

GLOBAL SOLUTION FOR THE COAGULATION EQUATION OF WATER DROPS IN FALL WITH THE HORIZONTAL WIND*

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Abstract

We consider the integro-differential equation describing the coagulation process of water drops falling in the air in a three-dimensional domain with presence of a horizontal wind. Under suitable hypothesis and some conditions we prove the existence of the stationary solution thus the global solution using the techniques developed in [10] and [2].

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1 Introduction

We consider the equation which describes the displacement of drops by the gravitational force and by the horizontal wind as well as the coagulation

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process. From a mathematical point of view, it is about the Smoluchowski coagulation equation (see [18], [13], [19]) with the displacement of drops determined by their mass; it is an integro-differential equation for an unknown function $\sigma = \sigma(m, t, x, y, z)$, that represents the density (compared to the air volume) of the liquid water contained in the drops of mass m at time t and at position $(x, y, z) \in \mathbb{R}^3$. The air motion in consideration is a horizontal wind in the direction of the x axis which depends on y (i.e $\bar{v} = \bar{v}(y)$). In [10] the authors proved the existence of the stationary solution with presence of a constant horizontal wind whereas in [2] the authors proved the existence and the uniqueness of the global solution of the same equation in a domain with one-dimensional space. In this work, we prove the existence and the uniqueness of the global solution in a three-dimensional domain with presence of a horizontal wind and with initial and boundary conditions (entry conditions) in a suitable spaces.

From a technical point of view, this work uses the techniques developed in [10] and [2], in particular the introduction of the curves family on which we consider the coagulation integral operator, and their properties, and on the construction of “cone of dependence” for the solution.

2 Position of the problem

Let’s consider the domain $\mathbb{R}^2 \times [0, 1]$, which represents a “horizontal” area in which the drops move due to the gravitational force and with the wind. Let’s indicate by $\sigma(m, t, x, y, z)$ the density of the water liquid contained in the drops of mass m at the point $(x, y, z) \in \mathbb{R}^2 \times]0, 1[$ at the moment $t \in \mathbb{R}_+$.

In the same way to [10] and [2], we suppose that the drops undergo the coagulation process and at the same time move by the gravitational force and the air motion in which they are undergoing the friction effect with this last; these considerations bring us to the following equation (see [1], [16], [10], [2])

$$\begin{aligned} & \partial_t \sigma(m, t, x, y, z) + \nabla_{(x,y,z)} \cdot (\sigma(m, t, x, y, z)u(m)) = & (1) \\ & = \frac{m}{2} \int_0^m \beta(m - m', m') \sigma(m', t, x, y, z) \sigma(m - m', t, x, y, z) dm' + \\ & \quad - m \int_0^\infty \beta(m, m') \sigma(m, t, x, y, z) \sigma(m', t, x, y, z) dm', \end{aligned}$$

where $\nabla_{(x,y,z)} = (\partial_x, \partial_y, \partial_z)$, while $\beta(m_1, m_2)$ represents the probability of meeting between a drop with mass m_1 and another with mass m_2 , and $u(m)$

indicates the velocity of drops with mass m . We suppose that

$$\beta(\cdot, \cdot) \in C(\mathbb{R}_+ \times \mathbb{R}_+), \quad \beta(m_1, m_2) \geq 0 \quad \forall (m_1, m_2) \in \mathbb{R}_+ \times \mathbb{R}_+,$$

$$\beta(m_1, m_2) = \beta(m_2, m_1)$$

and we admit that $u = u(m)$ is given by

$$u = u(m) = \left(\bar{v}(y), 0, -\frac{g}{\alpha(m)} \right), \quad (2)$$

where $\bar{v}(y)$ is the air velocity, g is a positive constant representing the gravitational acceleration and $\alpha(m)$ is the friction coefficient between drops and air. The relation (2) corresponds, in a good approximation, at the real velocity of drops in the atmosphere (see for example [17], [1], [16]).

As the small drops evaporate immediately due to the very high curve of surface (see [15], [8]) and on the other hand the very large drops fragment due to the friction with surrounding air, we consider that the drops are absent apart from an interval $[\bar{m}_a, \bar{m}_A]$ and consequently the function σ verifies

$$\sigma(m) = 0 \quad \text{for } m \in [0, \bar{m}_a[\cup]\bar{m}_A, \infty[.$$

This permit us to define the functions $\alpha(\cdot)$, $\beta(\cdot, \cdot)$ such that

$$0 < \inf_{m \in \mathbb{R}_+} \alpha(m) \leq \sup_{m \in \mathbb{R}_+} \alpha(m) < \infty$$

and

$$\beta(m_1, m_2) = 0 \quad \text{for } m_1 + m_2 > \bar{m}_A.$$

We pose

$$\bar{\alpha}_0 = \sup_{m \in \mathbb{R}_+} \alpha(m). \quad (3)$$

3 Stationary solution

We consider the following stationary equation of (1)

$$\begin{aligned} \nabla_{(x,y,z)} \cdot (\sigma(m, x, y, z)u(m)) = & \quad (4) \\ = \frac{m}{2} \int_0^m \beta(m - m', m') \sigma(m', x, y, z) \sigma(m - m', x, y, z) dm' + \\ & - m \int_0^\infty \beta(m, m') \sigma(m, x, y, z) \sigma(m', x, y, z) dm' \end{aligned}$$

with the boundary condition (entry condition)

$$\sigma(m, x, y, 1) = \bar{\sigma}(m, x, y). \quad (5)$$

3.1 Preliminaries

To solve the equation (4) with the condition (5), we will use the idea to transform it into an ordinary differential equation, by introducing the change of variables $(m, x, y, z) \mapsto (\tilde{m}, \xi, \tilde{y}, \tilde{z})$ defined by

$$\begin{cases} \tilde{m} = m, \\ \xi = x - \bar{v}(y)\frac{\alpha(m)}{g}(1-z), \\ \tilde{y} = y, \\ \tilde{z} = z \end{cases} \tag{6}$$

and let us define

$$\bar{\sigma}(\tilde{m}, \xi, \tilde{y}, \tilde{z}) = \sigma(m, x, y, z) = \sigma(m, \xi + \bar{v}(y)\frac{\alpha(m)}{g}(1-z), y, z).$$

In the following, we will simply write m, y, z and $\sigma(m, \xi, y, z)$ instead of $\tilde{m}, \tilde{y}, \tilde{z}$ and $\bar{\sigma}(\tilde{m}, \xi, \tilde{y}, \tilde{z})$, thus, the equation (4) will be

$$\begin{aligned} \frac{\partial}{\partial z}\sigma(m, \xi, y, z) &= \tag{7} \\ &= -\frac{m\alpha(m)}{2g} \int_0^m \beta(m-m', m')\sigma(m', \eta(m, m', \xi, y, z), y, z) \times \\ &\quad \times \sigma(m-m', \eta(m, m-m', \xi, y, z), y, z) dm' + \\ &+ \frac{m\alpha(m)}{g} \int_0^\infty \beta(m, m')\sigma(m, \xi, y, z)\sigma(m', \eta(m, m', \xi, y, z), y, z) dm', \end{aligned}$$

where

$$\eta(m, m', \xi, y, z) = \xi + \bar{v}(y)\frac{\alpha(m) - \alpha(m')}{g}(1-z)$$

and the condition (5) will be:

$$\sigma(m, \xi, y, 1) = \bar{\sigma}(m, \xi, y). \tag{8}$$

Consequently we will reformulate the equation (7) into an ordinary differential equation in a Banach space (or in a Frechet space). To suitably treat the integral operator in a functional framework, we introduce, for each fixed $y \in \mathbb{R}, z \in [0, 1]$, the curves family given by:

$$\gamma_\tau = \gamma_{\tau, y, z} = \{(m, \xi) \in \mathbb{R}_+ \times \mathbb{R} \mid \xi = \tau - \bar{v}(y)\frac{\alpha(m)}{g}(1-z)\}, \tau \in \mathbb{R}. \tag{9}$$

This curves family γ_τ is similar to that used in [10], however this last depends of y .

In a similar way to [10] we define a measure μ_γ on the curves γ_τ . More precisely, indicating by $P_{\mathbb{R}_+}$ the projection of γ_τ on \mathbb{R}_+ , we define the measurable sets of γ_τ and the measure μ_γ on γ_τ by the relations

- i) $A' \subset \gamma_\tau$ is measurable if and only if $P_{\mathbb{R}_+} A'$ is measurable according to Lebesgue on \mathbb{R}_+ ,
- ii) $\mu_\gamma(A') = \mu_{L, \mathbb{R}_+}(P_{\mathbb{R}_+} A')$, where $\mu_{L, \mathbb{R}_+}(\cdot)$ is the Lebesgue's measure on \mathbb{R}_+ .

As the curves γ_τ , $\tau \in \mathbb{R}$, are parallel, it is seen that the projection $P_{\mathbb{R}_+}$ and the measure $\mu_\gamma(\cdot)$ do not depend on $\tau \in \mathbb{R}$.

We remember that the measure $\mu_\gamma(\cdot)$ has the same properties with those proved in [10], indeed we have the following lemmas.

Lemma 1 *Let A a measurable set (according to Lebesgue) on $\mathbb{R}_+ \times \mathbb{R}$. We pose*

$$A_\tau = \{ m \in \mathbb{R}_+ \mid \exists \xi \in \mathbb{R} \text{ such that } (m, \xi) \in \gamma_\tau \cap A \},$$

$$A_m = \{ \tau \in \mathbb{R} \mid \exists \xi \in \mathbb{R} \text{ such that } (m, \xi) \in \gamma_\tau \cap A \}.$$

Then we have

$$\begin{aligned} \mu_{L, \mathbb{R}_+ \times \mathbb{R}}(A) &= \tilde{\mu}(A) = \int_{-\infty}^{\infty} \mu_\gamma(A_\tau) d\tau = \int_{\gamma_0} \mu_{L, \mathbb{R}}(A_m) \mu_\gamma(dm) = \\ &= \int_0^{\infty} \mu_{L, \mathbb{R}}(A_m) dm. \end{aligned} \quad (10)$$

(We indicate by dm , $d\tau$, $d\xi$ etc... instead of $\mu_{L, \mathbb{R}_+}(dm)$, $\mu_{L, \mathbb{R}}(d\tau)$, $\mu_{L, \mathbb{R}}(d\xi)$ etc...).

Lemma 2 *Let $\sigma(m, \xi) \in L^1(\mathbb{R}_+ \times \mathbb{R})$. Then, for almost any $\tau \in \mathbb{R}$ the restriction of $\sigma(m, \xi)$ to γ_τ belongs to $L^1(\gamma_\tau, \mu_\gamma)$.*

Lemma 3 *Let $\sigma(m, \xi) \in L^1(\mathbb{R}_+ \times \mathbb{R})$. Then we have*

$$\begin{aligned} \int_{\mathbb{R}_+ \times \mathbb{R}} \sigma(m, \xi) dm d\xi &= \int_{\mathbb{R}_+ \times \mathbb{R}} \sigma(m, \xi) d\tilde{\mu} = \\ &= \int_{-\infty}^{\infty} \left(\int_{\gamma_\tau} \sigma(m, \xi) \mu_\gamma(dm) \right) d\tau = \int_{\gamma_0} \left(\int_{-\infty}^{\infty} \sigma(m, \xi(m, \tau)) d\tau \right) \mu_\gamma(dm) = \\ &= \int_0^{\infty} \left(\int_{-\infty}^{\infty} \sigma(m, \xi) d\xi \right) dm = \int_{-\infty}^{\infty} \left(\int_0^{\infty} \sigma(m, \xi) dm \right) d\xi, \end{aligned}$$

where $\xi(m, \tau) = \tau - \bar{v}(y) \frac{\alpha(m)}{g} (1 - z)$.

Lemma 4 *Let f and g two functions belonging in $L^1(\gamma_\tau, \mu_\gamma)$. We pose*

$$(f * g)(m) = \int_{\gamma_\tau} f(m - m')g(m')\mu_\gamma(dm').$$

*Then we have $f * g \in L^1(\gamma_\tau, \mu_\gamma)$ and*

$$\|f * g\|_{L^1(\gamma_\tau, \mu_\gamma)} \leq \|f\|_{L^1(\gamma_\tau, \mu_\gamma)}\|g\|_{L^1(\gamma_\tau, \mu_\gamma)}.$$

For the proof of this lemmas see [10].

We pose

$$\tau(m, \xi, y, z) = \xi + \bar{v}(y) \frac{\alpha(m)}{g}(1 - z), \quad \gamma_\tau^{[0,m]} = \gamma_\tau \cap [0, m] \times \mathbb{R}. \quad (11)$$

Then we can write the equation (7) in the form

$$\frac{\partial}{\partial z} \sigma(z) = F_z(\sigma(z)), \quad \sigma(z) = \sigma(\cdot, \cdot, \cdot, z) \quad (12)$$

with

$$\begin{aligned} F_z(\sigma(z)) &= F_z(\sigma(z))(m, \xi, y) = \\ &= -\frac{m\alpha(m)}{2g} \int_{\gamma_{\tau(m, \xi, y, z)}^{[0,m]}} \beta(m - m', m') \sigma(m', \eta', y, z) \sigma(m - m', \eta'', y, z) \mu_\gamma(dm') + \\ &\quad + \frac{m\alpha(m)}{g} \int_{\gamma_{\tau(m, \xi, y, z)}} \beta(m, m') \sigma(m', \eta', y, z) \sigma(m, \xi, y, z) \mu_\gamma(dm'), \end{aligned}$$

where η' and η'' are defined such that

$$(m', \eta') \in \gamma_{\tau(m, \xi, y, z)}, \quad (m - m', \eta'') \in \gamma_{\tau(m, \xi, y, z)}.$$

3.2 Existence and uniqueness of the solution with the data in L^1

To prove the existence and the uniqueness for the solution of the equation (12) with the condition (8), we suppose that:

$$\bar{\sigma}(\cdot, \cdot, \cdot) \in L^1(\mathbb{R}_+ \times \mathbb{R}^2) \cap L^\infty(\mathbb{R}_+ \times \mathbb{R}^2), \quad (13)$$

$$\bar{\sigma}(m, \xi, y) \geq 0 \quad a.e. \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^2, \quad (14)$$

$$supp(\bar{\sigma}) \subset [\bar{m}_a, \bar{m}_A] \times \mathbb{R}^2, \quad (15)$$

$$\|\bar{\sigma}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^2)} < \frac{1}{M_1(\bar{m}_A - \bar{m}_a)}, \tag{16}$$

where

$$M_1 = \sup_{2\bar{m}_a \leq m \leq \bar{m}_A, \bar{m}_a \leq m' \leq m - \bar{m}_a} \frac{m\alpha(m)}{2g} \beta(m - m', m'). \tag{17}$$

Then we have the following result.

Proposition 1 *If $\bar{\sigma}(m, \xi, y)$ satisfies the conditions (13)–(16), then the equation (12) with the condition (8) admits one and only one solution σ verifying*

$$\sigma \in C([0, 1]; L^1(\mathbb{R}_+ \times \mathbb{R}^2)) \times L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times [0, 1]). \tag{18}$$

Proof. As in the equation (12) neither the derivative nor the integral compared to y arise, this circumstance implies that the equation for each fixed $y \in \mathbb{R}$ can be solved independently. That permit us to consider (12), (8) separately for each $y \in \mathbb{R}$. Therefore, we pose $\sigma(m, \xi, z) = \sigma(m, \xi, y, z)$, $\bar{\sigma}(m, \xi) = \bar{\sigma}(m, \xi, y)$ and we write \bar{v} instead of $\bar{v}(y)$, the proposition is proved in the same meaner as proposition 5.1 in [10]. \square

3.3 Existence and uniqueness of the solution with the data in L^∞

To prove the existence and the uniqueness of the solution for (12), (8) in a general case, we will use the ‘‘cone of dependance’’ property.

Let ω in $\mathbb{R}_+ \times \mathbb{R}^2$ a measurable set such that $0 < mes(\omega) < \infty$, we define

$$D[\omega] = \bigcup_{(m, \xi, y) \in \omega} D_{(m, \xi, y)}, \tag{19}$$

where

$$\begin{aligned} D_{(m, \xi, y)} &= \bigcup_{0 \leq z \leq 1} \left(\bigcup_{\tau_-(m, \xi, y, z) \leq \tau \leq \tau_+(m, \xi, y)} \gamma_{\tau, y, z} \right) = \tag{20} \\ &= \{(m', \eta', y', z') \in \mathbb{R}_+ \times \mathbb{R}^2 \times [0, 1] / \eta' = \tau - \bar{v}(y') \frac{\alpha(m')}{g} (1 - z'), \\ &\quad y' = y, \tau_-(m, \xi, y', z') \leq \tau \leq \tau_+(m, \xi, y')\} \end{aligned}$$

with

$$\begin{cases} \tau_+(m, \xi, y) = \tau(m, \xi, y, 0) = \xi + \bar{v}(y) \frac{\alpha(m)}{g}, \\ \tau_-(m, \xi, y, z) = \tau_+(m, \xi, y) - \bar{v}(y) \frac{\bar{\alpha}_0}{g} z = \xi + \bar{v}(y) \frac{\alpha(m)}{g} - \bar{v}(y) \frac{\bar{\alpha}_0}{g} z. \end{cases} \tag{21}$$

We also define $D_\omega(z)$ by

$$\begin{aligned} D_\omega(z) &= \bigcup_{(m,\xi,y) \in \omega} \left(\bigcup_{\tau_-(m,\xi,y,z) \leq \tau \leq \tau_+(m,\xi,y)} \gamma_{\tau,y,z} \right) = \\ &= \{(m', \eta', y', z') \in D[\omega] \mid z' = z\} \end{aligned} \quad (22)$$

thus $D_\omega(z_1)$ is the intersection of $\bigcup_{(m,\xi,y) \in \omega} D_{(m,\xi,y)}$ with the plan $z = z_1$.

According to the definition of the set $D_{(m,\xi,y)}$ we remark that

$$(m', \eta', y', z') \in D_{(m,\xi,y)} \Rightarrow \gamma_{\tau(m',\eta',y',z'),y',z'} \subset D_{(m,\xi,y)},$$

$$\tau(m_1, \xi_1, y_1, 0) = \tau(m_2, \xi_2, y_2, 0) \Rightarrow D_{(m_1,\xi_1,y_1)} = D_{(m_2,\xi_2,y_2)};$$

consequently, if (m_1, ξ_1) and (m_2, ξ_2) are on a curve $\gamma_{\tau,y,0}$, then they define the same set.

The ‘‘cone of dependence’’ property is given by the following lemma.

Lemma 5 *Let $\bar{\sigma}^{[1]}$ and $\bar{\sigma}^{[2]}$ be two functions defined on $\mathbb{R}_+ \times \mathbb{R}^2$ satisfying the conditions of proposition 1. Let $\sigma^{[1]}$ (resp. $\sigma^{[2]}$) be the solution of (12), (8) with $\bar{\sigma} = \bar{\sigma}^{[1]}$ (resp. $\bar{\sigma} = \bar{\sigma}^{[2]}$). If we have*

$$\bar{\sigma}^{[1]} = \bar{\sigma}^{[2]} \quad \text{on } D_\omega(1), \quad (23)$$

then

$$\sigma^{[1]} = \sigma^{[2]} \quad \text{a.e. in } D[\omega].$$

Proof. Writing the equation (12) into an integral form, we have

$$\begin{aligned} \sigma^{[i]}(m, \xi, y, z) &= \bar{\sigma}^{[i]}(m, \xi, y) + \\ &+ \frac{m\alpha(m)}{2g} \int_z^1 \int_{\gamma_{\tau(m,\xi,y,z'),y,z'}^{[0,m]}} \beta(m - m', m') \sigma^{[i]}(m', \eta', y, z') \sigma^{[i]}(m - m', \eta'', y, z') \\ &\mu_\gamma(dm') dz' - \frac{m\alpha(m)}{g} \int_z^1 \int_{\gamma_{\tau(m,\xi,y,z'),y,z'}} \beta(m, m') \sigma^{[i]}(m', \eta', y, z') \times \\ &\times \sigma^{[i]}(m, \xi, y, z') \mu_\gamma(dm') dz', \quad i = 1, 2. \end{aligned}$$

Making the difference for $i = 1$ and $i = 2$, we have

$$|\sigma^{[1]}(m, \xi, y, z) - \sigma^{[2]}(m, \xi, y, z)| \leq |\bar{\sigma}^{[1]}(m, \xi, y) - \bar{\sigma}^{[2]}(m, \xi, y)| +$$

$$\begin{aligned}
& + C_\beta \left[\int_z^1 \int_{\gamma_{\tau(m, \xi, y, z'), y, z'}^{[0, m]}} \left(|\sigma^{[1]}(m - m', \eta'', y, z') - \sigma^{[2]}(m - m', \eta'', y, z')| \right. \right. \\
& \sigma^{[2]}(m', \eta', y, z') + |\sigma^{[1]}(m', \eta', y, z') - \sigma^{[2]}(m', \eta', y, z')| \sigma^{[1]}(m - m', \eta'', y, z') \left. \left. \right) \right. \\
& \mu_\gamma(dm') dz' + \int_z^1 \int_{\gamma_{\tau(m, \xi, y, z'), y, z'}} \left(|\sigma^{[1]}(m, \xi, y, z') - \sigma^{[2]}(m, \xi, y, z')| \sigma^{[2]}(m', \eta', y, z') + \right. \\
& \left. \left. + |\sigma^{[1]}(m', \eta', y, z') - \sigma^{[2]}(m', \eta', y, z')| \sigma^{[1]}(m, \xi, y, z') \right) \mu_\gamma(dm') dz' \right],
\end{aligned}$$

where

$$C_\beta = \max \left[\sup_{0 < m' < m < \infty} \frac{m\alpha(m)}{2g} \beta(m - m', m'), \sup_{m, m' \in \mathbb{R}_+} \frac{m\alpha(m)}{g} \beta(m, m') \right].$$

We deduce from it that

$$\begin{aligned}
|\sigma^{[1]}(m, \xi, y, z) - \sigma^{[2]}(m, \xi, y, z)| & \leq |\bar{\sigma}^{[1]}((m, \xi, y) - \bar{\sigma}^{[2]}(m, \xi, y))| + \quad (24) \\
& + C_\beta \left[\int_z^1 \left(\|\sigma^{[1]}(\cdot, \cdot, y, z') - \sigma^{[2]}(\cdot, \cdot, y, z')\|_{L^\infty(\gamma_{\tau(m, \xi, y, z'), y, z'})} \right. \right. \\
& \|\sigma^{[2]}(\cdot, \cdot, y, z')\|_{L^1(\gamma_{\tau(m, \xi, y, z'), y, z'})} + \|\sigma^{[1]}(\cdot, \cdot, y, z')\|_{L^1(\gamma_{\tau(m, \xi, y, z'), y, z'})} \\
& \left. \left. \|\sigma^{[1]}(\cdot, \cdot, y, z') - \sigma^{[2]}(\cdot, \cdot, y, z')\|_{L^\infty(\gamma_{\tau(m, \xi, y, z'), y, z'})} \right) dz' + \right. \\
& + \int_z^1 \left(\|\sigma^{[1]}(\cdot, \cdot, y, z') - \sigma^{[2]}(\cdot, \cdot, y, z')\|_{L^\infty(\gamma_{\tau(m, \xi, y, z'), y, z'})} \right. \\
& \|\sigma^{[2]}(\cdot, \cdot, y, z')\|_{L^1(\gamma_{\tau(m, \xi, y, z'), y, z'})} + (\bar{m}_A - \bar{m}_a) \times \\
& \|\sigma^{[1]}(\cdot, \cdot, y, z') - \sigma^{[2]}(\cdot, \cdot, y, z')\|_{L^\infty(\gamma_{\tau(m, \xi, y, z'), y, z'})} \\
& \left. \left. \|\sigma^{[1]}(\cdot, \cdot, y, z') - \sigma^{[2]}(\cdot, \cdot, y, z')\|_{L^\infty(\gamma_{\tau(m, \xi, y, z'), y, z'})} \right) dz' \right].
\end{aligned}$$

Now let's consider a generic point (m, ξ, y, z) of $D[\omega]$, by virtue of (20)–(21) there exists $(m_0, \xi_0, y_0) \in \omega \subset \mathbb{R}_+ \times \mathbb{R}^2$ such that

$$\begin{aligned}
& \xi_0 + \bar{v}(y_0) \frac{\alpha(m_0)}{g} - \bar{v}(y_0) \frac{\bar{\alpha}_0}{g} z = \tau_-(m_0, \xi_0, y_0, z) \leq \\
& \leq \xi + \bar{v}(y) \frac{\alpha(m)}{g} (1 - z) \leq \tau_+(m_0, \xi_0, y_0) = \xi_0 + \bar{v}(y_0) \frac{\alpha(m_0)}{g}, \\
& \qquad \qquad \qquad y = y_0.
\end{aligned}$$

From this inequalities, the inequality $\bar{v}(y) \frac{\alpha(m)}{g} \leq \bar{v}(y) \frac{\bar{\alpha}_0}{g} < 0$, implies that for $0 \leq z \leq z' \leq 1$, we have

$$\begin{cases} \xi_0 + \bar{v}(y_0) \frac{\alpha(m_0)}{g} - \bar{v}(y_0) \frac{\bar{\alpha}_0}{g} z' \leq \xi + \bar{v}(y) \frac{\alpha(m)}{g} (1 - z') \leq \xi_0 + \bar{v}(y_0) \frac{\alpha(m_0)}{g}, \\ y = y_0, \end{cases}$$

by virtue of (11) and (21), we have

$$\begin{cases} \tau_-(m_0, \xi_0, y_0, z') \leq \tau(m, \xi, y, z') \leq \tau_+(m_0, \xi_0, y_0), \\ y = y_0 \end{cases}$$

and, according to the definition (22) of the set $D_\omega(z)$, we prove that

$$\gamma_{\tau(m, \xi, y, z'), y, z'} \subset D_\omega(z') \quad \text{for } 0 \leq z \leq z' \leq 1.$$

We recall that we have moreover, for $i = 1, 2$

$$\|\sigma^{[i]}(\cdot, \cdot, y, z)\|_{L^1(\gamma_{\tau(m, \xi, y, z), y, z, \mu_\gamma})} \leq (\bar{m}_A - \bar{m}_a) \|\sigma^{[i]}(\cdot, \cdot, y, z)\|_{L^\infty(D_\omega(z))},$$

for almost any $(m, \xi, y) \in \mathbb{R}_+ \times \mathbb{R}^2$.

From (24) we deduce that

$$\begin{aligned} & \|\sigma^{[1]}(\cdot, \cdot, \cdot, z) - \sigma^{[2]}(\cdot, \cdot, \cdot, z)\|_{L^\infty(D_\omega(z))} \leq \|\bar{\sigma}^{[1]} - \bar{\sigma}^{[2]}\|_{L^\infty(D_\omega(1))} \\ & + C \int_z^1 \left(\|\sigma^{[1]}(\cdot, \cdot, \cdot, z')\|_{L^\infty(D_\omega(z'))} + \|\sigma^{[2]}(\cdot, \cdot, \cdot, z')\|_{L^\infty(D_\omega(z'))} \right) \times \\ & \quad \times \|\sigma^{[1]}(\cdot, \cdot, \cdot, z') - \sigma^{[2]}(\cdot, \cdot, \cdot, z')\|_{L^\infty(D_\omega(z'))} dz', \end{aligned}$$

where C is a constant independent of z , using the Gronwall's lemma, we obtain

$$\begin{aligned} & \|\sigma^{[1]}(\cdot, \cdot, \cdot, z) - \sigma^{[2]}(\cdot, \cdot, \cdot, z)\|_{L^\infty(D_\omega(z))} \leq \|\bar{\sigma}^{[1]} - \bar{\sigma}^{[2]}\|_{L^\infty(D_\omega(1))} \times \quad (25) \\ & \times \exp \left(C \int_z^1 (\|\sigma^{[1]}(\cdot, \cdot, \cdot, z')\|_{L^\infty(D_\omega(z'))} + \|\sigma^{[2]}(\cdot, \cdot, \cdot, z')\|_{L^\infty(D_\omega(z'))}) dz' \right). \end{aligned}$$

However, under the assumption (23) we have

$$\|\bar{\sigma}^{[1]} - \bar{\sigma}^{[2]}\|_{L^\infty(D_\omega(1))} = 0,$$

that enables us to deduce from (25) that

$$\|\sigma^{[1]}(\cdot, \cdot, \cdot, z) - \sigma^{[2]}(\cdot, \cdot, \cdot, z)\|_{L^\infty(D_\omega(z))} \leq 0$$

and, taking into account the relation $D[\omega] = \bigcup_{0 \leq z \leq 1} D_\omega(z)$, we have

$$\sigma^{[1]}(m, \xi, y, z) = \sigma^{[2]}(m, \xi, y, z) \quad \text{a.e. in } D[\omega].$$

The lemma is proved. \square

Now we can prove the principal theorem.

Theorem 1 *If $\bar{\sigma}_1 \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$ satisfies the conditions*

$$\bar{\sigma}_1(m, \xi, y) \geq 0 \quad \text{a.e. on } \mathbb{R}_+ \times \mathbb{R}^2, \quad (26)$$

$$\bar{\sigma}_1(m, \xi, y) = 0 \quad \text{for } m \in [0, \bar{m}_a] \cup [\bar{m}_A, \infty[, \quad (27)$$

$$\|\bar{\sigma}_1\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^2)} < \frac{1}{M_1(\bar{m}_A - \bar{m}_a)}, \quad (28)$$

then the equation (12) with the condition (8) admits one and only one solution verifying

$$\sigma \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times [0, 1])$$

with

$$\sigma(m, \xi, y, z) \geq 0 \quad \text{a.e. in } \mathbb{R}_+ \times \mathbb{R}^2 \times [0, 1],$$

$$\sigma(m, \xi, y, z) = 0 \quad \text{for } m \in [0, \bar{m}_a] \cup [\bar{m}_A, \infty[.$$

Proof. We consider a measurable and bounded sets family ω_i , $i \in \mathbb{N}^*$, defined by

$$\omega_i = \{(m, \xi, y) \in \mathbb{R}_+ \times \mathbb{R}^2 / \bar{m}_a \leq m \leq \bar{m}_A, -i \leq \xi \leq i, -i \leq y \leq i\}. \quad (29)$$

The definition of $D[\omega]$ permits us to define a number N such that

$$D_{\omega_i}(1) \subset \{(m, \xi, y) \in \mathbb{R}_+ \times \mathbb{R}^2 / \bar{m}_a \leq m \leq \bar{m}_A, -i - N \leq \xi \leq i + N, \\ -i - 1 \leq y \leq i + 1\}.$$

We consider the function $\psi_i \in C^\infty(\mathbb{R}^2)$; $\psi_i \geq 0$ such that

$$\psi_i(\xi, y) = \begin{cases} 1 & \text{if } |\xi| \leq i + N \text{ and } |y| \leq i + 1, \\ 0 & \text{if } |\xi| \geq i + N + 1 \text{ and } |y| \geq i + 2, \end{cases} \quad (30)$$

then we have

$$D_{\omega_i}(1) \subset \{(m, \xi, y) \in \mathbb{R}_+ \times \mathbb{R}^2 / \psi_i(\xi, y) = 1\} \quad \text{for } i \in \mathbb{N}^*. \quad (31)$$

Let the equations family

$$\partial_z \sigma^{[i]}(m, \xi, y, z) = F(\sigma^{[i]}(z))(m, \xi, y), \quad i \in \mathbb{N}^* \tag{32}$$

(with $F(\cdot)$ defined in (12)), completed by the condition

$$\bar{\sigma}^{[i]} = \psi_i \bar{\sigma} \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^2. \tag{33}$$

According to the proposition 1, the problem (32)–(33) admits one solution

$$\sigma = \sigma^{[i]} \in C([0, 1]; L^1(\mathbb{R}_+ \times \mathbb{R}^2) \cap L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times [0, 1])),$$

such that

$$\begin{aligned} \sigma^{[i]} &\geq 0 \quad \text{a.e. in } \mathbb{R}_+ \times \mathbb{R}^2 \times [0, 1], \\ \sigma^{[i]}(m, \xi, y, z) &= 0 \quad \text{for } m \in [0, \bar{m}_a] \cup [\bar{m}_A, \infty[. \end{aligned}$$

In addition, according to the definition of the sets ω_i , we have

$$D[\omega_i] \subset D[\omega_{i'}] \quad \text{for } i \leq i',$$

therefore, by virtue of lemma 5 and of (33), we have

$$\sigma^{[i]} = \sigma^{[i']} \quad \text{a.e. in } D[\omega_i] \quad \text{for } i \leq i'.$$

Defining σ by

$$\sigma = \begin{cases} \sigma^{[1]} & \text{in } D[\omega_1], \\ \sigma^{[i]} & \text{in } D[\omega_i] \setminus D[\omega_{i-1}], \quad i = 2, \dots, \end{cases}$$

we have

$$\sigma = \sigma^{[i]} \quad \text{a.e. in } D_{\omega_i}(1) \quad \forall i \in \mathbb{N}^*$$

and from (32), (61) we obtain

$$\begin{aligned} \partial_z \sigma(m, \xi, y, z) &= F(\sigma(z))(m, \xi, y) \quad \text{in } D[\omega_i] \quad \forall i \in \mathbb{N}^*, \\ \sigma &= \sigma^{[i]} = \bar{\sigma} \quad \text{on } D_{\omega_i}(1). \end{aligned}$$

Remembering the relations $\mathbb{R}_+ \times \mathbb{R}^2 \times [0, 1] \subset \bigcup_{i \in \mathbb{N}^*} D[\omega_i]$ and $\mathbb{R}_+ \times \mathbb{R}^2 \subset \bigcup_{i \in \mathbb{N}^*} D_{\omega_i}(1)$ which result from the definition of ω_i , $D[\omega_i]$, $D_{\omega_i}(1)$, we can conclude that there exists a solution of (12), (8). To prove the uniqueness, let's consider two possible solutions σ_1 and σ_2 with $\sigma_1 \neq \sigma_2$ on a set of strictly positive measure, then we can choose a measurable set ω such that

$0 < \text{mes}(\omega) < \infty$ and that $\text{mes}(\{(m, \xi, y, z) \in D[\omega] / \sigma_1 \neq \sigma_2\}) > 0$. However as σ_1 and σ_2 are solutions of (12), (8), $\sigma_1 = \sigma_2$ on $\mathbb{R}_+ \times \mathbb{R}^2 \times \{1\}$ and in particular $\sigma_1 = \sigma_2$ on $\mathbb{R}_+ \times \mathbb{R}^2 \times \{1\} \cap D[\omega]$; consequently, according to lemma 5, we have $\sigma_1 = \sigma_2$ in $D[\omega]$, this proves that it is not possible to have two solutions σ_1 and σ_2 which are different on a set from strictly positive measure. The uniqueness of the solution is proved. \square

For the existence and the uniqueness of the solution in the (m, x, y, z) co-ordinates, we have the following theorem.

Theorem 2 *If $\bar{\sigma} \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$ satisfies the conditions*

$$\bar{\sigma}(m, x, y) \geq 0 \quad \text{a.e on } \mathbb{R}_+ \times \mathbb{R}^2,$$

$$\bar{\sigma}(m, x, y) = 0 \quad \text{for } m \in [0, \bar{m}_a] \cup [\bar{m}_A, \infty[,$$

$$\|\bar{\sigma}\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^2)} < \frac{1}{M_1(\bar{m}_A - \bar{m}_a)},$$

then the equation (4) with the condition (5) admits one solution σ and only one verifying

$$\sigma \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times [0, 1]),$$

such that

$$\sigma(m, x, y, z) \geq 0 \quad \text{a.e on } \mathbb{R}_+ \times \mathbb{R}^2 \times [0, 1],$$

$$\bar{\sigma}(m, x, y, z) = 0 \quad \text{for } m \in [0, \bar{m}_a] \cup [\bar{m}_A, \infty[.$$

Proof. We associate to the problem (4)-(5), where the unknown function to find is σ , the problem (12), (8) by a bijective mapping defined by the change of variables $(m, x, y, z) \mapsto (\tilde{m}, \xi, \tilde{y}, \tilde{z})$ introduced in (6) with

$$\sigma(m, x, y, z) = \tilde{\sigma}(m, \xi + \bar{v}(y) \frac{\alpha(m)}{g} (1 - z), y, z).$$

If $\tilde{\sigma}(m, \xi, y, z)$ is the solution of the problem (12), (8) in which the existence and the uniqueness have been proved in theorem 1, then, we obtain the existence and the uniqueness of the solution σ for (4)-(5) verifying the same conditions. \square

4 Global solution for the coagulation equation of the drops in fall with a horizontal wind

We will consider the problem to find a function $\sigma(m, t, x, y, z)$, that verifies the equation (1) for

$$(m, t, x, y, z) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times [0, 1]$$

and the following boundary condition (entry condition) and initial condition

$$\sigma(m, t, x, y, 1) = \bar{\sigma}_1(m, t, x, y), \tag{34}$$

$$\sigma(m, 0, x, y, z) = \bar{\sigma}_0(m, x, y, z). \tag{35}$$

In the same way to the stationary case, to solve the equation (1) with the conditions (34)-(35), we will transform it into an ordinary differential equation, by introducing the following variables $(m, t, x, y, z) \mapsto (\tilde{m}, \tilde{t}, \xi, \tilde{y}, \tilde{z})$

$$\begin{cases} \tilde{m} = m, \\ \xi = x - \bar{v}(y) \frac{\alpha(m)}{g} (1 - z), \\ \tilde{y} = y, \\ \tilde{z} = z, \\ \tilde{t} = t - \frac{\alpha(m)}{g} (1 - z) \end{cases} \tag{36}$$

and the unknown function to find would be

$$\begin{aligned} \tilde{\sigma}(\tilde{m}, \tilde{t}, \xi, \tilde{y}, \tilde{z}) &= \sigma(m, t, x, y, z) = \\ &= \sigma\left(m, \tilde{t} + \frac{\alpha(m)}{g} (1 - z), \xi + \bar{v}(y) \frac{\alpha(m)}{g} (1 - z), y, z\right), \end{aligned}$$

we will note by m, y, z and $\sigma(m, \tilde{t}, \xi, y, z)$ instead of $\tilde{m}, \tilde{y}, \tilde{z}$ and $\tilde{\sigma}(\tilde{m}, \tilde{t}, \xi, \tilde{y}, \tilde{z})$, the equation (1) is changed into

$$\begin{aligned} &\frac{\partial}{\partial z} \sigma(m, \tilde{t}, \xi, y, z) = \tag{37} \\ &= -\frac{m \alpha(m)}{2g} \int_0^m \beta(m - m', m') \sigma(m', \tilde{t}^*(m, m', \tilde{t}, z), \eta(m, m', \xi, y, z), y, z) \times \\ &\quad \times \sigma(m - m', \tilde{t}^*(m, m - m', \tilde{t}, z), \eta(m, m - m', \xi, y, z), y, z) dm' + \\ &\quad + \frac{m \alpha(m)}{g} \int_0^\infty \beta(m, m') \sigma(m, \tilde{t}, \xi, y, z) \times \end{aligned}$$

$$\times \sigma(m', \tilde{t}^*(m, m', \tilde{t}, z), \eta(m, m', \xi, y, z), y, z) dm',$$

where

$$\begin{cases} \tilde{t}^*(m, m', \tilde{t}, z) = \tilde{t} + \frac{\alpha(m) - \alpha(m')}{g}(1 - z), \\ \eta(m, m', \xi, y, z) = \xi + \bar{v}(y) \frac{\alpha(m) - \alpha(m')}{g}(1 - z). \end{cases}$$

We introduce for each fixed $y \in \mathbb{R}$, $z \in [0, 1]$, the curves family

$$\begin{aligned} \gamma_{\tau, \zeta} = \gamma_{\tau, \zeta, y, z} = \left\{ (m, \tilde{t}, \xi) \in \mathbb{R}_+ \times \mathbb{R}^2 / \tilde{t} = \tau - \frac{\alpha(m)}{g}(1 - z), \right. \\ \left. \xi = \zeta - \bar{v}(y) \frac{\alpha(m)}{g}(1 - z) \right\} \end{aligned} \quad (38)$$

with $\tau, \zeta \in \mathbb{R}$.

Let $\tau, \zeta, \gamma_{\tau, \zeta}^{[0, m]}$ such that

$$\begin{aligned} \tau(m, \tilde{t}, z) = \tilde{t} + \frac{\alpha(m)}{g}(1 - z), \quad \zeta(m, \xi, y, z) = \xi + \bar{v}(y) \frac{\alpha(m)}{g}(1 - z), \\ \gamma_{\tau, \zeta}^{[0, m]} = \gamma_{\tau, \zeta} \cap [0, m] \times \mathbb{R}^2. \end{aligned}$$

We note by

$$\kappa = (\tau, \zeta), \quad \vartheta = (\tilde{t}, \xi), \quad q = q(y) = (1, \bar{v}(y))^T,$$

then the curves defined in (38) can be written in the following form

$$\gamma_{\kappa} = \gamma_{\kappa, y, z} = \left\{ (m, \vartheta) \in \mathbb{R}_+ \times \mathbb{R}^2 / \vartheta = \kappa - q(y) \frac{\alpha(m)}{g}(1 - z) \right\} \quad (39)$$

with

$$\kappa(m, \vartheta, y, z) = \vartheta + q(y) \frac{\alpha(m)}{g}(1 - z), \quad \gamma_{\kappa}^{[0, m]} = \gamma_{\kappa} \cap [0, m] \times \mathbb{R}^2.$$

The curves family γ_{κ} is similar to that defined in (9) in the stationary case, so in the same way, we define a measure μ_{γ} on the curves γ_{κ} and the equation (37) will be

$$\frac{\partial}{\partial z} \sigma(z) = F_z(\sigma(z)), \quad \sigma(z) = \sigma(\cdot, \cdot, \cdot, \cdot, z), \quad (40)$$

where

$$F_z(\sigma(z)) = F_z(\sigma(z))(m, \vartheta, y) = \quad (41)$$

$$\begin{aligned}
 &= -\frac{m\alpha(m)}{2g} \int_{\gamma_{\kappa(m,\vartheta,y,z)}^{[0,m]}} \beta(m-m', m')\sigma(m', \vartheta', y, z)\sigma(m-m', \vartheta'', y, z)\mu_\gamma(dm') + \\
 &\quad + \frac{m\alpha(m)}{g} \int_{\gamma_{\kappa(m,\vartheta,y,z)}} \beta(m, m')\sigma(m', \vartheta', y, z)\sigma(m, \vartheta, y, z)\mu_\gamma(dm')
 \end{aligned}$$

with ϑ' and ϑ'' are defined by the relation

$$(m', \vartheta') \in \gamma_{\kappa(m,\vartheta,y,z)}, \quad (m - m', \vartheta'') \in \gamma_{\kappa(m,\vartheta,y,z)}^{[0,m]}.$$

We remark that this equation is in the same type of the equation (12) in the stationary case and that the integral operator appearing in (41) verifies the same properties in lemmas 1, 2, 3 and 4.

In the same way, the boundary and the initial conditions will be changed into

$$\sigma(m, \tilde{t}, \xi, y, 1) = \bar{\sigma}_1^*(m, \tilde{t}, \xi, y) = \bar{\sigma}_1^*(m, \vartheta, y) \tag{42}$$

and

$$\sigma(m, -\frac{\alpha(m)}{g}(1-z), \xi, y, z) = \bar{\sigma}_0^*(m, \xi, y, z), \tag{43}$$

where $\bar{\sigma}_0^*$ and $\bar{\sigma}_1^*$ are the functions obtained of $\bar{\sigma}_0$ and $\bar{\sigma}_1$ by the change of variables introduced in (36).

4.1 Solution with an entry condition in class L^1

We define the domain in which we will consider the equation (40) by

$$\Omega = \bigcup_{\kappa \in \mathbb{R}_+^* \times \mathbb{R}, y \in \mathbb{R}, 0 < z < 1} \gamma_{\kappa,y,z} = \tag{44}$$

$$= \left\{ (m, \vartheta, y, z) = (m, \tilde{t}, \xi, y, z) \in \mathbb{R}_+ \times \mathbb{R}^3 \times]0, 1[/ \tilde{t} > \frac{\alpha(m)}{g}(z-1) \right\}$$

and we pose

$$\Gamma_a = \left\{ (m, \vartheta, y, z) = (m, \tilde{t}, \xi, y, z) \in \mathbb{R}_+ \times \mathbb{R}^3 \times [0, 1] / \tilde{t} = \frac{\alpha(m)}{g}(z-1) \right\},$$

$$\Gamma_b = \{z = 1\} \cap \bar{\Omega}.$$

The conditions (42)–(43) can be written in the form

$$\sigma = \bar{\sigma}_1^* \quad \text{on } \Gamma_b, \quad \sigma = \bar{\sigma}_0^* \quad \text{on } \Gamma_a. \tag{45}$$

Proposition 2 Let $\bar{\sigma}_{(a)} \in L^1(\Gamma_a) \cap L^\infty(\Gamma_a)$ and $\bar{\sigma}_{(b)} \in L^1(\Gamma_b) \cap L^\infty(\Gamma_b)$ such that

$$\bar{\sigma}_{(a)}(m, \vartheta, y, z) \geq 0 \quad \text{a.e. on } \Gamma_a, \quad \bar{\sigma}_{(b)}(m, \vartheta, y) \geq 0 \quad \text{a.e. on } \Gamma_b,$$

$$\bar{\sigma}_{(a)}(m, \vartheta, y, z) = \bar{\sigma}_{(b)}(m, \vartheta, y) = 0 \quad \text{for } m \in [0, \bar{m}_a] \cup [\bar{m}_A, \infty[.$$

If

$$\max(\|\bar{\sigma}_{(a)}\|_{L^\infty(\Gamma_a)}, \|\bar{\sigma}_{(b)}\|_{L^\infty(\Gamma_b)}) < \frac{1}{M_1(\bar{m}_A - \bar{m}_a)},$$

then there exists unique solution σ of the equation (40) satisfying to the conditions

$$\sigma = \bar{\sigma}_{(b)} \quad \text{on } \Gamma_b, \quad \sigma = \bar{\sigma}_{(a)} \quad \text{on } \Gamma_a, \quad (46)$$

with

$$\sigma \in C([0, 1]; L^1(\Omega_z) \cap L^\infty(\Omega)), \quad (47)$$

where

$$\Omega_z = \{(m, \vartheta, y) = (m, \tilde{t}, \xi, y) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R} / \tilde{t} > \frac{\alpha(m)}{g}(z - 1)\}. \quad (48)$$

Proof. In (40) and (41), the absence of derivative and integral compared to y is remarked, as in (12), this implies that the equation (40) can be solved separately for each $y \in \mathbb{R}$.

We define for each point $(m, \vartheta) \in \mathbb{R}_+ \times \mathbb{R}^2$ the number $\zeta_1(m, \vartheta) \in [0, 1]$ such that

$$\zeta_1(m, \vartheta) = \zeta_1(m, \tilde{t}, \xi) = \zeta_1(m, \tilde{t}) = \begin{cases} \max(0, 1 + \frac{\tilde{t}}{\alpha(m)}g) & \text{if } \tilde{t} \leq 0, \\ 1 & \text{if } \tilde{t} > 0 \end{cases} \quad (49)$$

and we have

$$(m, \vartheta, y, \zeta_1(m, \vartheta)) \in \Gamma_b \cup \Gamma_a \quad \forall (m, \vartheta, y) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}, \quad \tilde{t} \geq -\frac{\alpha(m)}{g},$$

these permit us to replace $(t, x) \in \mathbb{R}^2$ by the time axis, then we find the conditions for the proof of proposition 4.1 in [2], consequently renewing the stages of the proof of this one, we prove the proposition. \square

4.2 Existence and uniqueness of the global solution in time with a horizontal wind

In the same way to the stationary case, to obtain the existence and the uniqueness of the global solution with an horizontal wind in a general case, we use the “cone of dependence” property and the proposition 2.

We consider a set $\omega \in \mathbb{R}_+ \times \mathbb{R}^3$ such that $0 < mes(\omega) < \infty$ and we define

$$D[\omega] = \bigcup_{(m,\vartheta,y) \in \omega} D_{(m,\vartheta,y)}, \tag{50}$$

where

$$\begin{aligned} D_{(m,\vartheta,y)} &= \left(\bigcup_{0 \leq z \leq 1} \left(\bigcup_{\kappa_-(m,\vartheta,y,z) \leq \kappa \leq \kappa_+(m,\vartheta,y)} \gamma_{\kappa,y,z} \right) \right) = \tag{51} \\ &= \{(m', \vartheta', y', z') \in \mathbb{R}_+ \times \mathbb{R}^3 \times [0, 1] / \vartheta' = \kappa - q(y') \frac{\alpha(m')}{g} (1 - z'), \\ &\quad y' = y, \kappa_-(m, \vartheta, y, z') \leq \kappa \leq \kappa_+(m, \vartheta, y)\} \end{aligned}$$

with

$$\begin{cases} \kappa_+(m, \vartheta, y) = \kappa(m, \vartheta, y, 0) = \vartheta + q(y) \frac{\alpha(m)}{g}, \\ \kappa_-(m, \vartheta, y, z) = \kappa_+(m, \vartheta, y) - q(y) \frac{\bar{\alpha}_0}{g} z = \vartheta + q(y) \frac{\alpha(m)}{g} - q(y) \frac{\bar{\alpha}_0}{g} z. \end{cases} \tag{52}$$

We define $D_\omega(z)$ by

$$\begin{aligned} D_\omega(z) &= \bigcup_{(m,\vartheta,y) \in \omega} \left(\bigcup_{\kappa_-(m,\vartheta,y,z) \leq \kappa \leq \kappa_+(m,\vartheta,y)} \gamma_{\kappa,y,z} \right) = \\ &= \{(m', \vartheta', y', z') \in D[\omega] \mid z' = z\}. \end{aligned}$$

We remark that $D[\omega]$ in the evolution case is defined in a similar way to the stationary case (see (19), (20), (21)) then we have the following lemma.

Lemma 6 *Let $\bar{\sigma}_{(a)}^{[1]}$ and $\bar{\sigma}_{(a)}^{[2]}$ two functions defined on Γ_a , $\bar{\sigma}_{(b)}^{[1]}$ and $\bar{\sigma}_{(b)}^{[2]}$ two functions defined on Γ_b . We suppose that $\bar{\sigma}_{(a)}^{[1]}$, $\bar{\sigma}_{(a)}^{[2]}$, $\bar{\sigma}_{(b)}^{[1]}$, $\bar{\sigma}_{(b)}^{[2]}$ satisfy the conditions of the proposition 2. Let $\sigma^{[1]}$ (resp. $\sigma^{[2]}$) the solution of the equation (40) with the condition (46) and $\bar{\sigma}_{(a)} = \bar{\sigma}_{(a)}^{[1]}$, $\bar{\sigma}_{(b)} = \bar{\sigma}_{(b)}^{[1]}$ (resp. $\bar{\sigma}_{(a)} = \bar{\sigma}_{(a)}^{[2]}$, $\bar{\sigma}_{(b)} = \bar{\sigma}_{(b)}^{[2]}$). If we have*

$$\bar{\sigma}_{(b)}^{[1]} = \bar{\sigma}_{(b)}^{[2]} \quad \text{on } \Gamma_b \cap D[\omega], \quad \bar{\sigma}_{(a)}^{[1]} = \bar{\sigma}_{(a)}^{[2]} \quad \text{on } \Gamma_a \cap D[\omega], \tag{53}$$

then

$$\sigma^{[1]} = \sigma^{[2]} \quad \text{a.e. in } D[\omega].$$

Proof. Writing the equation (40) into an integral form, we have

$$\begin{aligned} \sigma^{[i]}(m, \vartheta, y, z) &= \sigma^{[i]}(m, \vartheta, y, \zeta_1(m, \vartheta)) + \\ &+ \frac{m\alpha(m)}{2g} \int_z^{\zeta_1} \int_{\gamma_{\kappa(m, \vartheta, y, z'), y, z'}^{[0, m]}} \beta(m - m', m') \sigma^{[i]}(m', \vartheta', y, z') \sigma^{[i]}(m - m', \vartheta'', z') \\ &\mu_\gamma(dm') dz' - \frac{m\alpha(m)}{g} \int_z^{\zeta_1} \int_{\gamma_{\kappa(m, \vartheta, y, z'), y, z'}} \beta(m, m') \sigma^{[i]}(m', \vartheta', y, z') \sigma^{[i]}(m, \vartheta, y, z') \\ &\mu_\gamma(dm') dz', \quad i = 1, 2. \end{aligned}$$

From (45) it results that

$$\sigma^{[i]}(m, \vartheta, y, \zeta_1(m, \vartheta)) = \begin{cases} \bar{\sigma}_{(a)}^{[i]} & \text{on } \Gamma_a, \\ \bar{\sigma}_{(b)}^{[i]} & \text{on } \Gamma_b, \end{cases}$$

$\zeta_1(m, \vartheta)$ is the number defined in (49).

Making the difference for $i = 1$ and $i = 2$, we have

$$\begin{aligned} &|\sigma^{[1]}(m, \vartheta, y, z) - \sigma^{[2]}(m, \vartheta, y, z)| \leq |\sigma^{[1]}(m, \vartheta, y, \zeta_1) - \sigma^{[2]}(m, \vartheta, y, \zeta_1)| + \\ &+ C_\beta \left[\int_z^{\zeta_1} \int_{\gamma_{\kappa(m, \vartheta, y, z'), y, z'}^{[0, m]}} (|\sigma^{[1]}(m - m', \vartheta'', y, z') - \sigma^{[2]}(m - m', \vartheta'', y, z')| \sigma^{[2]}(m', \vartheta', y, z') + \right. \\ &+ |\sigma^{[1]}(m', \vartheta', y, z') - \sigma^{[2]}(m', \vartheta', y, z')| \sigma^{[1]}(m - m', \vartheta'', y, z')) \mu_\gamma(dm') dz' + \\ &+ \int_z^{\zeta_1} \int_{\gamma_{\kappa(m, \vartheta, y, z'), y, z'}} (|\sigma^{[1]}(m, \vartheta, y, z') - \sigma^{[2]}(m, \vartheta, y, z')| \sigma^{[2]}(m', \vartheta', y, z') + \\ &\left. + |\sigma^{[1]}(m', \vartheta', y, z') - \sigma^{[2]}(m', \vartheta', y, z')| \sigma^{[1]}(m, \vartheta, y, z')) \mu_\gamma(dm') dz' \right], \end{aligned}$$

we deduce from it that

$$\begin{aligned} &|\sigma^{[1]}(m, \vartheta, y, z) - \sigma^{[2]}(m, \vartheta, y, z)| \leq |\sigma^{[1]}(m, \vartheta, y, \zeta_1) - \sigma^{[2]}(m, \vartheta, y, \zeta_1)| + \quad (54) \\ &+ C_\beta \left[\int_z^1 (\|\sigma^{[1]}(\cdot, \cdot, \cdot, z') - \sigma^{[2]}(\cdot, \cdot, \cdot, z')\|_{L^\infty(\gamma_{\kappa(m, \vartheta, y, z'), y, z'})} \right. \\ &\left. \|\sigma^{[2]}(\cdot, \cdot, \cdot, z')\|_{L^1(\gamma_{\kappa(m, \vartheta, y, z'), y, z'})} + \|\sigma^{[1]}(\cdot, \cdot, \cdot, z')\|_{L^1(\gamma_{\kappa(m, \vartheta, y, z'), y, z'})} \right] \end{aligned}$$

$$\begin{aligned}
 & \|\sigma^{[1]}(\cdot, \cdot, \cdot, z') - \sigma^{[2]}(\cdot, \cdot, \cdot, z')\|_{L^\infty(\gamma_{\kappa(m, \vartheta, y, z'), y, z'})} dz' + \\
 & + \int_z^1 \left(\|\sigma^{[1]}(\cdot, \cdot, \cdot, z') - \sigma^{[2]}(\cdot, \cdot, \cdot, z')\|_{L^\infty(\gamma_{\kappa(m, \vartheta, y, z'), y, z'})} \times \right. \\
 & \quad \times \|\sigma^{[2]}(\cdot, \cdot, \cdot, z')\|_{L^1(\gamma_{\kappa(m, \vartheta, y, z'), y, z'})} + \\
 & \quad + (\bar{m}_A - \bar{m}_a) \|\sigma^{[1]}(\cdot, \cdot, \cdot, z')\|_{L^\infty(\gamma_{\kappa(m, \vartheta, y, z'), y, z'})} \times \\
 & \quad \left. \times \|\sigma^{[1]}(\cdot, \cdot, \cdot, z') - \sigma^{[2]}(\cdot, \cdot, \cdot, z')\|_{L^\infty(\gamma_{\kappa(m, \vartheta, y, z'), y, z'})} \right) dz'.
 \end{aligned}$$

We remark that this inequality is similar to the inequality (24) in the proof of lemma 5 and by the same way we obtain the result. \square

Now we can prove the principal theorem.

Theorem 3 *If $\bar{\sigma}_0^* \in L^\infty(\Gamma_a)$ and $\bar{\sigma}_1^* \in L^\infty(\Gamma_b)$ satisfy to the conditions*

$$\bar{\sigma}_0^*(m, \xi, y, z) \geq 0 \quad \text{a.e. on } \Gamma_a, \quad \bar{\sigma}_1^*(m, \vartheta, y) \geq 0 \quad \text{a.e. on } \Gamma_b, \quad (55)$$

$$\bar{\sigma}_0^*(m, \xi, y, z) = 0, \quad \bar{\sigma}_1^*(m, \vartheta, y) = 0 \quad \text{for } m \in [0, \bar{m}_a] \cup [\bar{m}_A, \infty[, \quad (56)$$

$$\max \left(\|\bar{\sigma}_0^*\|_{L^\infty(\Gamma_a)}; \|\bar{\sigma}_1^*\|_{L^\infty(\Gamma_b)} \right) < \frac{1}{M_1(\bar{m}_A - \bar{m}_a)}, \quad (57)$$

then the equation (40) with the condition (45) admits one solution σ and only one verifying

$$\sigma \in L^\infty(\Omega)$$

with

$$\sigma(m, \vartheta, y, z) \geq 0 \quad \text{a.e. in } \Omega,$$

$$\sigma(m, \vartheta, y, z) = 0, \quad \text{for } m \in [0, \bar{m}_a] \cup [\bar{m}_A, \infty[.$$

Proof. We consider a measurable and bounded sets family ω_i , $i \in \mathbb{N}^*$, defined by

$$\omega_i = \left\{ (m, \vartheta, y) = (m, \tilde{t}, \xi, y) \in \mathbb{R}_+ \times \mathbb{R}^3 / \bar{m}_a \leq m \leq \bar{m}_A, \quad (58) \right.$$

$$\left. -\frac{\alpha(m)}{g} \leq \tilde{t} \leq i, \quad -i \leq \xi \leq i, \quad -i \leq y \leq i \right\}$$

$$= \Omega_0 \cap \left\{ (m, \vartheta, y) = (m, \tilde{t}, \xi, y) \in [\bar{m}_a, \bar{m}_A] \times \mathbb{R}^3 / \tilde{t} \leq i \right\},$$

where Ω_0 is the set defined in (48) with $z = 0$. The definition of $D[\omega]$ (see (50)) permits us to define a number N such that

$$D_{\omega_i}(1) \subset \left\{ (m, \vartheta, y) = (m, \tilde{t}, \xi, y) \in \mathbb{R}_+ \times \mathbb{R}^3 / \bar{m}_a \leq m \leq \bar{m}_A, \right. \\ \left. \tilde{t} \leq i + N, -i - N \leq \xi \leq i + N, -i - 1 \leq y \leq i + 1 \right\} \quad (59)$$

and we consider a function $\psi_i \in C^\infty(\mathbb{R}^3)$; $\psi_i \geq 0$ such that

$$\psi_i(\vartheta, y) = \psi_i(\tilde{t}, \xi, y) = \begin{cases} 1 & \text{if } \tilde{t} \leq i + N, |\xi| \leq i + N, |y| \leq i + 1 \\ 0 & \text{if } \tilde{t} \geq i + N + 1, |\xi| \geq i + N + 1, |y| \geq i + 2, \end{cases} \quad (60)$$

then we have

$$D_{\omega_i}(1) \subset \{(m, \vartheta, y) \in \mathbb{R}_+ \times \mathbb{R}^3 / \psi_i(\vartheta, y) = 1\} \quad i \in \mathbb{N}^*. \quad (61)$$

The theorem will be proved in the same way to theorem 5.1 of [2] (see also theorem 1 of the stationary case) by renewing the same stages. \square

The existence and the uniqueness of the solution in the (m, t, x, y, z) co-ordinates is given in the following theorem.

Theorem 4 *If $\bar{\sigma}_0 \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times [0, 1])$ and $\bar{\sigma}_1 \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2)$ satisfy the conditions*

$$\bar{\sigma}_0(m, x, y, z) \geq 0 \quad \text{a.e. on } \mathbb{R}_+ \times \mathbb{R}^2 \times [0, 1], \\ \bar{\sigma}_1(m, t, x, y) \geq 0 \quad \text{a.e. on } \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2, \\ \bar{\sigma}_0(m, x, y, z) = \bar{\sigma}_1(m, t, x, y) = 0 \quad \text{for } m \in [0, \bar{m}_a] \cup [\bar{m}_A, \infty[, \\ \max \left(\|\bar{\sigma}_0\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times [0, 1])}; \|\bar{\sigma}_1\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2)} \right) < \frac{1}{M_1(\bar{m}_A - \bar{m}_a)},$$

then the equation (1) with the conditions (34) and (35) admits one solution σ and only one verifying

$$\sigma \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times]0, 1[),$$

where

$$\sigma(m, t, x, y, z) \geq 0 \quad \text{a.e. in } \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times]0, 1[, \\ \sigma(m, t, x, y, z) = 0 \quad \text{for } m \in [0, \bar{m}_a] \cup [\bar{m}_A, \infty[.$$

Proof. We associate to the problem (1), (34),(35), where the unknown function to find is σ , the problem (40), (45) by a bijective mapping defined by the change of variables $(m, t, x, y, z) \mapsto (\tilde{m}, \tilde{t}, \xi, \tilde{y}, \tilde{z})$ introduced in (36) with

$$\sigma(m, t, x, y, z) = \tilde{\sigma}\left(m, t - \frac{\alpha(m)}{g}(1-z), x + \bar{v}(y)\frac{\alpha(m)}{g}(1-z), y, z\right).$$

If $\tilde{\sigma}(m, \tilde{t}, \xi, y, z)$ is the solution of the problem (40), (45) in which the existence and the uniqueness have been proved in theorem 3, then, we obtain the existence and the uniqueness of the solution σ of (1), (34),(35) verifying the same conditions. \square

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