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Coefficient bounds for a subclass of Bi-univalent functions using differential operators*

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Abstract

In the present paper, we introduce new subclass $ST_{\Sigma}(b,\phi)$ of bi-univalent functions defined in the open disk. Furthermore, we find upper bounds for the second and third coefficients for functions in these new subclass using differential operator.
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Keywords: bi-univalent functions, coefficient estimates, starlike function, convex function, differential operator.

1 Introduction. Definitions And Preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathcal{C} : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$.

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. However, the

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famous Koebe one-quarter theorem ensures that the image of the unit disk $\mathbb{U}$ under every function $f \in \mathcal{A}$ contains a disk of radius $1/4$. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$, $(z \in \mathbb{U})$ and $f\left(f^{-1}(w)\right) = w$, $\left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right)$ where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \quad (1.2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. We let Σ to denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). If $f(z)$ is bi-univalent, it must be analytic in the boundary of the domain and such that it can be continued across the boundary of the domain so that $f^{-1}(z)$ is defined and analytic throughout $|w| < 1$. Examples of functions in the class Σ are

$$\frac{z}{1-z}, -log\,(1-z)$$

and so on.

The coefficient estimate problem for the class $\mathcal{S}$, known as the Bieberbach conjecture, is settled by de-Branges [4], who proved that for a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the class $\mathcal{S}$, $|a_n| \leq n$, for $n = 2, 3, \cdots$, with equality only for the rotations of the Koebe function

$$K_0(z) = \frac{z}{(1-z)^2}.$$

In 1967, Lewin [9] introduced the class Σ of bi-univalent functions and showed that $|a_2| < 1.51$ for the functions belonging to Σ. It was earlier believed that for $f \in \Sigma$, the bound was $|a_n| < 1$ for every n and the extremal function in the class was $\frac{z}{1-z}$. E.Netanyahu [11] in 1969, ruined this conjecture by proving that in the set Σ, $\max_{f \in \Sigma} |a_2| \leq 4/3$. In 1969, Suffridge [15] gave an example of $f \in \Sigma$ for which $a_2 = 4/3$ and conjectured that $|a_2| \leq 4/3$. In 1981, Styer and Wright [14] disproved the conjecture that $|a_2| > 4/3$. Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}$. Kedzierawski [7] in 1985 proved this conjecture for a special case when the function f and f^{-1} are starlike functions. Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}$. Tan [16] in proved that $|a_2| \leq 1.485$ which is the best known estimate for functions in the class of bi-univalent functions.

Brannan and Taha [3] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $S^*(\alpha)$ and $C(\alpha)$ of the univalent function class Σ. Recently, Ali et al.[1] extended the results of Brannan and Taha [3] by generalising their classes using subordination.

An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, provided there is a Schwarz function w defined on $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. Ma and Minda [10], unified various subclasses of starlike and convex functions for which either of the quantity $\dfrac{zf'(z)}{f(z)}$ or $1 + \dfrac{zf''(z)}{f'(z)}$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function ϕ with positive real part in the unit disk U, $\phi(0) = 1$, $\phi'(0) > 0$ and ϕ maps U onto a region starlike with respect to 1 and symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, (B_1 > 0). \tag{1.3}$$

Recently Selvaraj and Karthikeyan [8] defined the following operator $D_\lambda^m(\alpha_1, \beta_1) f : \mathbb{U} \to \mathbb{U}$ by

$$\begin{aligned} D_\lambda^0(\alpha_1; \beta_1)f(z) &= f(z) * \mathcal{G}_{q,s}(\alpha_1, \beta_1; z), \\ D_\lambda^1(\alpha_1; \beta_1)f(z) &= (1-\lambda)(f(z) * \mathcal{G}_{q,s}(\alpha_1, \beta_1; z)) + \lambda z (f(z) * \mathcal{G}_{q,s}(\alpha_1, \beta_1; z))', \\ D_\lambda^m(\alpha_1; \beta_1)f(z) &= D_\lambda^1(D_\lambda^{m-1}(\alpha_1; \beta_1)f(z)), \end{aligned} \tag{1.4}$$

where $m \in \mathbb{N}_0$, $\lambda \geq 0$.

If $f \in \mathcal{A}$, then from (1.4) we may easily deduce that

$$D_\lambda^m(\alpha_1; \beta_1)f(z) = z + \sum_{n=2}^{\infty} \left[1 + (n-1)\lambda\right]^m \frac{(\alpha_1)_{n-1} \ldots (\alpha_q)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_s)_{n-1}} \frac{a_n z^n}{(n-1)!}. \tag{1.5}$$

Special cases of the operator $D_\lambda^m(\alpha_1; \beta_1)f$ includes various other linear operators which were considered in many earlier work on the subject of analytic and univalent functions. If we let $m = 0$ in $D_\lambda^m(\alpha_1; \beta_1)f$, we have

$$D_\lambda^0(\alpha_1; \beta_1)f(z) = \mathcal{H}_q^1(\alpha_1; \beta_1)f(z)$$

where $\mathcal{H}_{q,s}^1(\alpha_1; \beta_1)$ is Dziok-Srivastava operator for functions in $\mathcal{A}$ (see [6]) and for $q = 2, s = 1$ $\alpha_1 = \beta_1, \alpha_2 = 1$ and $\lambda = 1$, we get the operator introduced by Salagean([13]). It can be easily verified from the definition of (1.5),

$$z\left(D_\lambda^m(\alpha_1, \beta_1) f(z)\right)' = (\alpha_1 + 1) D_\lambda^m(\alpha_1 + 1, \beta_1) f(z) - \alpha_1 D_\lambda^m(\alpha_1, \beta_1) f(z). \tag{1.6}$$

Definition 1.1 *Let b be a non-zero complex number. A function $f(z)$ given by (1.1) is said to be in the class $ST_{\Sigma}(b,\phi)$ if the following conditions are satisfied:*

$$f \in \Sigma \quad and \quad 1+\frac{1}{b}\left(\frac{D_{\lambda}^{m+1}(\alpha_1,\beta_1)\,f(z)}{D_{\lambda}^{m}(\alpha_1,\beta_1)\,f(z)}-1\right) \prec \phi(z), \quad z \in \mathbb{U} \tag{1.7}$$

$$and \quad 1+\frac{1}{b}\left(\frac{D_{\lambda}^{m+1}(\alpha_1,\beta_1)\,g(w)}{D_{\lambda}^{m}(\alpha_1,\beta_1)\,g(w)}-1\right) \prec \phi(z), \quad z \in \mathbb{U} \tag{1.8}$$

where the function g is given by (1.2).

Definition 1.2 *Let b be a non-zero complex number. A function $f(z)$ given by (1.1) is said to be in the class $ST_{\Sigma}(\alpha_1,\beta_1,b,\phi)$ if the following conditions are satisfied:*

$$f \in \Sigma \quad and \quad 1+\frac{1}{b}\left(\frac{D_{\lambda}^{m}(\alpha_1+1,\beta_1)\,f(z)}{D_{\lambda}^{m}(\alpha_1,\beta_1)\,f(z)}-1\right) \prec \phi(z), \quad z \in \mathbb{U} \tag{1.9}$$

$$and \quad 1+\frac{1}{b}\left(\frac{D_{\lambda}^{m}(\alpha_1+1,\beta_1)\,g(w)}{D_{\lambda}^{m}(\alpha_1,\beta_1)\,g(w)}-1\right) \prec \phi(w), \quad w \in \mathbb{U}, \tag{1.10}$$

where the function g is given by (1.2).

2 Coefficient estimates

Lemma 2.1 *[12] If $p \in \wp$, then $|c_k| \le 2$ for each k, where $\wp$ is the family of functions p analytic in $\mathbb{U}$ for which $Re\,p(z) > 0$, $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ for $z \in \mathbb{U}$.*

Theorem 2.2 *Let the function $f(z) \in \mathcal{A}$ be given by (1.1). If $f \in ST_{\Sigma}(b,\phi)$, then*

$$|a_2| \le \frac{B_1\sqrt{B_1}\,|b|}{\sqrt{\left(4(1+2\lambda)^m-(1+\lambda)^{2m}\right)B_1^2 b\lambda+(B_1-B_2)\,\lambda^2(1+\lambda)^{2m}}} \tag{2.1}$$

and

$$|a_3| \le \frac{(B_1+|B_2-B_1|)\,|b|}{\lambda\left(4(1+2\lambda)^m-(1+\lambda)^{2m}\right)}.$$

Proof. Since $f \in ST_{\Sigma}(b, \phi)$, there exists two analytic functions $r, s : \mathbb{U} \to \mathbb{U}$, with $r(0) = 0 = s(0)$, such that

$$1 + \frac{1}{b}\left(\frac{D_{\lambda}^{m+1}(\alpha_1, \beta_1) f(z)}{D_{\lambda}^{m}(\alpha_1, \beta_1) f(z)} - 1\right) = \phi(r(z)) \tag{2.2}$$

and

$$1 + \frac{1}{b}\left(\frac{D_{\lambda}^{m+1}(\alpha_1, \beta_1) g(w)}{D_{\lambda}^{m}(\alpha_1, \beta_1) g(w)} - 1\right) = \phi(s(z)).$$

It is also written as

$$\begin{aligned} 1 + \frac{1}{b}\left(\frac{D_{\lambda}^{m+1}(\alpha_1, \beta_1) f(z) - D_{\lambda}^{m}(\alpha_1, \beta_1) f(z)}{D_{\lambda}^{m}(\alpha_1, \beta_1) f(z)}\right) &= \phi(r(z)) \quad \text{and} \\ 1 + \frac{1}{b}\left(\frac{D_{\lambda}^{m+1}(\alpha_1, \beta_1) g(w) - D_{\lambda}^{m}(\alpha_1, \beta_1) g(w)}{D_{\lambda}^{m}(\alpha_1, \beta_1) g(w)}\right) &= \phi(s(z)). \end{aligned} \tag{2.3}$$

Define the functions p and q by

$$p(z) = \frac{1 + r(z)}{1 - r(z)} = 1 + p_1 z + p_2 z^2 + \cdots \text{ and } q(z) = \frac{1 + s(z)}{1 - s(z)} = 1 + q_1 z + q_2 z^2 + \cdots. \tag{2.4}$$

Or equivalently,

$$\begin{aligned} r(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2}\Bigg(p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \\ \left(p_3 + \frac{p_1}{2}\left(\frac{p_1^2}{2} - p_2\right) - \frac{p_1 p_2}{2}\right) z^3 + \cdots \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} s(z) = \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2}\Bigg(q_1 z + \left(q_2 - \frac{q_1^2}{2}\right) z^2 + \\ \left(q_3 + \frac{q_1}{2}\left(\frac{q_1^2}{2} - q_2\right) - \frac{q_1 q_2}{2}\right) z^3 + \cdots. \end{aligned} \tag{2.6}$$

It is clear that p and q are analytic in $\mathbb{U}$ and $p(0) = 1 = q(0)$. Also p and q have positive real part in $\mathbb{U}$ and hence $|p_i| \leq 2$ and $|q_i| \leq 2$. In the view of (2.3), (2.4)and (2.5), clearly, Using (2.5) and (2.6), one can easily verify that

$$\phi\left(\frac{p(z)-1}{p(z)+1}\right)=1+\frac{B_1p_1}{2}z+\left(\frac{B_1}{2}\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}B_2p_1^2\right)z^2+\cdots \quad (2.7)$$

and

$$\phi\left(\frac{q(w)-1}{q(w)+1}\right)=1+\frac{B_1q_1}{2}w+\left(\frac{B_1}{2}\left(q_2-\frac{q_1^2}{2}\right)+\frac{B_2q_1^2}{4}\right)w^2+\cdots. \quad (2.8)$$

Since $f\in\Sigma$ has the Maclaurin series given by (1.1), computation shows that its inverse $g=f^{-1}$ has the expansion given by (1.2). It follows from (2.6), (2.7) and (2.8) that

$$(1+\lambda)^m a_2=\frac{1}{2\lambda}B_1p_1b, \quad (2.9)$$

$$4\lambda(1+2\lambda)^m a_3-\lambda(1+\lambda)^{2m}a_2^2=\frac{1}{2}bB_1\left(p_2-\frac{1}{2}p_1^2\right)+\frac{1}{4}bB_2p_1^2 \quad (2.10)$$

and

$$-(1+\lambda)^m a_2=\frac{1}{2\lambda}B_1bq_1, \quad (2.11)$$

$$\lambda\left(8\lambda(1+2\lambda)^m-(1+\lambda)^{2m}\right)a_2^2-4\lambda(1+2\lambda)^m a_3=\frac{1}{2}bB_1\left(q_2-\frac{1}{2}q_1^2\right)$$
$$+\frac{1}{4}bB_2q_1^2. \quad (2.12)$$

From (2.9) and (2.11), it follows that

$$p_1=-q_1. \quad (2.13)$$

Now (2.10), (2.12) and (2.13) gives

$$a_2^2=\frac{B_1^3b^2(p_2+q_2)}{4\left[\left(4(1+2\lambda)^m-(1+\lambda)^{2m}\right)B_1^2b\lambda+(B_1-B_2)\lambda^2(1+\lambda)^{2m}\right]}. \quad (2.14)$$

Using the fact that $|p_2|\le 2$ and $|q_2|\le 2$ gives the desired estimate on $|a_2|$,

$$|a_2| \leq \frac{B_1\sqrt{B_1}\,|b|}{\sqrt{\left(4\,(1+2\lambda)^m-(1+\lambda)^{2m}\right)B_1^2 b\lambda+(B_1-B_2)\,\lambda^2\,(1+\lambda)^{2m}}}.$$

From (2.10)-(2.12), gives

$$a_3 = \frac{\frac{bB_1}{2}\left[8\left(1+2\lambda)^m-(1+\lambda)^{2m}\right)p_2+(1+\lambda)^{2m}q_2\right]}{8\lambda\left[4(1+2\lambda)^{2m}-(1+\lambda)^{2m}(1+2\lambda)^m\right]}$$

$$+\frac{2(1+2\lambda)^m p_1^2\,(B_2-B_1)\,b}{8\lambda\left[4(1+2\lambda)^{2m}-(1+\lambda)^{2m}(1+2\lambda)^m\right]}$$

Using the inequalities $|p_1| \leq 2$, $|p_2| \leq 2$ and $|q_2| \leq 2$ for functions with positive real part yields the desired estimation of $|a_3|$.

■ For a choice of $\phi(z) = \dfrac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, we have the following corollary.

Corollary 2.3 *Let $-1 \leq B < A \leq 1$. If $f \in ST_\Sigma\left(b, \frac{1+Az}{1+Bz}\right)$, then*

$$|a_2| \leq \frac{|b|\,(A-B)}{\sqrt{\left(4\,(1+2\lambda)^m-(1+\lambda)^{2m}\right)(A-B)\,b\lambda+(1+B)\,\lambda^2\,(1+\lambda)^{2m}}}$$

and

$$|a_3| \leq \frac{|A-B|\,(1+|1+B|)\,|b|}{\lambda\left(4\,(1+2\lambda)^m-(1+\lambda)^{2m}\right)}.$$

Theorem 2.4 *Let the function $f(z) \in \mathcal{A}$ be given by (1.1). If $ST_\Sigma(\alpha_1, \beta_1, b, \phi)$, then*

$$|a_2| \leq \frac{(\alpha_1+1)\,B_1\sqrt{B_1}\,|b|}{\sqrt{\left(4\,(1+2\lambda)^m-(1+\lambda)^{2m}\right)B_1^2 b\,(\alpha_1+1)+(B_1-B_2)\,(1+\lambda)^{2m}}} \tag{2.15}$$

and

$$|a_3| \leq \frac{(\alpha_1+1)\,(B_1+|B_2-B_1|)\,|b|}{\left(4\,(1+2\lambda)^m-(1+\lambda)^{2m}\right)}.$$

Proof. Since $ST_{\Sigma}(\alpha_1, \beta_1, b, \phi)$, there exists two analytic functions $r, s : \mathbb{U} \to \mathbb{U}$, with $r(0) = 0 = s(0)$, such that

$$1 + \frac{1}{b}\left(\frac{D_{\lambda}^{m}(\alpha_1 + 1, \beta_1) f(z)}{D_{\lambda}^{m}(\alpha_1, \beta_1) f(z)} - 1\right) = \phi(r(z)) \tag{2.16}$$

and

$$1 + \frac{1}{b}\left(\frac{D_{\lambda}^{m}(\alpha_1 + 1, \beta_1) g(w)}{D_{\lambda}^{m}(\alpha_1, \beta_1) g(w)} - 1\right) = \phi(s(z)).$$

Using (2.3), (2.4), (2.7) and (2.8), one can easily verified that

$$(1 + \lambda)^{m} a_2 = \frac{(\alpha_1 + 1)}{2} B_1 p_1 b, \tag{2.17}$$

$$4(1 + 2\lambda)^{m} a_3 - (1 + \lambda)^{2m} a_2^2 = (\alpha_1 + 1)\left[\frac{1}{2} b B_1\left(p_2 - \frac{1}{2} p_1^2\right) + \frac{1}{4} b B_2 p_1^2\right] \tag{2.18}$$

and

$$-(1 + \lambda)^{m} a_2 = \frac{(\alpha_1 + 1)}{2} B_1 p_1 b, \tag{2.19}$$

$$\left(8(1 + 2\lambda)^{m} - (1 + \lambda)^{2m}\right) a_2^2 - 4(1 + 2\lambda)^{m} a_3 =$$

$$= (\alpha_1 + 1)\left[\frac{1}{2} b B_1\left(q_2 - \frac{1}{2} q_1^2\right) + \frac{1}{4} b B_2 q_1^2\right]. \tag{2.20}$$

From (2.17) and (2.19), it follows that

$$p_1 = -q_1. \tag{2.21}$$

Now (2.18), (2.20), (2.21) and using the fact that $|p_2| \leq 2$ and $|q_2| \leq 2$,

$$|a_2| \leq \frac{|\alpha_1 + 1| B_1 \sqrt{B_1} |b|}{\sqrt{\left(4(1 + 2\lambda)^{m} - (1 + \lambda)^{2m}\right) B_1^2 b(\alpha_1 + 1) + (B_1 - B_2)(1 + \lambda)^{2m}}}.$$

From (2.18)-(2.20), gives

$$|a_3| \leq \frac{|\alpha_1 + 1| (B_1 + |B_2 - B_1|) |b|}{\left(4(1 + 2\lambda)^{m} - (1 + \lambda)^{2m}\right)}.$$

■

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