downarrow\not\Uparrow & & \Downarrow\not\Uparrow & & \Downarrow\not\Uparrow \\
\boxed{s.e.d} & \overset{\not\Leftarrow}{\Longrightarrow} & \boxed{e.d} & \Longrightarrow & \boxed{w.e.d} & \overset{\not\Rightarrow}{\Longleftarrow} & \boxed{s.e.d}
\end{array}$$

6 Examples and counterexamples

The aim of this section is to give some illustrative examples and counterexamples which show that the converse of the implications presented in the previous sections are not valid. We begin with some notations used in what follows.

Let $\mathcal{P}$ be the set of all families of projections $P : \mathbb{R}_+ \to \mathcal{B}(X)$ satisfying the equality

$$P(t)P(s) = P(s) \quad \text{for all } t, s \geq 0.$$

We observe that if $P \in \mathcal{P}$ then its complementary Q verifies the relations

$$Q(t)Q(s) = Q(t) \quad \text{and} \quad Q(t)P(s) = 0 \quad \text{for all } t, s \geq 0.$$

We shall denote by $\mathcal{U}_1$ the set of all $u : \mathbb{R}_+ \to (0, \infty)$ with the property that there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that

$$e^{\alpha(t-s)}u(s) \leq Ne^{\beta s}u(t) \quad \text{for all } (t, s) \in \Delta.$$

As a remarkable subset of $\mathcal{U}_1$ we point out the set denoted by $\mathcal{U}_0$, defined as the set of all functions $u : \mathbb{R}_+ \to (0, \infty)$ with the property that there are $N \geq 1$ and $\alpha > 0$ such that

$$e^{\alpha(t-s)}u(s) \leq Nu(t) \quad \text{for all } (t, s) \in \Delta.$$

As examples, we give $u_1, u_2, u_3 : \mathbb{R}_+ \to (0, \infty)$ defined by

$$u_1(t) = e^{\frac{3t}{2+\cos(3\pi t)}}, \quad u_2(t) = e^{\frac{2t}{1+\{2t\}}}, \quad u_3(t) = e^{2t}$$

where $\{t\}$ denotes the fractional part of t.
It is easy to see that $u_1 \in \mathcal{U}_1 \setminus \mathcal{U}_0$ (with $N = \alpha = 1$, $\beta = 2$), $u_2 \in \mathcal{U}_1 \setminus \mathcal{U}_0$ (with $N = \alpha = \beta = 1$ and $u_3 \in \mathcal{U}_0$ (with $N = \alpha = 1$).

An example of a dichotomic pair (U, P) with $P \in \mathcal{P}$ is presented by the following example.

Example 1. *Let $X = l^\infty$ the Banach space of all bounded real-valued sequences, endowed with the norm*

$$\|x\| = \sup_{n\geq 0} |x_n|, \qquad \textit{where } x = (x_0, x_1, \ldots, x_n, \ldots) \in X.$$

For every nondecreasing function $p : \mathbb{R}_+ \to \mathbb{R}_+$ we define $P : \mathbb{R}_+ \to \mathcal{B}(X)$ by

$$P(t)x = (x_0 + p(t)x_1, 0, x_2 + p(t)x_3, 0, \ldots)$$

for all $t \geq 0$ and $x = (x_0, x_1, \ldots) \in X$.
Then P is a family of projections which belongs to $\mathcal{P}$ and its complementary is given by

$$Q(t)x = (-p(t)x_1, x_1, -p(t)x_3, x_3, \ldots).$$

Moreover, for all $(t, s, x) \in T$ we have

$$\|P(t)\| = 1 + p(t) \quad \textit{and} \quad \|Q(s)x\| = \max\{1, p(s)\} \sup_{n\geq 0} |x_{2n+1}| \leq \|Q(t)x\|.$$

In particular:

- *for $p(t) = e^t - 1$ we have that P is exponentially bounded;*
- *for $p(t) = e^{t^2} - 1$ it results that P is not exponentially bounded.*

For every $u : \mathbb{R}_+ \to (0, \infty)$ we define $U : \Delta \to \mathcal{B}(X)$ by

$$U(t, s) = \frac{u(s)}{u(t)} P(s) + \frac{u(t)}{u(s)} Q(t)$$

for all $(t, s) \in \Delta$ where Q is the complementary family of P.
It is easy to verify that (U, P) is a dichotomic pair and the skew-evolution operator associated to (U, P) is given by

$$V(t, s)Q(t) = \frac{u(s)}{u(t)} Q(s) \quad \textit{for } (t, s) \in \Delta.$$

Moreover

$$U(t, s)P(s) = \frac{u(s)}{u(t)} P(s) \quad \textit{and} \quad U(t, s)Q(s) = \frac{u(t)}{u(s)} Q(t) \quad \textit{for all } (t, s) \in \Delta.$$

By Definitions 5, 6, 7, in the particular case of the above defined dichotomic pair (U, P), we obtain the following conclusions:

(i) (U, P) is e.d if and only if $u \in \mathcal{U}_1$;

(ii) (U, P) *is u.e.d if and only if* $u \in \mathcal{U}_0$*;*

(iii) (U, P) *is s.e.d if and only if* $u \in \mathcal{U}_1$ *and* P *is exponentially bounded;*

(iv) (U, P) *is u.s.e.d if and only if* $u \in \mathcal{U}_0$ *and* P *is bounded;*

(v) (U, P) *is w.e.d if and only if* $u \in \mathcal{U}_1$*;*

(vi) (U, P) *is u.w.e.d if and only if* $u \in \mathcal{U}_0$*;*

From these characterizations we obtain, with the aid of functions u *and* p *from the definition of* (U, P)*, that*

(vii) if $u \in \mathcal{U}_1 \setminus \mathcal{U}_0$ *then* (U, P) *is e.d (hence also w.e.d) although* (U, P) *is not u.w.e.d (hence not u.e.d). Thus we obtain that e.d* $\not\Rightarrow$ *u.e.d and w.e.d* $\not\Rightarrow$ *u.w.e.d;*

(viii) if $u \in \mathcal{U}_0$ *and* P *is not exponentially bounded (for example, if* $p(t) = e^{t^2} - 1$*) then* (U, P) *is u.e.d (hence e.d) but* (U, P) *is not s.e.d (hence not u.s.e.d). Thus we have that e.d* $\not\Rightarrow$ *s.e.d and u.e.d* $\not\Rightarrow$ *u.s.e.d;*

(ix) if $u \in \mathcal{U}_1 \setminus \mathcal{U}_0$ *and* P *is exponentially bounded and not bounded then* (U, P) *is s.e.d and it is not u.s.e.d. Hence s.e.d* $\not\Rightarrow$ *u.s.e.d;*

(x) for $u = u_1 \in \mathcal{U}_1$*, with* $\beta = 2 \notin [0, \alpha) = [0, 1)$*, we have that* (U, P) *is e.d with*

$$\lim_{t\to\infty} \|U(t,s)P(s)x\| = 0 \quad \text{and} \quad \lim_{t\to\infty} \|U(t,s)Q(s)x\| = \infty$$

for every $x \in X$ *with* $Q(s)x \neq 0$*. Thus, it results that the condition* $\beta \in [0, \alpha)$ *is not necessary for the validity of Corollary 1.*

Example 2. *Let* $u, v : \mathbb{R}_+ \to (0, \infty)$ *be two nondecreasing functions such that there exist* $N \geq 1$*,* $\alpha > 0$ *and* $\gamma > 0$ *with the following properties:*

$$Nu(t) \geq e^{\alpha(t-s)} u(s) \quad \text{and} \quad v(t) \geq e^{\gamma t^2}$$

for all $(t, s) \in \Delta$*.*

On $X = l^\infty$*, the Banach space of bounded real-valued sequences endowed with the sup-norm, we consider the family of projections* $P : \mathbb{R}_+ \to \mathcal{B}(X)$ *defined by* $P(s)x = y$*, where* $x = (x_0, x_1, \ldots, x_n, \ldots)$ *and* $y = (y_0, y_1, \ldots, y_n, \ldots)$ *with*

$$y_n = \begin{cases} x_n, & n = 3k \\ 0, & \text{otherwise} \end{cases}.$$

The complementary family of P is given by $Q(s)x = z = (z_0, z_1, \dots, z_n, \dots)$ with

$$z_n = \begin{cases} 0, & n = 3k \\ x_n, & \text{otherwise} \end{cases}.$$

Consider $U : \Delta \to \mathcal{B}(X)$ defined by $U(t,s)x = w = (w_0, w_1, \dots, w_n, \dots)$, where

$$w_n = \begin{cases} \frac{u(s)}{u(t)} x_n, & n = 3k \\ \frac{u(t)}{u(s)} x_n, & n = 3k+1 \\ \frac{v(s)}{v(t)} x_n, & n = 3k+2 \end{cases}$$

It is easy to check that P is compatible with U. Moreover, for all $(t,s,x) \in T$ we have that

$$\|U(t,s)P(s)x\| = \frac{u(s)}{u(t)} \|P(s)x\| \leq N e^{-\alpha(t-s)} \|P(s)x\| \tag{1}$$

and

$$\begin{aligned} \|U(t,s)Q(s)x\| &= \sup_{n \in \mathbb{N}} \left\{ \frac{u(t)}{u(s)} |x_{3n+1}|, \frac{v(s)}{v(t)} |x_{3n+2}| \right\} \leq \\ &\leq \max_{n \in \mathbb{N}} \left\{ \frac{u(t)}{u(s)}, \frac{v(s)}{v(t)} \right\} \|Q(s)x\| = \frac{u(t)}{u(s)} \|Q(s)x\|. \end{aligned} \tag{2}$$

By choosing $x' = (x'_0, x'_1, \dots, x'_n, \dots)$ with

$$x'_n = \begin{cases} 0, & n = 3k \\ 1, & \text{otherwise} \end{cases}$$

we have that

$$\|U(t,s)Q(s)x'\| = \frac{u(t)}{u(s)} \|Q(s)x'\|$$

hence

$$\|U(t,s)Q(s)\| = \frac{u(t)}{u(s)} \|Q(s)\| \geq \frac{1}{N} e^{\alpha(t-s)} \|Q(s)\|. \tag{3}$$

From relations (1) and (3) we have that the pair (U,P) satisfies the conditions (wed'_1) and (wed'_2) from Open Problem 2. Taking into account that P is bounded, from (1) and (3) we get that for all $(t,s) \in \Delta$,

$$\|U(t,s)P(s)\| \leq N e^{-\alpha(t-s)} \quad \text{and} \quad \|U(t,s)Q(s)\| \geq \frac{1}{N} e^{\alpha(t-s)} \tag{4}$$

hence the pair (U,P) satisfies the conditions (sed_1'') and (sed_2'') from Proposition 1.

On the other hand, for $(t,s,x) \in T$ we have that

$$\|V(t,s)Q(t)x\| = \sup_{n\in\mathbb{N}} \left\{ \frac{u(s)}{u(t)} |x_{3n+1}|, \frac{v(t)}{v(s)} |x_{3n+2}| \right\}. \tag{5}$$

Assume by a contradiction that the pair (U,P) is w.e.d. Then there exist $\alpha > 0$, $\beta \geq 0$ and $N \geq 1$ such that

$$\|V(t,s)Q(t)\| \leq Ne^{\beta t} e^{-\alpha(t-s)} \|Q(t)\| = Ne^{\beta t} e^{-\alpha(t-s)}. \tag{6}$$

By choosing $x_0 = (0,0,1,0,0,1,\ldots) \in X$ with $\|Q(t)x_0\| = 1$ we get from (5) *that for all $(t,s) \in \Delta$,*

$$\|V(t,s)Q(t)\| \geq \frac{v(t)}{v(s)}. \tag{7}$$

From (6) *and* (7)*, by taking $s = 0$, we obtain the contradiction*

$$e^{\gamma t^2} \leq v(t) \leq v(0)Ne^{(\beta-\alpha)t}, \quad \textit{for all } t \geq 0.$$

Hence the pair (U,P) is not w.e.d and by Theorem 3 it is not s.e.d.

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