Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 6, No. 2/2014

QUANTUM AND CLASSICAL CORRELATIONS IN GAUSSIAN OPEN QUANTUM SYSTEMS*

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Abstract

In the framework of the theory of open systems based on completely positive quantum dynamical semigroups, we give a description of the continuous-variable quantum correlations (quantum entanglement and quantum discord) for a system consisting of two non-interacting bosonic modes embedded in a thermal environment. We solve the Kossakowski-Lindblad master equation for the time evolution of the considered system and describe the entanglement and discord in terms of the covariance matrix for Gaussian input states. For all values of the temperature of the thermal reservoir, an initial separable Gaussian state remains separable for all times. We study the time evolution of logarithmic negativity, which characterizes the degree of entanglement, and show that in the case of an entangled initial squeezed thermal state, entanglement suppression takes place for all temperatures of the environment, including zero temperature. We analyze the time evolution of the Gaussian quantum discord, which is a measure of all quantum correlations in the bipartite state, including entanglement, and show that it decays asymptotically in time under the effect of the thermal bath. This is in contrast with the sudden death of entanglement. Before the suppression of the entanglement, the qualitative evolution of quantum discord is very similar to that of the entanglement. We describe also the time evolution of the degree of

*Accepted for publication in revised form on June 5-th, 2013

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classical correlations and of quantum mutual information, which measures the total correlations of the quantum system.

MSC: 81P40, 81R30, 81S22

keywords: quantum correlations, entanglement, discord, squeezed states, open systems, master equations.

1 Introduction

The study of quantum correlations is a key issue in quantum information theory [1] and quantum entanglement represents the indispensable physical resource for the description and performance of quantum information processing tasks, like quantum teleportation, cryptography, superdense coding and quantum computation [2]. However, entanglement does not describe all the non-classical properties of quantum correlations. Recent theoretical and experimental results indicate that some non-entangled mixed states can improve performance in some quantum computing tasks [3]. Zurek [4, 5] defined the quantum discord as a measure of quantum correlations which includes entanglement of bipartite systems and it can also exist in separable states. The total amount of correlations contained in a quantum state is given by the quantum mutual information which is equal to the sum of the quantum discord and classical correlations [6].

In recent years there is an increasing interest in using non-classical entangled states of continuous variable systems in applications of quantum information processing, communication and computation [7]. In this respect, Gaussian states, in particular two-mode Gaussian states, play a key role since they can be easily created and controlled experimentally. Due to the unavoidable interaction with the environment, in order to describe realistically quantum information processes it is necessary to take decoherence and dissipation into consideration. Decoherence and dynamics of quantum entanglement in continuous variable open systems have been intensively studied in the last years [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21].

In this review paper we describe, in the framework of the theory of open systems based on completely positive quantum dynamical semigroups, the dynamics of continuous variable quantum entanglement and quantum discord of a subsystem consisting of two uncoupled bosonic modes (harmonic oscillators) interacting with a common thermal environment. We are interested in discussing the correlation effect of the environment, therefore we assume that the two modes are independent, i.e. they do not interact directly. The initial state of the open system is taken of Gaussian form and the evolution under the quantum dynamical semigroup assures the preservation in time of the Gaussian form of the state. In particular, we consider unimodal squeezed states, squeezed vacuum states, and symmetric and non-symmetric squeezed thermal states as initial states [22, 23, 24]. We show that entanglement suppression (entanglement sudden death) takes place for all temperatures of the environment, including zero temperature. We analyze the time evolution of Gaussian quantum discord, which is a measure of all quantum correlations in the bipartite state, including entanglement, and show that discord decays asymptotically in time under the effect of the thermal bath. This is contrast with the sudden death of entanglement. Before the suppression of the entanglement, the qualitative evolution of quantum discord is very similar to that of the entanglement.

The paper is organized as follows. In Sect. 2 the notion of the quantum dynamical semigroup is defined using the concept of a completely positive map. Then we give the general form of the Kossakowski-Lindblad quantum mechanical master equation describing the evolution of open quantum systems in the Markovian approximation. We mention the role of complete positivity in connection with the quantum entanglement of systems interacting with an external environment. In Sec. 3 we write the equations of motion in the Heisenberg picture for two independent bosonic modes interacting with a general environment and give the general solution of the evolution equation for the covariance matrix, i.e. we derive the variances and covariances of coordinates and momenta corresponding to a generic two-mode Gaussian state. Then, by using the Peres-Simon necessary and sufficient condition for separability of two-mode Gaussian states [25, 26], we investigate in Sec. 4 the dynamics of quantum correlations (quantum entanglement and Gaussian quantum discord) for the considered subsystem. We describe also the time evolution of the degree of classical correlations and of quantum mutual information. A summary and conclusions are given in Sec. 5. In Appendix we present some elementary notions and examples of quantum correlations (entanglement) in quantum information theory, and describe the influence of diffusion and dissipation on the dynamics of a harmonic oscillator interacting with an environment, in particular with a thermal bath.

2 Axiomatic theory of open quantum systems

The time evolution of a closed physical system is given by a dynamical group U_t, uniquely determined by its generator H, which is the Hamiltonian operator of the system. The action of the dynamical group U_t on any density matrix ρ from the set $\mathcal{D}(\mathcal{H})$ of all density matrices in the Hilbert space $\mathcal{H}$ of the quantum system is defined by

$$\rho(t) = U_t(\rho) = e^{-\frac{i}{\hbar}Ht}\rho e^{\frac{i}{\hbar}Ht} \tag{1}$$

for all $t \in (-\infty, \infty)$. According to von Neumann, density operators $\rho \in \mathcal{D}(\mathcal{H})$ are trace class (Tr $\rho < \infty$), self-adjoint ($\rho^\dagger = \rho$), positive ($\rho > 0$) operators with Tr $\rho = 1$. All these properties are conserved by the time evolution defined by U_t.

In the case of open quantum systems, the time evolution Φ_t of the density operator $\rho(t) = \Phi_t(\rho)$ has to preserve the von Neumann conditions for all times. It follows that Φ_t must have the following properties:

(i) $\Phi_{\rm t}(\lambda_1\rho_1 + \lambda_2\rho_2) = \lambda_1\Phi_{\rm t}(\rho_1) + \lambda_2\Phi_{\rm t}(\rho_2)$ for $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$, i. e. Φ_t must preserve the convex structure of $\mathcal{D}(\mathcal{H})$,

(ii) $\Phi_{\rm t}(\rho^\dagger) = \Phi_{\rm t}^\dagger(\rho)$,

(iii) $\Phi_{\rm t}(\rho) > 0$,

(iv) Tr $\Phi_{\rm t}(\rho) = 1$.

The time evolution U_t for closed systems must be a group $U_{t+s} = U_tU_s$. We have also $U_0(\rho) = \rho$ and $U_t(\rho) \to \rho$ in the trace norm when $t \to 0$. The dual group $\widetilde{U}_t$ acting on the observables $A \in \mathcal{B}(\mathcal{H})$, i.e. on the bounded operators on $\mathcal{H}$, is given by

$$\widetilde{U}_t(A) = e^{\frac{i}{\hbar}Ht}Ae^{-\frac{i}{\hbar}Ht}. \tag{2}$$

Then $\widetilde{U}_t(AB) = \widetilde{U}_t(A)\widetilde{U}_t(B)$ and $\widetilde{U}_t(I) = I$, where I is the identity operator on $\mathcal{H}$. Also, $\widetilde{U}_t(A) \to A$ ultraweakly when $t \to 0$ and $\widetilde{U}_t$ is an ultraweakly continuous mapping [27, 28, 29]. These mappings have a strong positivity property called complete positivity:

$$\sum_{i,j} B_i^\dagger \widetilde{U}_t(A_i^\dagger A_j)B_j \geq 0, \quad A_i, B_i \in \mathcal{B}(\mathcal{H}). \tag{3}$$

In the axiomatic approach to the description of the evolution of open quantum systems [27, 28, 29], one supposes that the time evolution Φ_t of open systems is not very different from the time evolution of closed systems. The simplest dynamics Φ_t which introduces a preferred direction in time, characteristic for dissipative processes, is that in which the group condition is replaced by the semigroup condition [27, 30, 31]

$$\Phi_{t+s} = \Phi_t \Phi_s,\ \ t, s \geq 0. \tag{4}$$

The complete positivity condition has the form:

$$\sum_{i,j} B_i^\dagger \widetilde{\Phi}_t(A_i^\dagger A_j) B_j \geq 0, \quad A_i, B_i \in \mathcal{B}(\mathcal{H}), \tag{5}$$

where $\widetilde{\Phi}_t$ denotes the dual of Φ_t acting on $\mathcal{B}(\mathcal{H})$, defined by the duality condition

$$\mathrm{Tr}(\Phi_t(\rho)A) = \mathrm{Tr}(\rho \widetilde{\Phi}_t(A)). \tag{6}$$

Then the conditions $\mathrm{Tr}\Phi_t(\rho) = 1$ and $\widetilde{\Phi}_t(I) = I$ are equivalent. Also the conditions $\widetilde{\Phi}_t(A) \to A$ ultraweakly when $t \to 0$ and $\Phi_t(\rho) \to \rho$ in the trace norm when $t \to 0$, are equivalent. For the semigroups with these properties and with a more weak property of positivity than Eq. (5), namely

$$A \geq 0 \to \widetilde{\Phi}_t(A) \geq 0, \tag{7}$$

it is well known that there exists a (generally unbounded) mapping $\widetilde{L}$ – the generator of $\widetilde{\Phi}_t$, and $\widetilde{\Phi}_t$ is uniquely determined by $\widetilde{L}$. The dual generator of the dual semigroup Φ_t is denoted by L:

$$\mathrm{Tr}(L(\rho)A) = \mathrm{Tr}(\rho \widetilde{L}(A)). \tag{8}$$

The evolution equations by which L and $\widetilde{L}$ determine uniquely Φ_t and $\widetilde{\Phi}_t$, respectively, are given in the Schrödinger and Heisenberg picture by

$$\frac{d\Phi_t(\rho)}{dt} = L(\Phi_t(\rho)) \tag{9}$$

and

$$\frac{d\widetilde{\Phi}_t(A)}{dt} = \widetilde{L}(\widetilde{\Phi}_t(A)). \tag{10}$$

These equations replace in the case of open systems the von Neumann-Liouville equations

$$\frac{dU_t(\rho)}{dt} = -\frac{i}{\hbar}[H, U_t(\rho)] \tag{11}$$

and

$$\frac{d\widetilde{U}_t(A)}{dt} = \frac{i}{\hbar}[H, \widetilde{U}_t(A)], \tag{12}$$

respectively. For applications, Eqs. (9) and (10) are only useful if the detailed structure of the generator $L(\widetilde{L})$ is known and can be related to the concrete properties of the open systems described by such equations. For the class of dynamical semigroups which are completely positive and norm continuous, the generator $\widetilde{L}$ is bounded. In many applications the generator is unbounded.

According to Lindblad [29], the following argument can be used to justify the complete positivity of $\widetilde{\Phi}_t$: if the open system is extended in a trivial way to a larger system described in a Hilbert space $\mathcal{H}\otimes\mathcal{K}$ with the time evolution defined by

$$\widetilde{W}_t(A \otimes B) = \widetilde{\Phi}_t(A) \otimes B, \quad A \in \mathcal{B}(\mathcal{H}), \; B \in \mathcal{B}(\mathcal{K}), \tag{13}$$

then the positivity of the states of the compound system will be preserved by $\widetilde{W}_t$ only if $\widetilde{\Phi}_t$ is completely positive. With this observation a new equivalent definition of the complete positivity is obtained: $\widetilde{\Phi}_t$ is completely positive if $\widetilde{W}_t$ is positive for any finite dimensional Hilbert space $\mathcal{K}$. The physical meaning of complete positivity can mainly be understood in relation to the existence of entangled states, the typical example being given by a vector state with a singlet-like structure that cannot be written as a tensor product of vector states. Positivity property guarantees the physical consistency of evolving states of single systems, while complete positivity prevents inconsistencies in entangled composite systems, and therefore the existence of entangled states makes the request of complete positivity necessary [32].

A bounded mapping $\widetilde{L} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ which satisfies $\widetilde{L}(I) = 0$, $\widetilde{L}(A^\dagger) = \widetilde{L}^\dagger(A)$ and

$$\widetilde{L}(A^\dagger A) - \widetilde{L}(A^\dagger)A - A^\dagger\widetilde{L}(A) \geq 0 \tag{14}$$

is called dissipative. The 2-positivity property of the completely positive mapping $\widetilde{\Phi}_t$:

$$\widetilde{\Phi}_t(A^\dagger A) \geq \widetilde{\Phi}_t(A^\dagger)\widetilde{\Phi}_t(A), \tag{15}$$

with equality at $t = 0$, implies that $\widetilde{L}$ is dissipative. Conversely, the dissipativity of $\widetilde{L}$ implies that $\widetilde{\Phi}_t$ is 2-positive. $\widetilde{L}$ is called completely dissipative if all trivial extensions of $\widetilde{L}$ to a compound system described by $\mathcal{H} \otimes \mathcal{K}$

with any finite dimensional Hilbert space $\mathcal{K}$ are dissipative. There exists a one-to-one correspondence between the completely positive norm continuous semigroups $\widetilde{\Phi}_t$ and completely dissipative generators $\widetilde{L}$. The following structural theorem gives the most general form of a completely dissipative mapping $\widetilde{L}$ [29].

Theorem. $\widetilde{L}$ is completely dissipative and ultraweakly continuous if and only if it is of the form

$$\widetilde{L}(A) = \frac{i}{\hbar}[H, A] + \frac{1}{2\hbar}\sum_j (V_j^\dagger [A, V_j] + [V_j^\dagger, A]V_j), \tag{16}$$

where $V_j,\ \sum_j V_j^\dagger V_j \in \mathcal{B}(\mathcal{H}),\ H \in \mathcal{B}(\mathcal{H})_{\text{s.a.}}$.

The dual generator on the state space (Schrödinger picture) is of the form

$$L(\rho) = -\frac{i}{\hbar}[H, \rho] + \frac{1}{2\hbar}\sum_j ([V_j\rho, V_j^\dagger] + [V_j, \rho V_j^\dagger]). \tag{17}$$

Eqs. (9) and (17) give the explicit form of the Kossakowski-Lindblad master equation, which is the most general time-homogeneous quantum mechanical Markovian master equation with a bounded Liouville operator [29, 31, 33, 34]:

$$\frac{d\Phi_t(\rho)}{dt} = -\frac{i}{\hbar}[H, \Phi_t(\rho)] + \frac{1}{2\hbar}\sum_j ([V_j\Phi_t(\rho), V_j^\dagger] + [V_j, \Phi_t(\rho)V_j^\dagger]). \tag{18}$$

The assumption of a semigroup dynamics is only applicable in the limit of weak coupling of the subsystem with its environment, i.e. for long relaxation times [35]. We mention that the majority of Markovian master equations found in the literature are of this form after some rearrangement of terms, even for unbounded generators. It is also an empirical fact for many physically interesting situations that the time evolutions Φ_t drive the system towards a unique final state $\rho(\infty) = \lim_{t\to\infty} \Phi_t(\rho(0))$ for all $\rho(0) \in \mathcal{D}(\mathcal{H})$.

3 Time evolution of two independent bosonic modes interacting with an environment

We are interested in the dynamics of quantum correlations in a subsystem composed of two non-interacting (independent) bosonic modes (harmonic oscillators) in weak interaction with a thermal environment, so that their reduced time evolution can be described by a Markovian, completely positive quantum dynamical semigroup. If $\widetilde{\Phi}_t$ is the dynamical semigroup describing the irreversible time evolution of the open quantum system in the Heisenberg representation, then the Kossakowski-Lindblad master equation has the following form for an operator A (see Eqs. (10), (16)) [29, 31, 33, 34]:

$$\frac{d\widetilde{\Phi}_t(A)}{dt} = \frac{i}{\hbar}[H, \widetilde{\Phi}_t(A)] + \frac{1}{2\hbar}\sum_j (V_j^\dagger[\widetilde{\Phi}_t(A), V_j] + [V_j^\dagger, \widetilde{\Phi}_t(A)]V_j). \qquad (19)$$

Here, H denotes the Hamiltonian of the open system and the operators $V_j, V_j^\dagger$, defined on the Hilbert space of H, represent the interaction of the open system with the environment. We are interested in the set of Gaussian states, therefore we introduce quantum dynamical semigroups which preserve this set and in this case our model represents a Gaussian noise channel. Consequently H is chosen as a polynomial of second degree in the coordinates x, y and momenta p_x, p_y of the two quantum oscillators and $V_j, V_j^\dagger$ are taken polynomials of first degree in these canonical observables. Then in the linear space spanned by the coordinates and momenta there exist only four linearly independent operators $V_{j=1,2,3,4}$ [36]:

$$V_j = a_{xj}p_x + a_{yj}p_y + b_{xj}x + b_{yj}y, \qquad (20)$$

where $a_{xj}, a_{yj}, b_{xj}, b_{yj}$ are complex coefficients. The Hamiltonian H of the two uncoupled non-resonant modes of identical mass m and frequencies ω_1 and ω_2 is given by

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{m}{2}(\omega_1^2 x^2 + \omega_2^2 y^2). \qquad (21)$$

The fact that $\widetilde{\Phi}_t$ is a dynamical semigroup implies the positivity of the following matrix formed by the scalar products of the four vectors $\mathbf{a}_x, \mathbf{a}_y, \mathbf{b}_x, \mathbf{b}_y$, whose entries are the components $a_{xj}, a_{yj}, b_{xj}, b_{yj}$, respectively:

$$\frac{1}{2}\hbar = \begin{pmatrix} (\mathbf{a}_x\mathbf{a}_x) & (\mathbf{a}_x\mathbf{b}_x) & (\mathbf{a}_x\mathbf{a}_y) & (\mathbf{a}_x\mathbf{b}_y) \\ (\mathbf{b}_x\mathbf{a}_x) & (\mathbf{b}_x\mathbf{b}_x) & (\mathbf{b}_x\mathbf{a}_y) & (\mathbf{b}_x\mathbf{b}_y) \\ (\mathbf{a}_y\mathbf{a}_x) & (\mathbf{a}_y\mathbf{b}_x) & (\mathbf{a}_y\mathbf{a}_y) & (\mathbf{a}_y\mathbf{b}_y) \\ (\mathbf{b}_y\mathbf{a}_x) & (\mathbf{b}_y\mathbf{b}_x) & (\mathbf{b}_y\mathbf{a}_y) & (\mathbf{b}_y\mathbf{b}_y) \end{pmatrix} \qquad (22)$$

Its matrix elements have to be chosen appropriately to suit various physical models of the environment. For a quite general environment able to induce noise and damping effects, we take this matrix of the following form, where all the coefficients D_{xx}, D_{xp_x},... and λ are real quantities, representing the diffusion coefficients and, respectively, the dissipation constant:

$$\left(\begin{array}{cccc} D_{xx} & -D_{xp_x}-i\hbar\lambda/2 & D_{xy} & -D_{xp_y} \\ -D_{xp_x}+i\hbar\lambda/2 & D_{p_xp_x} & -D_{yp_x} & D_{p_xp_y} \\ D_{xy} & -D_{yp_x} & D_{yy} & -D_{yp_y}-i\hbar\lambda/2 \\ -D_{xp_y} & D_{p_xp_y} & -D_{yp_y}+i\hbar\lambda/2 & D_{p_yp_y}. \end{array}\right) \tag{23}$$

It follows that the principal minors of this matrix are positive or zero. From the Cauchy-Schwarz inequality the following relations hold for the coefficients defined in Eq. (23) (from now on we put, for simplicity, $\hbar = 1$):

$$\begin{gathered} D_{xx}D_{p_xp_x}-D^2_{xp_x}\geq\frac{\lambda^2}{4},\ D_{yy}D_{p_yp_y}-D^2_{yp_y}\geq\frac{\lambda^2}{4}, \\ D_{xx}D_{yy}-D^2_{xy}\geq 0,\ D_{p_xp_x}D_{p_yp_y}-D^2_{p_xp_y}\geq 0, \\ D_{xx}D_{p_yp_y}-D^2_{xp_y}\geq 0,\ D_{yy}D_{p_xp_x}-D^2_{yp_x}\geq 0. \end{gathered} \tag{24}$$

The matrix of the coefficients (23) can be conveniently written as (T denotes the transposed matrix)

$$\left(\begin{array}{cc} C_1 & C_3 \\ C_3{}^{\mathrm{T}} & C_2 \end{array}\right), \tag{25}$$

in terms of 2×2 matrices $C_1 = C_1{}^\dagger$, $C_2 = C_2{}^\dagger$ and C_3. This decomposition has a direct physical interpretation: the elements containing the diagonal contributions C_1 and C_2 represent diffusion and dissipation coefficients corresponding to the first, respectively the second, system in absence of the other, while the elements in C_3 represent environment generated couplings between the two modes, taken initially independent.

We introduce the following 4×4 bimodal covariance matrix:

$$\sigma(t)=\left(\begin{array}{cccc} \sigma_{xx}(t) & \sigma_{xp_x}(t) & \sigma_{xy}(t) & \sigma_{xp_y}(t) \\ \sigma_{xp_x}(t) & \sigma_{p_xp_x}(t) & \sigma_{yp_x}(t) & \sigma_{p_xp_y}(t) \\ \sigma_{xy}(t) & \sigma_{yp_x}(t) & \sigma_{yy}(t) & \sigma_{yp_y}(t) \\ \sigma_{xp_y}(t) & \sigma_{p_xp_y}(t) & \sigma_{yp_y}(t) & \sigma_{p_yp_y}(t) \end{array}\right) \tag{26}$$

where the correlations of operators R_i and $R_j, i, j = 1,..,4$, with $\mathbf{R} = \{x, p_x, y, p_y\}$, are defined by using the density operator ρ of the initial state of the quantum system, as follows:

$$\sigma_{R_iR_j}(t)=\frac{1}{2}\mathrm{Tr}[\rho(R_iR_j+R_jR_i)(t)]-\mathrm{Tr}[\rho R_i(t)]\mathrm{Tr}[\rho R_j(t)]. \tag{27}$$

The problem of solving the master equation for the operators in Heisenberg representation can be transformed into a problem of solving first-order in time, coupled linear differential equations for the covariance matrix elements. Namely, from Eq. (19) we obtain by direct calculation the following systems of equations for the quantum correlations of the canonical observables [36]:

$$\frac{d\sigma(t)}{dt} = Y\sigma(t) + \sigma(t)Y^{\mathrm{T}} + 2D, \tag{28}$$

where

$$Y = \begin{pmatrix} -\lambda & 1/m & 0 & 0 \\ -m\omega_1^2 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 1/m \\ 0 & 0 & -m\omega_2^2 & -\lambda \end{pmatrix} \tag{29}$$

$$D = \begin{pmatrix} D_{xx} & D_{xp_x} & D_{xy} & D_{xp_y} \\ D_{xp_x} & D_{p_xp_x} & D_{yp_x} & D_{p_xp_y} \\ D_{xy} & D_{yp_x} & D_{yy} & D_{yp_y} \\ D_{xp_y} & D_{p_xp_y} & D_{yp_y} & D_{p_yp_y} \end{pmatrix} \tag{30}$$

Introducing the notation $\sigma(\infty) \equiv \lim_{t\to\infty} \sigma(t)$, the time-dependent solution of Eq. (28) is given by [36]

$$\sigma(t) = M(t)[\sigma(0) - \sigma(\infty)]M^{\mathrm{T}}(t) + \sigma(\infty), \tag{31}$$

where the matrix $M(t) = \exp(Yt)$ has to fulfill the condition $\lim_{t\to\infty} M(t) = 0$. In order that this limit exists, Y must only have eigenvalues with negative real parts. The values at infinity are obtained from the equation

$$Y\sigma(\infty) + \sigma(\infty)Y^{\mathrm{T}} = -2D. \tag{32}$$

4 Dynamics of quantum correlations

To describe the dynamics of quantum correlations, we use two types of measures: logarithmic negativity for entanglement, and quantum discord.

4.1 Time evolution of entanglement and logarithmic negativity

A well-known sufficient condition for inseparability is the so-called Peres-Horodecki criterion [25, 37], which is based on the observation that the non-completely positive nature of the partial transposition operation of the density matrix for a bipartite system (transposition with respect to degrees of freedom of one subsystem only) may turn an inseparable state into a nonphysical state. The signature of this non-physicality, and thus of quantum entanglement, is the appearance of a negative eigenvalue in the eigenspectrum of the partially transposed density matrix of a bipartite system. The characterization of the separability of continuous variable states using second-order moments of quadrature operators was given in Refs. [26, 38]. For Gaussian states, whose statistical properties are fully characterized by just second-order moments, this criterion was proven to be necessary and sufficient: a Gaussian continuous variable state is separable if and only if the partial transpose of its density matrix is non-negative (positive partial transpose (PPT) criterion).

The two-mode Gaussian state is entirely specified by its covariance matrix (30), which is a real, symmetric and positive matrix with the following block structure:

$$\sigma(t) = \begin{pmatrix} A & C \\ C^{\mathrm{T}} & B \end{pmatrix}, \tag{33}$$

where A, B and C are 2×2 Hermitian matrices. A and B denote the symmetric covariance matrices for the individual reduced one-mode states, while the matrix C contains the cross-correlations between modes. When these correlations have non-zero values, then the states with $\det C \geq 0$ are separable states, while for $\det C < 0$ it may be possible that the states are entangled.

The 4×4 covariance matrix (33) (where all first moments can be set to zero by means of local unitary operations which do not affect the entanglement) contains four local symplectic invariants in form of the determinants of the block matrices A, B, C and covariance matrix σ. Based on the above invariants, Simon [26] derived the following PPT criterion for bipartite Gaussian continuous variable states: the necessary and sufficient condition for separability is $S(t) \geq 0$, where

$$\begin{aligned} S(t) \equiv \det A \det B + (\frac{1}{4} - |\det C|)^2 \\ -\mathrm{Tr}[AJCJBJC^{\mathrm{T}}J] - \frac{1}{4}(\det A + \det B) \end{aligned} \tag{34}$$

and J is the 2×2 symplectic matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{35}$$

For Gaussian states, the measures of entanglement of bipartite systems are based on the invariants constructed from the elements of the covariance matrix [8, 12]. In order to quantify the degree of entanglement of the two-mode states it is suitable to use the logarithmic negativity. For a Gaussian density operator, the logarithmic negativity is completely defined by the symplectic spectrum of the partial transpose of the covariance matrix. It is given by $E_N = \max\{0, -\log_2 2\tilde{\nu}_-\}$, where $\tilde{\nu}_-$ is the smallest of the two symplectic eigenvalues of the partial transpose $\tilde{\sigma}$ of the two-mode covariance matrix σ [11]:

$$2\tilde{\nu}_{\mp}^2 = \tilde{\Delta} \mp \sqrt{\tilde{\Delta}^2 - 4\det\sigma} \tag{36}$$

and $\tilde{\Delta}$ is the symplectic invariant (seralian), given by $\tilde{\Delta} = \det A + \det B - 2\det C$.

In our model, the logarithmic negativity is calculated as [39, 40]

$$E_N(t) = \max\{0, -\frac{1}{2}\log_2[4g(\sigma(t))]\}, \tag{37}$$

where

$$\begin{aligned} g(\sigma(t)) &= \frac{1}{2}(\det A + \det B) - \det C \\ &- \left(\left[\frac{1}{2}(\det A + \det B) - \det C\right]^2 - \det\sigma(t)\right)^{1/2}. \end{aligned} \tag{38}$$

It determines the strength of entanglement for $E_N(t) > 0$, and if $E_N(t) \leq 0$, then the state is separable.

We suppose that the asymptotic state of the considered open system is a Gibbs state corresponding to two independent bosonic modes in thermal equilibrium at temperature T. Then the quantum diffusion coefficients have the following form [34]:

$$\begin{aligned} m\omega_1 D_{xx} &= \frac{D_{p_xp_x}}{m\omega_1} = \frac{\lambda}{2}\coth\frac{\omega_1}{2kT}, \\ m\omega_2 D_{yy} &= \frac{D_{p_yp_y}}{m\omega_2} = \frac{\lambda}{2}\coth\frac{\omega_2}{2kT}, \\ D_{xp_x} = D_{yp_y} = D_{xy} &= D_{p_xp_y} = D_{xp_y} = D_{yp_x} = 0. \end{aligned} \tag{39}$$

The elements of the covariance matrix can be calculated from Eqs. (31), (32).

In the following, we analyze the dependence of the Simon function $S(t)$ and of the logarithmic negativity $E_N(t)$ on time t and temperature T of the thermal bath, with the diffusion coefficients given by Eqs. (39). We consider two types of the initial Gaussian states: separable and entangled.

1) We consider a separable initial Gaussian state, with the two modes initially prepared in their single-mode squeezed states (unimodal squeezed state) and with its initial covariance matrix taken of the form

$$\sigma_s(0) = \frac{1}{2}\begin{pmatrix} \cosh 2r & \sinh 2r & 0 & 0 \\ \sinh 2r & \cosh 2r & 0 & 0 \\ 0 & 0 & \cosh 2r & \sinh 2r \\ 0 & 0 & \sinh 2r & \cosh 2r \end{pmatrix} \tag{40}$$

where r denotes the squeezing parameter. In this case $S(t)$ becomes strictly positive after the initial moment of time ($S(0) = 0$), so that the initial separable state remains separable for all values of the temperature T and for all times.

2) We take an entangled initial Gaussian state of the form of a two-mode vacuum squeezed state, with the initial covariance matrix given by

$$\sigma_e(0) = \frac{1}{2}\begin{pmatrix} \cosh 2r & 0 & \sinh 2r & 0 \\ 0 & \cosh 2r & 0 & -\sinh 2r \\ \sinh 2r & 0 & \cosh 2r & 0 \\ 0 & -\sinh 2r & 0 & \cosh 2r \end{pmatrix} \tag{41}$$

We observe that for all temperatures T, at certain finite moment of time, which depends on T, $E_N(t)$ becomes zero and therefore the state becomes separable. This is the so-called phenomenon of entanglement sudden death [23, 41]. It is in contrast to the quantum decoherence, during which the loss of quantum coherence is usually gradual [17, 42].

3) We assume that the initial Gaussian state is a two-mode squeezed thermal state, with the covariance matrix of the form [43]

$$\sigma_{st}(0) = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & -c \\ c & 0 & b & 0 \\ 0 & -c & 0 & b \end{pmatrix} \tag{42}$$

with the matrix elements given by

$$\begin{aligned} a &= n_1 \cosh^2 r + n_2 \sinh^2 r + \frac{1}{2}\cosh 2r, \\ b &= n_1 \sinh^2 r + n_2 \cosh^2 r + \frac{1}{2}\cosh 2r, \\ c &= \frac{1}{2}(n_1 + n_2 + 1)\sinh 2r, \end{aligned} \tag{43}$$

where n_1, n_2 are the average number of thermal photons associated with the two modes and r denotes the squeezing parameter. In the particular case $n_1 = 0$ and $n_2 = 0$, (42) becomes the covariance matrix of the two-mode squeezed vacuum state (41). A two-mode squeezed thermal state is entangled when the squeezing parameter r satisfies the inequality $r > r_s$ [43], where

$$\cosh^2 r_s = \frac{(n_1+1)(n_2+1)}{n_1+n_2+1}. \tag{44}$$

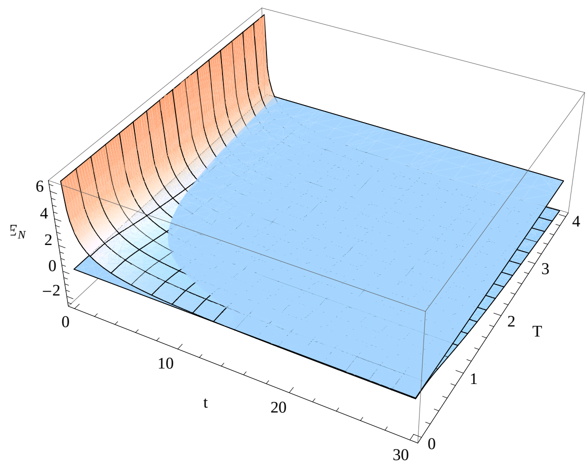

Figure 1: Logarithmic negativity E_N versus time t and temperature T for an entangled initial non-symmetric squeezed thermal state with squeezing parameter $r = 3$, $n_1 = 3, n_2 = 1$ and $\lambda = 0.1, \omega_1 = 1, \omega_2 = 2$. We take $m = \hbar = k = 1$.

The evolution of entangled initial squeezed thermal states with the covariance matrix given by Eq. (42) is illustrated in Fig. 1, where we represent the dependence of the logarithmic negativity $E_N(t)$ on time t and temperature T for the case of an initial non-symmetric Gaussian state ($a \neq b$). For all temperatures T, including zero temperature, at certain finite moment of time, which depends on T, $E_N(t)$ becomes zero and therefore the state becomes separable. One can show that the dissipation favors the phenomenon of entanglement sudden death – with increasing the dissipation parameter λ, the entanglement suppression happens earlier. The same qualitative behaviour of the time evolution of entanglement was obtained previously in the particular case $n_1 = 0$ and $n_2 = 0$ corresponding to an initial two-mode squeezed vacuum state and in the case of symmetric initial squeezed thermal states.

One can assert that the asymmetry ($a \neq b$) of the initial Gaussian state favors the suppression of entanglement. The most robust under the influence of the environment is the entanglement of symmetric ($a = b$) initial squeezed thermal states. An even stronger influence on the entanglement has the non-resonant character of the two modes: by increasing the ratio of the frequencies of the two modes, the entanglement sudden death happens earlier in time. The longest surviving entanglement takes place when the modes are resonant ($\omega_1 = \omega_2$). This effect due to the non-resonance of the modes is stronger for small values of the frequencies, and it diminishes, for the same ratio of frequencies, by increasing the values of frequencies.

In our model, in which we suppose that the asymptotic state of the considered open system is a Gibbs state corresponding to two independent bosonic modes in thermal equilibrium, a separable initial state remains separable in time, and it is not possible to generate entanglement. This is in contrast with the possibility of entanglement generation starting, for instance, with a separable state in the case of two non-interacting two-level systems immersed in a common bath [32]. At the same time we notice that in the case of two identical harmonic oscillators interacting with a general environment, characterized by general diffusion and dissipation coefficients, we obtain that for separable initial states and for definite values of these coefficients, entanglement generation or a periodic generation and collapse of entanglement take place [40, 44]. In discussing the entanglement decay, it is interesting to mention that models have been elaborated to realize quantum feedback control of continuous variable entanglement for a system consisting of two interacting bosonic modes plunged into an environment, based on a local technique [45], or on a nonlocal homodyne measurement [46].

The dynamics of entanglement of the two modes strongly depends on the initial states and the coefficients describing the interaction of the system with the thermal environment (dissipation constant and temperature). As expected, the logarithmic negativity has a behaviour similar to that one of the Simon function in what concerns the characteristics of the state of being separable or entangled [39, 40, 42, 44].

4.2 Asymptotic entanglement

On general grounds, one expects that the effects of decoherence is dominant in the long-time regime, so that no quantum correlations (entanglement) is expected to be left at infinity. Indeed, using the diffusion coefficients given by Eqs. (39), we obtain from Eq. (32) the following elements of the asymptotic matrices $A(\infty)$ and $B(\infty)$:

$$m\omega_1\sigma_{xx}(\infty)=\frac{\sigma_{p_xp_x}(\infty)}{m\omega_1}=\frac{1}{2}\coth\frac{\omega_1}{2kT},\quad \sigma_{xp_x}(\infty)=0,$$
$$m\omega_2\sigma_{yy}(\infty)=\frac{\sigma_{p_yp_y}(\infty)}{m\omega_2}=\frac{1}{2}\coth\frac{\omega_2}{2kT},\quad \sigma_{yp_y}(\infty)=0 \tag{45}$$

and of the entanglement matrix $C(\infty)$:

$$\sigma_{xy}(\infty)=\sigma_{xp_y}(\infty)=\sigma_{yp_x}(\infty)=\sigma_{p_xp_y}(\infty)=0. \tag{46}$$

Then the Simon expression (34) takes the following form in the limit of large times:

$$S(\infty)=\frac{1}{16}\left(\coth^2\frac{\omega_1}{2kT}-1\right)\left(\coth^2\frac{\omega_2}{2kT}-1\right), \tag{47}$$

and, correspondingly, the equilibrium asymptotic state is always separable in the case of two non-interacting bosonic modes immersed in a common thermal reservoir.

In Refs. [20, 21, 39, 40, 42, 44] we described the dependence of the logarithmic negativity $E_N(t)$ on time and mixed diffusion coefficient for two modes interacting with a general environment. In the present case of a thermal bath, the asymptotic logarithmic negativity is given by (for $\omega_1\le\omega_2$)

$$E_N(\infty)=-\log_2\coth\frac{\omega_2}{2kT}. \tag{48}$$

It depends only on temperature, and does not depend on the initial Gaussian state. $E_N(\infty)<0$ for $T\neq 0$ and $E_N(\infty)=0$ for $T=0$, and this confirms the previous statement that the asymptotic state is always separable.

4.3 Gaussian quantum discord

The separability of quantum states has often been described as a property synonymous with the classicality. However, recent studies have shown that separable states, usually considered as being classically correlated, might also contain quantum correlations. Quantum discord was introduced [4, 5] as a measure of all quantum correlations in a bipartite state, including – but not restricted to – entanglement. Quantum discord has been defined as the difference between two quantum analogues of classically equivalent expression of the mutual information, which is a measure of total correlations in a quantum state. For pure entangled states quantum discord coincides with the entropy of entanglement. Quantum discord can be different from zero also for some mixed separable state and therefore the correlations in such separable states with positive discord are an indicator of quantumness. States with zero discord represent essentially a classical probability distribution embedded in a quantum system.

For an arbitrary bipartite state ρ_{12}, the total correlations are expressed by quantum mutual information [47]

$$I(\rho_{12}) = \sum_{i=1,2} S(\rho_i) - S(\rho_{12}), \tag{49}$$

where ρ_i represents the reduced density matrix of subsystem i and $S(\rho) = -\mathrm{Tr}(\rho \ln \rho)$ is the von Neumann entropy. Henderson and Vedral [6] proposed a measure of bipartite classical correlations $C(\rho_{12})$ based on a complete set of local projectors $\{\Pi_2^k\}$ on the subsystem 2: the classical correlation in the bipartite quantum state ρ_{12} can be given by

$$C(\rho_{12}) = S(\rho_1) - \inf_{\{\Pi_2^k\}}\{S(\rho_{1|2})\}, \tag{50}$$

where $S(\rho_{1|2}) = \sum_k p^k S(\rho_1^k)$ is the conditional entropy of subsystem 1 and $\inf\{S(\rho_{1|2})\}$ represents the minimal value of the entropy with respect to a complete set of local measurements $\{\Pi_2^k\}$. Here, p^k is the measurement probability for the kth local projector and ρ_1^k denotes the reduced state of subsystem 1 after the local measurements. Then the quantum discord is defined by

$$D(\rho_{12}) = I(\rho_{12}) - C(\rho_{12}). \tag{51}$$

Originally the quantum discord was defined and evaluated mainly for finite dimensional systems. Recently [48, 49] the notion of discord has been extended to the domain of continuous variable systems, in particular to the analysis of bipartite systems described by two-mode Gaussian states. Closed formulas have been derived for bipartite thermal squeezed states [48] and for all two-mode Gaussian states [49].

The Gaussian quantum discord of a general two-mode Gaussian state ρ_{12} can be defined as the quantum discord where the conditional entropy is restricted to generalized Gaussian positive operator valued measurements (POVM) on the mode 2 and in terms of symplectic invariants it is given by (the symmetry between the two modes 1 and 2 is broken) [49]

$$D = f(\sqrt{\beta}) - f(\nu_-) - f(\nu_+) + f(\sqrt{\varepsilon}), \tag{52}$$

where

$$f(x) = \frac{x+1}{2}\log\frac{x+1}{2} - \frac{x-1}{2}\log\frac{x-1}{2}, \tag{53}$$

$$\varepsilon = \begin{cases} \dfrac{2\gamma^2 + (\beta-1)(\delta-\alpha) + 2|\gamma|\sqrt{\gamma^2 + (\beta-1)(\delta-\alpha)}}{(\beta-1)^2}, \\ \quad \text{if } \; (\delta-\alpha\beta)^2 \le (\beta+1)\gamma^2(\alpha+\delta) \\ \\ \dfrac{\alpha\beta - \gamma^2 + \delta - \sqrt{\gamma^4 + (\delta-\alpha\beta)^2 - 2\gamma^2(\delta+\alpha\beta)}}{2\beta}, \\ \quad \text{otherwise,} \end{cases} \tag{54}$$

$$\alpha = 4\det A, \quad \beta = 4\det B, \quad \gamma = 4\det C, \quad \delta = 16\det\sigma, \tag{55}$$

and $\nu_\mp$ are the symplectic eigenvalues of the state, given by

$$2\nu_\mp^2 = \Delta \mp \sqrt{\Delta^2 - 4\det\sigma}, \tag{56}$$

where $\Delta = \det A + \det B + 2\det C$. Notice that Gaussian quantum discord only depends on $|\det C|$, i.e., entangled ($\det C < 0$) and separable states are treated on equal footing.

The evolution of the Gaussian quantum discord D is illustrated in Fig. 2, where we represent the dependence of D on time t and temperature T for an entangled initial non-symmetric Gaussian state, taken of the form of a two-mode squeezed thermal state (42), for such values of the parameters which satisfy for all times the first condition in formula (54). The Gaussian discord has nonzero values for all finite times and this fact certifies the existence of non-classical correlations in two-mode Gaussian states, either separable or entangled. Gaussian discord asymptotically decreases in time, compared to the case of logarithmic negativity, which has an evolution leading to a sudden suppression of entanglement. For entangled initial states the Gaussian discord remains strictly positive in time and in the limit of infinite

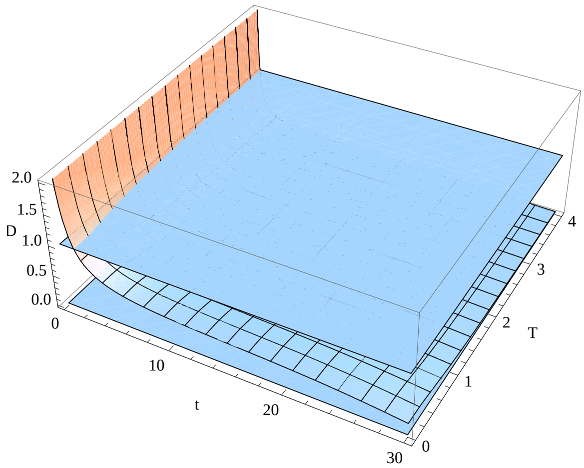

Figure 2: Gaussian quantum discord D versus time t and temperature T for an entangled initial non-symmetric squeezed thermal state with squeezing parameter $r = 3$, $n_1 = 3, n_2 = 1$ and $\lambda = 0.1, \omega_1 = 1, \omega_2 = 2$. We take $m = \hbar = k = 1$.

time it tends asymptotically to zero, corresponding to the thermal product (separable) state, with no correlations at all. One can easily show that for a separable initial Gaussian state with covariance matrix (42) the quantum discord is zero and it keeps this values during the whole time evolution of the state.

From Fig. 2 we notice that, in agreement with the general properties of the Gaussian quantum discord [49], the states can be either separable or entangled for $D \leq 1$ and all the states above the threshold $D = 1$ are entangled. We also notice that the decay of quantum discord is stronger when the temperature T is increasing. It should be remarked that the decay of quantum discord is very similar to that of the entanglement before the time of the sudden death of entanglement. Near the threshold of zero logarithmic negativity ($E_N = 0$), the nonzero values of the discord can quantify the non-classical correlations for separable mixed states and one considers that this fact could make possible some tasks in quantum computation [50]. The discord is increasing with the squeezing parameter r and it is decreasing with increasing the ratio of the frequencies ω_1 and ω_2 of the two modes and the difference of parameters a and b.

4.4 Classical corellations and quantum mutual information

The measure of classical correlations for a general two-mode Gaussian state ρ_{12} can also be calculated and it is given by [49]

$$C = f(\sqrt{\alpha}) - f(\sqrt{\varepsilon}), \tag{57}$$

while the expression of the quantum mutual information, which measures the total correlations, is given by

$$I = f(\sqrt{\alpha}) + f(\sqrt{\beta}) - f(\nu_-) - f(\nu_+). \tag{58}$$

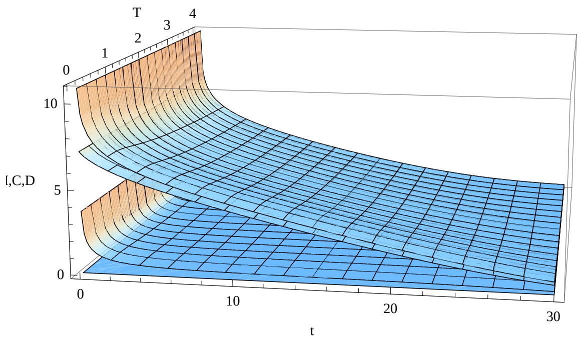

Figure 3: Quantum mutual information I versus time t and temperature T for an entangled initial non-symmetric squeezed thermal state with squeezing parameter $r = 3$, $n_1 = 3, n_2 = 1$ and $\lambda = 0.1, \omega_1 = 1, \omega_2 = 2$. We take $m = \hbar = k = 1$. There are also represented the Gaussian quantum discord and classical correlations.

In Fig. 3 we illustrate the evolution of classical correlations C and quantum mutual information I, as functions of time t and temperature T for an entangled initial Gaussian state, taken of the form of a two-mode squeezed thermal state (42). These two quantities manifest a qualitative behaviour similar to that one of the Gaussian discord: they have nonzero values for all finite times and in the limit of infinite time they tend asymptotically to zero, corresponding to the thermal product (separable) state, with no correlations at all. One can also see that the classical correlations and quantum mutual information decrease with increasing the temperature of the thermal bath. One can show that the classical correlations and quantum mutual information increase with increasing the squeezing parameter r and the difference of parameters a and b. At the same time classical correlations increase with the ratio of the frequencies ω_1 and ω_2 of the two modes, while quantum mutual information is decreasing with increasing this ratio. For comparison these quantities as well as quantum discord are represented on the same graphic. In the considered case the value of classical correlations is larger than that of quantum correlations, represented by the Gaussian quantum discord.

5 Conclusion

We have given a brief review of the theory of open quantum systems based on completely positive quantum dynamical semigroups and mentioned the necessity of the complete positivity for the existence of entangled states of systems interacting with an external environment. In the framework of this theory, by using the Peres-Simon necessary and sufficient condition for separability of two-mode Gaussian states, we investigated the Markovian dynamics of quantum correlations for a subsystem composed of two non-interacting bosonic modes embedded in a thermal bath. We have analyzed the influence of the environment on the dynamics of quantum entanglement and quantum discord for Gaussian initial states. We have described the time evolution of the logarithmic negativity, which characterizes the degree of entanglement of the quantum state, in terms of the covariance matrix for squeezed vacuum states and squeezed thermal states, for the case when the asymptotic state of the considered open system is a Gibbs state corresponding to two independent quantum harmonic oscillators in thermal equilibrium. For all values of the temperature of the thermal reservoir, an initial separable Gaussian state remains separable for all times. The dynamics of the quantum entanglement strongly depends on the initial states and the parameters characterizing the environment (temperature and dissipation constant). For an entangled initial squeezed vacuum state and squeezed thermal state, entanglement suppression (entanglement sudden death) takes place for all values of the temperatures of the environment, including zero temperature. The time when the entanglement is suppressed decreases with increasing the temperature and dissipation.

We described also the time evolution of Gaussian quantum discord, which is a measure of all quantum correlations in the bipartite state, including entanglement. The values of quantum discord decrease asymptotically in time. This is in contrast to the sudden death of entanglement. The time evolution of quantum discord is very similar to that of entanglement before the sudden suppression of the entanglement. Quantum discord is decreasing with increasing the temperature. After the sudden death of entanglement the nonzero values of discord manifest the existence of quantum correlations for separable mixed states. One considers that the robustness of quantum discord could favorize the realization of scalable quantum computing in contrast to the fragility of the entanglement [50]. We described also the time evolution of classical correlations and quantum mutual information, which measures the total correlations of the quantum system.

The existence of quantum correlations between the two bosonic modes interacting with a common environment is the result of the competition between entanglement and quantum decoherence. From the formal point of view, entanglement suppression corresponds to the finite time vanishing of the Simon separability function or, respectively, of the logarithmic negativity.

Presently there is a large debate relative to the physical interpretation existing behind the fascinating phenomena of quantum decoherence and existence of quantum correlations - quantum entanglement and quantum discord. Due to the increased interest manifested towards the continuous variables approach [7, 51] to quantum information theory, the present results, in particular the existence of quantum discord and the possibility of maintaining a bipartite entanglement in a thermal environment for long times, might be useful in controlling entanglement and discord in open systems and also for applications in the field of quantum information processing and communication.

6 Appendix

1. **Quantum information** is the study of the information processing tasks that can be accomplished using quantum mechanical systems [1].

Quantum theory, formalized in the first few decades of the 20th century, contains elements that are radically different from the classical description of Nature. An important aspect in these fundamental differences is the existence of quantum correlations in the quantum formalism. In the classical description of Nature, if a system is formed by different subsystems, complete knowledge of the whole system implies that the sum of the information of the subsystems makes up the complete information for the whole system. This is no longer true in the quantum formalism. In the quantum world, there exist states of composite systems for which we might have the complete information, while our knowledge about the subsystems might be completely random. One may reach some paradoxical conclusions if one applies a classical description to states which have characteristic quantum signatures. During the last decades, it was realized that these fundamentally nonclassical states, also denoted as entangled states, can provide us with something else than just paradoxes. They may be used to perform tasks that cannot be achieved with classical states. As benchmarks of this turning point in our view of such nonclassical states, one might mention the spectacular discoveries of (entanglement-based) quantum cryptography, quantum dense coding, and quantum teleportation [52].

Let us consider a bipartite system, which is traditionally supposed to be in possession of Alice (A) and Bob (B), who can be located in distant regions. Let Alice's physical system be described by the Hilbert space $\mathcal{H}_\mathcal{A}$ and that of Bob by $\mathcal{H}_\mathcal{B}$. Then the joint physical system of Alice and Bob is described by the tensor product Hilbert space $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$.

A pure state, i.e., a projector $|\psi_{AB} >< \psi_{AB}|$ on a vector $|\psi_{AB} >\in \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$, is a product state if the states of local subsystems are also pure states, that is, if $|\psi_{AB} >= |\psi_A > \otimes |\psi_B >$. However, there are states that cannot be written in this form. These states are called entangled states.

An example of entangled state is the well-known singlet state $(|01 > -|10 >)/\sqrt{2}$ (Bell state), where $|0 >$ and $|1 >$ are two orthonormal states.

A mixed state described by a density operator ρ_{AB} of a two-party system is separable if and only if it can be represented as a convex combination of the product states:

$$\rho_{AB} = \sum_i p_i \rho_A^i \otimes \rho_B^i,$$

where p_i is a probability distribution. Otherwise, the mixed state is said to be inseparable (entangled).

An important operational entanglement criterion is the positive partial transposition (PPT) criterion for detecting entanglement: given a bipartite state ρ_{AB}, find the eigenvalues of any of its partial transpositions with respect to one of the subsystems (transposition is equivalent to time reversal, or, expressed in terms of continuous variables, sign change of the momenta). A negative eigenvalue immediately implies that the state is entangled. Examples of states for which the partial transposition has negative eigenvalues include the singlet state.

The notion of entanglement appeared explicitly in the literature first in 1935, long before the dawn of the relatively young field of quantum information, and without any reference to discrete-variable qubit states. In fact, the entangled states treated in this 1935 paper by Einstein, Podolsky, and Rosen (EPR) were two-particle states quantum mechanically correlated with respect to their positions and momenta. The concept of entanglement has played an important role in quantum physics ever since its discovery last century and has now been recognized as a key resource in quantum information science.

The superposition principle leads to the existence of entangled states of two or more quantum systems and such states are characterized by the existence of correlations between the systems, the form of which cannot be satisfactorily accounted for by any classical theory. These have played a central role in the development of quantum theory since early in its development, starting with the famous paradox or dilemma of EPR. No less disturbing than the EPR dilemma is the problem of Schrödinger cat, an example of the apparent absurdity of following entanglement into the macroscopic world. It was Schrödinger who gave us the name entanglement (in German, "Verschränkung"); he emphasized its fundamental significance when he wrote, "I would call this not one but the characteristic trait of quantum mechanics, the one that enforces the entire departure from classical thought".

The prime example for an entangled Gaussian state is the pure two-mode squeezed (vacuum) state, described by the Gaussian Wigner function

$$W_{svs} = \frac{4}{\pi^2} \times \exp\{-e^{-2r}[(x_A + x_B)^2 + (p_A - p_B)^2] - e^{+2r}[(x_A - x_B)^2 + (p_A + p_B)^2]\},$$

where x_A, p_A, x_B, p_B are the coordinates and momenta of the the two-mode system and r is the squeezing parameter.

A unique measure of bipartite entanglement for pure states is given by the partial von Neumann entropy. This is the von Neumann entropy, of the reduced system after tracing out either subsystem: $\mathrm{Tr}\rho_A \ln \rho_A = \mathrm{Tr}\rho_B \ln \rho_B$, where $\rho_A = \mathrm{Tr}_B\rho_{AB},\ \rho_B = \mathrm{Tr}_A\rho_{AB}$.

In order to quantify the degrees of entanglement of an infinite-dimensional bipartite system states it is suitable to use the logarithmic negativity. The logarithmic negativity of a bipartite system consisting of two subsystems A and B is $E_N = \log_2 \|\rho^{\mathrm{T_B}}\|_1$, where $\rho^{\mathrm{T_B}}$ means the partial transpose of a mixed state density matrix operator ρ_{AB} with respect to subsystem B. The operation $\|\cdot\|_1$ denotes the trace norm, which for any Hermitian operator O is defined as $\|O\|_1 \equiv \mathrm{Tr}|O| \equiv \mathrm{Tr}\sqrt{O^\dagger O}$ and it is calculated as the sum of absolute values of the eigenvalues of O.

Logarithmic negativity quantifies the degree of violation of PPT criterion for separability, i.e. how much the partial transposition of ρ fails to be positive and it is based on negative eigenvalues of the partial transpose of the subsystem density matrix. For a Gaussian density operator, the negativity is completely defined by the symplectic spectrum of the partial transpose of the covariance matrix.

2. The **damped quantum harmonic oscillator** is considered in the framework of the theory of open systems based on completely positive quantum dynamical semigroups [33, 34]. The basic assumption is that the general form of a bounded mapping L given by Lindblad theorem is also valid for an unbounded completely dissipative mapping L:

$$L(\rho) = -\frac{i}{\hbar}[H, \rho] + \frac{1}{2\hbar}\sum_j([V_j\rho, V_j^\dagger] + [V_j, \rho V_j^\dagger]).$$

This assumption gives one of the simplest way to construct an appropriate model for this quantum dissipative system. Another simple condition imposed to the operators $H, V_j, V_j^\dagger$ is that they are functions of the basic observables of the one-dimensional quantum mechanical system q and p with $[q, p] = i\hbar I$, where I is the identity operator on $\mathcal{H}$ of such kind that the obtained model is exactly solvable. A precise version for this last condition is that linear spaces spanned by the first degree (respectively second degree) noncommutative polynomials in p and q are invariant to the action of the completely dissipative mapping L. This condition implies that V_j are at most first degree polynomials in p and q and H is at most a second degree polynomial in p and q.

Beacause in the linear space of the first degree polynomials in p and q the operators p and q give a basis, there exist only two C-linear independent operators V_1, V_2 which can be written in the form

$$V_i = a_i p + b_i q, \; i = 1, 2,$$

with a_i, b_i complex numbers. The constant term is omitted because its contribution to the generator L is equivalent to terms in H linear in p and q which for simplicity are chosen to be zero. Then H is chosen of the form

$$H = H_0 + \frac{\mu}{2}(pq + qp), \quad H_0 = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}q^2.$$

With these choices the Markovian master equation can be written:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_0, \rho] - \frac{i}{2\hbar}(\lambda + \mu)[q, \rho p + p\rho] + \frac{i}{2\hbar}(\lambda - \mu)[p, \rho q + q\rho]$$

$$-\frac{D_{pp}}{\hbar^2}[q, [q, \rho]] - \frac{D_{qq}}{\hbar^2}[p, [p, \rho]] + \frac{D_{pq}}{\hbar^2}([q, [p, \rho]] + [p, [q, \rho]]).$$

Here we used the notations:

$$D_{qq}=\frac{\hbar}{2}\sum_{j=1,2}|a_j|^2, D_{pp}=\frac{\hbar}{2}\sum_{j=1,2}|b_j|^2,$$
$$D_{pq}=D_{qp}=-\frac{\hbar}{2}\mathrm{Re}\sum_{j=1,2}a_j^*b_j, \lambda=-\mathrm{Im}\sum_{j=1,2}a_j^*b_j,$$

where D_{pp}, D_{qq} and D_{pq} are the diffusion coefficients and λ the friction constant. They satisfy the following fundamental constraints:

i) $D_{pp}>0$

ii) $D_{qq}>0$

iii) $D_{pp}D_{qq}-{D_{pq}}^2\geq\lambda^2\hbar^2/4.$

We introduce the following notations:

$$\sigma_q(t)=\mathrm{Tr}(\rho(t)q),$$

$$\sigma_p(t)=\mathrm{Tr}(\rho(t)p),$$
$$\sigma_{qq}=\mathrm{Tr}(\rho(t)q^2)-\sigma_q^2(t),$$
$$\sigma_{pp}=\mathrm{Tr}(\rho(t)p^2)-\sigma_p^2(t),$$
$$\sigma_{pq}(t)=\mathrm{Tr}(\rho(t)\frac{pq+qp}{2})-\sigma_p(t)\sigma_q(t).$$

In the Heisenberg picture the master equation has the following symmetric form:

$$\frac{d\widetilde{\Phi}_t(A)}{dt}=\widetilde{L}(\widetilde{\Phi}_t(A))=\frac{i}{\hbar}[H_0,\widetilde{\Phi}_t(A)]-\frac{i}{2\hbar}(\lambda+\mu)([\widetilde{\Phi}_t(A),q]p+p[\widetilde{\Phi}_t(A),q])$$
$$+\frac{i}{2\hbar}(\lambda-\mu)(q[\widetilde{\Phi}_t(A),p]+[\widetilde{\Phi}_t(A),p]q)-\frac{D_{pp}}{\hbar^2}[q,[q,\widetilde{\Phi}_t(A)]]$$
$$-\frac{D_{qq}}{\hbar^2}[p,[p,\widetilde{\Phi}_t(A)]]+\frac{D_{pq}}{\hbar^2}([p,[q,\widetilde{\Phi}_t(A)]]+[q,[p,\widetilde{\Phi}_t(A)]]).$$

Denoting by A any selfadjoint operator we have

$$\sigma_A(t)=\mathrm{Tr}(\rho(t)A),\ \sigma_{AA}(t)=\mathrm{Tr}(\rho(t)A^2)-\sigma_A^2(t).$$

It follows that

$$\frac{d\sigma_A(t)}{dt}=\mathrm{Tr}L(\rho(t))A=\mathrm{Tr}\rho(t)\widetilde{L}(A)$$

and

$$\frac{d\sigma_{AA}(t)}{dt} = \mathrm{Tr}L(\rho(t))A^2 - 2\frac{d\sigma_A(t)}{dt}\sigma_A(t) = \mathrm{Tr}\rho(t)\widetilde{L}(A^2) - 2\sigma_A(t)\mathrm{Tr}\rho(t)\widetilde{L}(A).$$

An important consequence of the precise version of solvability condition is the fact that when A is put equal to p or q, then $d\sigma_p(t)/dt$ and $d\sigma_q(t)/dt$ are functions only of $\sigma_p(t)$ and $\sigma_q(t)$ and $d\sigma_{pp}(t)/dt, d\sigma_{qq}(t)/dt$ and $d\sigma_{pq}(t)/dt$ are functions only of $\sigma_{pp}(t), \sigma_{qq}(t)$ and $\sigma_{pq}(t)$. This fact allows an immediate determination of the functions of time $\sigma_p(t), \sigma_q(t), \sigma_{pp}(t), \sigma_{qq}(t), \sigma_{pq}(t)$. Indeed we obtain:

$$\frac{d\sigma_q(t)}{dt} = -(\lambda - \mu)\sigma_q(t) + \frac{1}{m}\sigma_p(t),$$

$$\frac{d\sigma_p(t)}{dt} = -m\omega^2\sigma_q(t) - (\lambda + \mu)\sigma_p(t)$$

and

$$\frac{d\sigma_{qq}(t)}{dt} = -2(\lambda - \mu)\sigma_{qq}(t) + \frac{2}{m}\sigma_{pq}(t) + 2D_{qq},$$

$$\frac{d\sigma_{pp}}{dt} = -2(\lambda + \mu)\sigma_{pp}(t) - 2m\omega^2\sigma_{pq}(t) + 2D_{pp},$$

$$\frac{d\sigma_{pq}(t)}{dt} = -m\omega^2\sigma_{qq}(t) + \frac{1}{m}\sigma_{pp}(t) - 2\lambda\sigma_{pq}(t) + 2D_{pq}.$$

The integration of these systems of equations of motion is straightforward. There are two cases: *a)* $\mu > \omega$ (overdamped) and *b)* $\mu < \omega$ (underdamped). In the case *a)* with the notation $\nu^2 = \mu^2 - \omega^2$ we obtain:

$$\sigma_q(t) = e^{-\lambda t}((\cosh\nu t + \frac{\mu}{\nu}\sinh\nu t)\sigma_q(0) + \frac{1}{m\nu}\sinh\nu t\sigma_p(0)),$$

$$\sigma_p(t) = e^{-\lambda t}(-\frac{m\omega^2}{\nu}\sinh\nu t\sigma_q(0) + (\cosh\nu t - \frac{\mu}{\nu}\sinh\nu t)\sigma_p(0)).$$

If $\lambda > \nu$, then $\sigma_q(\infty) = \sigma_p(\infty) = 0$. If $\lambda < \nu$, then $\sigma_q(\infty) = \sigma_p(\infty) \to \infty$. In the case *b)* with the notation $\Omega^2 = \omega^2 - \mu^2$, we obtain:

$$\sigma_q(t) = e^{-\lambda t}((\cos\Omega t + \frac{\mu}{\Omega}\sin\Omega t)\sigma_q(0) + \frac{1}{m\Omega}\sin\Omega t\sigma_p(0)),$$

$$\sigma_p(t) = e^{-\lambda t}(-\frac{m\omega^2}{\Omega}\sin\Omega t\sigma_q(0) + (\cos\Omega t - \frac{\mu}{\Omega}\sin\Omega t)\sigma_p(0))$$

and $\sigma_q(\infty) = \sigma_p(\infty) = 0$.

In order to integrate the system of equations for the covariances it is convenient to consider the vector

$$X(t) = \left(m\omega\sigma_{qq}(t) \frac{1}{m\omega}\sigma_{pp}(t)\sigma_{pq}(t) \right).$$

Introducing the following matrices: in the overdamped case ($\mu > \omega,\ \nu^2 = \mu^2 - \omega^2$)

$$T = \frac{1}{2\nu} \left(\mu + \nu\mu - \nu 2\omega\mu - \nu\mu + \nu 2\omega - \omega - \omega - 2\mu \right),$$

$$K = \left(-2(\lambda - \nu)000 - 2(\lambda + \nu)000 - 2\lambda \right),$$

and in the underdamped case ($\mu < \omega,\ \Omega^2 = \omega^2 - \mu^2$)

$$T = \frac{1}{2i\Omega} \left(\mu + i\Omega\mu - i\Omega 2\omega\mu - i\Omega\mu + i\Omega 2\omega - \omega - \omega - 2\mu \right),$$

$$K = \left(-2(\lambda - i\Omega)000 - 2(\lambda + i\Omega)000 - 2\lambda \right),$$

the solution can be written in the form [33, 34]

$$X(t) = (Te^{Kt}T)(X(0) - X(\infty)) + X(\infty).$$

Between the asymptotic values of $\sigma_{qq}(t), \sigma_{pp}(t), \sigma_{pq}(t)$ and the diffusion coefficients D_{qq}, D_{pp}, D_{pq} there exist the following connection, which is the same for both cases, underdamped and overdamped:

$$D_{qq} = (\lambda - \mu)\sigma_{qq}(\infty) - \frac{1}{m}\sigma_{pq}(\infty),$$

$$D_{pp} = (\lambda + \mu)\sigma_{pp}(\infty) + m\omega^2\sigma_{pq}(\infty),$$

$$D_{pq} = \frac{1}{2}(m\omega^2\sigma_{qq}(\infty) - \frac{1}{m}\sigma_{pp}(\infty) + 2\lambda\sigma_{pq}(\infty)).$$

These relations show that the asymptotic values $\sigma_{qq}(\infty), \sigma_{pp}(\infty), \sigma_{pq}(\infty)$ do not depend on the initial values $\sigma_{qq}(0), \sigma_{pp}(0), \sigma_{pq}(0)$.

If the asymptotic state is a Gibbs state (T denotes the temperature of the thermal bath)

$$\rho_G(\infty) = e^{-\frac{H_0}{kT}} / \mathrm{Tr}(e^{-\frac{H_0}{kT}}),$$

then

$$\sigma_{qq}(\infty) = \frac{\hbar}{2m\omega} \coth \frac{\hbar\omega}{2kT},\ \ \sigma_{pp}(\infty) = \frac{\hbar m\omega}{2} \coth \frac{\hbar\omega}{2kT},\ \ \sigma_{pq}(\infty) = 0$$

and

$$D_{pp} = \frac{\lambda+\mu}{2}\hbar m\omega \coth\frac{\hbar\omega}{2kT},\ D_{qq} = \frac{\lambda-\mu}{2}\frac{\hbar}{m\omega}\coth\frac{\hbar\omega}{2kT},\ D_{pq}=0$$

and the fundamental constraints are satisfied only if $\lambda > \mu$ and

$$(\lambda^2-\mu^2)\coth^2\frac{\hbar\omega}{2kT} \geq \lambda^2.$$

If the initial state is the ground state of the harmonic oscillator, then

$$\sigma_{qq}(0) = \frac{\hbar}{2m\omega},\ \sigma_{pp}(0) = \frac{m\hbar\omega}{2},\ \sigma_{pq}(0) = 0.$$

The explicit time dependence of $\sigma_{qq}(t), \sigma_{pp}(t), \sigma_{pq}(t)$ can be given for both under- and overdamped cases if we have the matrix elements of $Te^{Kt}T$. In the overdamped case ($\mu > \omega,\ \nu^2 = \mu^2-\omega^2$) we have (in this case the restriction $\lambda > \nu$ is necessary):

$$Te^{Kt}T = \frac{e^{-2\lambda t}}{2\nu^2}\left(a_{11}a_{12}a_{13}a_{21}a_{22}a_{23}a_{31}a_{32}a_{33}\right),$$

with

$a_{11} = (\mu^2+\nu^2)\cosh 2\nu t + 2\mu\nu\sinh 2\nu t - \omega^2,$
$a_{12} = (\mu^2-\nu^2)\cosh 2\nu t - \omega^2,$
$a_{13} = 2\omega(\mu\cosh 2\nu t + \nu\sinh 2\nu t - \mu),$
$a_{21} = (\mu^2-\nu^2)\cosh 2\nu t - \omega^2,$
$a_{22} = (\mu^2+\nu^2)\cosh 2\nu t - 2\mu\nu\sinh 2\nu t - \omega^2,$
$a_{23} = 2\omega(\mu\cosh 2\nu t - \nu\sinh 2\nu t - \mu),$
$a_{31} = -\omega(\mu\cosh 2\nu t + \nu\sinh 2\nu t - \mu),$
$a_{32} = -\omega(\mu\cosh 2\nu t - \nu\sinh 2\nu t - \mu),$
$a_{33} = -2(\omega^2\cosh 2\nu t - \mu^2).$

In the underdamped case ($\mu < \omega,\ \Omega^2 = \omega^2-\mu^2$) we have

$$Te^{Kt}T = -\frac{e^{-2\lambda t}}{2\Omega^2}\left(b_{11}b_{12}b_{13}b_{21}b_{22}b_{23}b_{31}b_{32}b_{33}\right)$$

with

$b_{11} = (\mu^2-\Omega^2)\cos 2\Omega t - 2\mu\Omega\sin 2\Omega t - \omega^2,$
$b_{12} = (\mu^2+\Omega^2)\cos 2\Omega t - \omega^2,$
$b_{13} = 2\omega(\mu\cos 2\Omega t - \Omega\sin 2\Omega t - \mu),$
$b_{21} = (\mu^2+\Omega^2)\cos 2\Omega t - \omega^2,$
$b_{22} = (\mu^2-\Omega^2)\cos 2\Omega t + 2\mu\Omega\sin 2\Omega t - \omega^2,$

$b_{23} = 2\omega(\mu \cos 2\Omega t + \Omega \sin 2\Omega t - \mu),$
$b_{31} = -\omega(\mu \cos 2\Omega t - \Omega \sin 2\Omega t - \mu),$
$b_{32} = -\omega(\mu \cos 2\Omega t + \Omega \sin 2\Omega t - \mu),$
$b_{33} = -2(\omega^2 \cos 2\Omega t - \mu^2).$

Acknowledgments. The author acknowledges the financial support received from the Romanian Ministry of Education and Research, through the Projects CNCS-UEFISCDI PN-II-ID-PCE-2011-3-0083 and PN 09 37 01 02/2009.

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