

MINIMAL INVARIANT SUBSPACES AND REACHABILITY OF 2D HYBRID LTI SYSTEMS*

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Abstract

An algorithm is provided to determine the minimal subspace which is invariant with respect to some commutative matrices and which includes a given subspace. Reachability criteria are obtained for 2D continuous-discrete time-variable Attasi type systems by using a suitable 2D reachability Gramian. Necessary and sufficient conditions of reachability are derived for LTI 2D systems. The presented algorithm is used to determine the subspace of the reachable states of a 2D system.

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1 Introduction

The multidimensional (nD) systems form a distinct and important branch of Systems and Control Theory.

In various problems such as signal and image processing, seismology and geophysics, control of multipass processes, iterative learning control synthesis

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[8] or repetitive processes [5], the suitable mathematical model is represented by the hybrid (continuous-discrete) multidimensional systems ([7], [10], [11]).

The concepts of controllability and observability, introduced by Kalman for 1D systems were extended to 2D systems for Roesser [13], Fornasini and Marchesini [4], and Attasi [1] models; in order to keep their relationship with minimality, new concepts of modal controllability and modal observability were introduced in [6].

The Geometric Approach is a trend in Systems and Control Theory developed to achieve a better and neater investigation of the structural properties of the linear dynamical systems and to provide elegant solutions of problems of controller synthesis such as decoupling and pole-assignment problems for linear time-invariant multivariable systems. The geometric approach leads to a very clear notion of minimality and to geometric conditions for controllability, reachability, observability, constructibility and minimality of linear systems. The cornerstone of this approach is the concept of invariance of a subspace with respect to a linear transformation.

In 1969 Basile and Marro [2] introduced and studied the basic geometric tools called controlled and conditioned invariant subspaces which were applied to disturbance rejection or unknown-input observability. In 1970 Wonham and Morse [15] applied a maximal controlled invariant method to decoupling and noninteracting control problems and later on Wonham's book [14] imposed the name of "(A,B)-invariant" instead of "(A,B)-controlled invariant". Basile and Marro, opened the way to new applications by the robust controlled invariant and the emphasis of the duality [3], [9]. The LQ problem was also studied in a geometric framework by Silverman, Hautus, Willems. Further contributions are due to numerous researchers among which Anderson, Akashi, Bhattacharyya, Kucera, Malabre, Molinari, Pearson, Francis and Schumacher.

In this paper a class of 2D continuous-discrete time-variable linear systems is studied, which is related to Attasi's 2D discrete model and represents the extension to time-variable framework of the hybrid systems introduced in [12].

In Section 2 an algorithm is proposed which determines the minimal (A_1, A_2) -invariant subspace which includes a given subspace \mathcal{B} of a space \mathbf{K}^n , where A_1 and A_2 are commutative matrices over a field \mathbf{K} .

The state and output formulæ for these systems are established in Section 3 and the notions of complete reachability and complete controllability are

defined. These properties are characterized by means of the full rank of a suitable 2D reachability Gramian.

Section 4 is devoted to time-invariant 2D continuous-discrete systems and a list of criteria of reachability is provided. The subspace of the reachable states of a system $\Sigma = (A_1, A_2; B)$ is characterized as the minimal (A_1, A_2) -invariant subspace which includes the subspace $\text{Im}B$.

In Section 5 the algorithm presented in Section 2 is used to determine the subspace of reachable states. A MATLAB Program and an example illustrate the proposed algorithm.

2 Minimal invariant subspaces with respect to two matrices

Let \mathbf{K} be a field and $A_1, A_2 \in \mathbf{K}^{n \times n}$ commutative matrices.

Definition 1 *A subspace \mathcal{V} of \mathbf{K}^n is said to be (A_1, A_2) -invariant if*

$$\forall v \in \mathcal{V}, A_1 v \in \mathcal{V} \text{ and } A_2 v \in \mathcal{V}. \tag{1}$$

Let \mathcal{B} be a proper subspace of \mathbf{K}^n . The intersection of the (A_1, A_2) -invariant subspaces which include \mathcal{B} is the minimal (A_1, A_2) -invariant subspace which includes \mathcal{B} . We denote it $\text{min}I(A_1, A_2; \mathcal{B})$.

We consider the subspaces $A_1^k A_2^l \mathcal{V} = \{A_1^k A_2^l v \mid v \in \mathcal{V}, k, l \in \mathbf{N}\}$, $A_i^0 \mathcal{V} = \mathcal{V}$ and $\sum_{i \in I} \mathcal{V}_i = \{\sum_{j \in J} v_j \mid v_j \in \mathcal{V}_j, J \subset I, J \text{ finite sets}\}$, where $\mathcal{V}_i, i \in I$ are subspaces of \mathbf{K}^n , and the set I is at most countable.

Proposition 1 *The minimal (A_1, A_2) -invariant subspace which includes \mathcal{B} is*

$$\text{min}I(A_1, A_2; \mathcal{B}) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} A_1^i A_2^j \mathcal{B}. \tag{2}$$

Proof: Let us denote by \mathcal{W} the subspace $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} A_1^i A_2^j \mathcal{B}$. Then $v \in \mathcal{W}$ if

and only if $v = \sum_{i \in I} \sum_{j \in J} A_1^i A_2^j v_{ij}$ with $v_{ij} \in \mathcal{B}$ and $I, J \subset \mathbf{N}$ finite sets. Then

$A_1 v = \sum_{i \in I} \sum_{j \in J} A_1^{i+1} A_2^j v_{ij}$ hence $A_1 v \in \mathcal{W}$ and similarly $A_2 v \in \mathcal{W}$. Obviously

$\mathcal{B} \subset \mathcal{W}$, hence \mathcal{W} is an (A_1, A_2) -invariant subspace which includes \mathcal{B} .

Now, let \mathcal{V} be an (A_1, A_2) -invariant subspace which includes \mathcal{B} . Then $A_1^i A_2^j \mathcal{B} \subset \mathcal{V}$, $\forall i, j \in \mathbf{N}$, hence $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} A_1^i A_2^j \mathcal{B} \subset \mathcal{V}$, i.e. the subspace $\mathcal{W} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} A_1^i A_2^j \mathcal{B}$ is the minimal such subspace.

Proposition 2 *The minimal (A_1, A_2) -invariant subspace which includes \mathcal{B} is*

$$\min I(A_1, A_2; \mathcal{B}) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} A_1^i A_2^j \mathcal{B}. \quad (3)$$

Proof: Let us denote by \mathcal{V} the subspace in the right-hand member of (3). Obviously, by Proposition 1, $\mathcal{V} \subset \min I(A_1, A_2; \mathcal{B})$.

Let $p_k(s) = \det(sI - A_k) = s^n + a_{n-1,k}s^{n-1} + \dots + a_{1,k}s + a_{0,k}$, $k = 1, 2$ be the characteristic polynomial of the matrix A_k , $k = 1, 2$. By Hamilton-Cayley Theorem, each matrix verifies its characteristic equation, hence

$$A_k^n = -a_{n-1,k}A_k^{n-1} - \dots - a_{1,k}A_k - a_{0,k}I, \quad k = 1, 2. \quad (4)$$

Then, for any vector $v \in \mathcal{B}$, $A_k^n v = -a_{n-1,k}A_k^{n-1}v - \dots - a_{1,k}A_k v - a_{0,k}v$, $k = 1, 2$. We can prove by induction and by right-multiplication of (4) by $A_1^l A_2^q v$, $l, q \in \mathbf{N}$ that $A_1^i A_2^j v \in \mathcal{V}$, $\forall i, j \geq n$, i.e. $\min I(A_1, A_2; \mathcal{B}) \subset \mathcal{V}$, hence $\min I(A_1, A_2; \mathcal{B}) = \mathcal{V}$.

The following algorithm determines recurrently the subspace $\min I(A_1, A_2; \mathcal{B})$.

Algorithm 1

Stage 1. Construct the sequence of subspaces $(S_{0,j})_{0 \leq j \leq n}$ of the space $X = \mathbf{K}^n$:

$$S_{0,0} = \mathcal{B}; \quad (5)$$

$$S_{0,j} = \mathcal{B} + A_2 S_{0,j-1}, \quad j = 1, \dots, n; \quad (6)$$

Stage 2. Determine j_0 , the first index in $\{0, 1, \dots, n-1\}$ which verifies

$$S_{0,j_0+1} = S_{0,j_0}. \quad (7)$$

If $j_0 = n-1$, then $\min I(A_1, A_2; \mathcal{B}) = \mathbf{K}^n$. STOP

If $j_0 < n - 1$, GO TO Stage 3.

Stage 3. Construct the sequence of subspaces $(S_{i,j_0})_{0 \leq i \leq n}$ of the space $X = \mathbf{K}^n$:

$$S_{i,j_0} = S_{i-1,j_0} + A_1 S_{i-1,j_0}. \quad (8)$$

Stage 4. Determine i_0 , the first index in $\{0, 1, \dots, n - 1\}$ which verifies

$$S_{i_0+1,j_0} = S_{i_0,j_0}. \quad (9)$$

Then $\min I(A_1, A_2; \mathcal{B}) = S_{i_0,j_0}$. STOP

Proof. Let us consider the doubly-indexed sequence of subspaces

$$\tilde{S}_{i,j} = \left(\sum_{k=0}^i A_1^k \right) \left(\sum_{l=0}^j A_2^l \right) \mathcal{B}, \quad i, j \in \{0, 1, \dots, n\}, \quad (10)$$

where $A_k^0 \mathcal{B} = \mathcal{B}$, $k = 1, 2$.

By Proposition 1, $\tilde{S}_{i,j} \subseteq \min I(A_1, A_2; \mathcal{B}) \quad \forall i, j$ and by Proposition 2, $\tilde{S}_{n-1,n-1} = \min I(A_1, A_2; \mathcal{B})$. Moreover

$$\tilde{S}_{i,j} \subseteq \tilde{S}_{k,l}, \quad \forall k \geq i, l \geq j. \quad (11)$$

Obviously $\tilde{S}_{0,0} = A_1^0 A_2^0 \mathcal{B} = \mathcal{B} = S_{0,0}$ and, for any $j \in \{0, 1, \dots, n - 1\}$, $\tilde{S}_{0,j} = \sum_{l=0}^j A_2^l \mathcal{B} = \mathcal{B} + A_2 \sum_{l=0}^{j-1} A_2^l \mathcal{B} = \mathcal{B} + A_2 \tilde{S}_{0,j-1}$.

Then, if we assume that $\tilde{S}_{0,j-1} = S_{0,j-1}$, we get by (6) the following equality:

$$\tilde{S}_{0,j} = S_{0,j}, \quad \forall j \in \{0, 1, \dots, n - 1\}, \quad (12)$$

hence

$$S_{0,j} \subseteq S_{0,j+1}, \quad \forall j \in \{0, 1, \dots, n - 1\}, \quad (13)$$

Using Hamilton-Cayley Theorem (see (4)) and (13) one obtains $S_{0,n} = \tilde{S}_{0,n} = \tilde{S}_{0,n-1} = S_{0,n-1}$, hence $j_0 \leq n - 1$.

In the chain of subspaces

$$\{0\} \subset S_{0,0} \subseteq S_{0,1} \subseteq \dots \subseteq S_{0,k-1} \subseteq S_{0,k} \subseteq \dots \subseteq S_{0,n-1} = S_{0,n},$$

$\dim S_{0,0} = \dim \mathcal{B} \geq 1$ since \mathcal{B} is a proper subspace of $X = \mathbf{K}^n$. If $j_0 = n - 1$ is the first index which verifies (7), we have

$$1 \leq \dim S_{0,0} < \dim S_{0,1} < \dots < \dim S_{0,n-1} \leq n$$

hence $\dim S_{0,n-1} = n$. Therefore $\mathbf{K}^n = S_{0,n-1} \subseteq \min I(A_1, A_2; \mathcal{B}) \subseteq \mathbf{K}^n$, hence $\min I(A_1, A_2; \mathcal{B}) = \mathbf{K}^n$.

If $j_0 < n - 1$ and $S_{0,j_0+1} = S_{0,j_0}$ one obtains by (6) $S_{0,j_0+2} = \mathcal{B} + A_2 S_{0,j_0+1} = \mathcal{B} + A_2 S_{0,j_0} = S_{0,j_0+1} = S_{0,j_0}$.

Let us assume that $S_{0,j} = S_{0,j_0}$ for some $j \in \{j_0 + 1, j_0 + 2, \dots, n - 1\}$. Then $S_{0,j+1} = \mathcal{B} + A_2 S_{0,j} = \mathcal{B} + A_2 S_{0,j_0} = S_{0,j_0+1} = S_{0,j_0}$ hence

$$S_{0,j} = S_{0,j_0}, \quad \forall j \in \{j_0 + 1, j_0 + 2, \dots, n\}. \quad (14)$$

Now, if we assume that $\tilde{S}_{i-1,j_0} = S_{i-1,j_0}$, for some $i \in \{1, 2, \dots, n - 1\}$, we get by (10),(8) and by the equality $\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3 = (\mathcal{V}_1 + \mathcal{V}_2) + (\mathcal{V}_2 + \mathcal{V}_3)$ for any subspaces \mathcal{V}_k , $k = 1, 2, 3$ the following equalities:

$$\begin{aligned} \tilde{S}_{i,j_0} &= \sum_{k=0}^i \sum_{l=0}^{j_0} A_1^k A_2^l \mathcal{B} = \sum_{k=0}^{i-1} \sum_{l=0}^{j_0} A_1^k A_2^l \mathcal{B} + \sum_{k=1}^i \sum_{l=0}^{j_0} A_1^k A_2^l \mathcal{B} = \sum_{k=0}^{i-1} \sum_{l=0}^{j_0} A_1^k A_2^l \mathcal{B} + \\ &A_1 \sum_{k=0}^{i-1} \sum_{l=0}^{j_0} A_1^k A_2^l \mathcal{B} = \tilde{S}_{i-1,j_0} + A_1 \tilde{S}_{i-1,j_0} = S_{i-1,j_0} + A_1 S_{i-1,j_0} = S_{i,j_0}, \end{aligned}$$

hence $\tilde{S}_{i,j_0} = S_{i,j_0}$, $\forall i \in \{0, 1, \dots, n\}$. It follows by (11) that $S_{i,j_0} \subseteq S_{i+1,j_0}$ and by Hamilton-Cayley Theorem that $S_{n-1,j_0} = S_{n,j_0}$.

Now, let us consider the chain of subspaces

$$\{0\} \subset S_{0,j_0} \subseteq S_{1,j_0} \subseteq \dots \subseteq S_{n-1,j_0} = S_{n,j_0} \subset \mathbf{R}^n.$$

Since $\dim S_{0,j_0} \geq \dim S_{0,0} \geq 1$, if $i_0 = n - 1$ we obtain as above that $\dim S_{n-1,j_0} = n$, hence $S_{n-1,j_0} = \min I(A_1, A_2; \mathcal{B}) = \mathbf{K}^n$.

If $i_0 < n - 1$, we have by $\tilde{S}_{i_0+1,j_0} = S_{i_0+1,j_0} = S_{i_0,j_0} = \tilde{S}_{i_0,j_0}$ and by (9) $\tilde{S}_{i_0+2,j_0} = \tilde{S}_{i_0+1,j_0} + A_1 \tilde{S}_{i_0+1,j_0} = \tilde{S}_{i_0,j_0} + A_1 \tilde{S}_{i_0,j_0} = \tilde{S}_{i_0+1,j_0} = \tilde{S}_{i_0,j_0}$. Let us assume that $\tilde{S}_{i,j_0} = \tilde{S}_{i_0,j_0}$ for some $i \in \{i_0 + 1, i_0 + 2, \dots, n - 1\}$. Then, again by (9) $\tilde{S}_{i+1,j_0} = \tilde{S}_{i,j_0} + A_1 \tilde{S}_{i,j_0} = \tilde{S}_{i_0,j_0} + A_1 \tilde{S}_{i_0,j_0} = \tilde{S}_{i_0+1,j_0} = \tilde{S}_{i_0,j_0}$, therefore

$$\tilde{S}_{i,j_0} = S_{i_0,j_0}, \quad \forall i \in \{i_0 + 1, i_0 + 2, \dots, n - 1\}. \quad (15)$$

From (9), (14) and (15) we get $\sum_{l=0}^j A_2^l \mathcal{B} = \sum_{l=0}^{j_0} A_2^l \mathcal{B}$, $\forall j \in \{j_0 + 1, j_0 + 2, \dots, n\}$ and $\sum_{k=0}^i \sum_{l=0}^j A_1^k A_2^l \mathcal{B} = \sum_{k=0}^i \sum_{l=0}^{j_0} A_1^k A_2^l \mathcal{B} \quad \forall i \in \{i_0 + 1, i_0 + 2, \dots, n\}$.

Then, $\forall i \in \{i_0 + 1, i_0 + 2, \dots, n\}$ and $\forall j \in \{j_0 + 1, j_0 + 2, \dots, n\}$, one obtains $\tilde{S}_{i,j} = \sum_{k=0}^i \sum_{l=0}^j A_1^k A_2^l \mathcal{B} = \sum_{k=0}^i A_1^k \left(\sum_{l=0}^j A_2^l \mathcal{B} \right) = \sum_{k=0}^i A_1^k \left(\sum_{l=0}^{j_0} A_2^l \mathcal{B} \right) = \sum_{k=0}^{i_0} \sum_{l=0}^{j_0} A_1^k A_2^l \mathcal{B} = S_{i_0, j_0}$, hence

$$\tilde{S}_{i,j} = S_{i_0, j_0}, \quad \forall i \in \{i_0 + 1, i_0 + 2, \dots, n\}, \quad \forall j \in \{j_0 + 1, j_0 + 2, \dots, n\}.$$

By Proposition 2, we obtain

$$\min I(A_1, A_2; \mathcal{B}) = \tilde{S}_{n-1, n-1} = S_{i_0, j_0}, \quad \text{q.e.d.}$$

The equality above proves the following result:

Proposition 3 *The minimal (A_1, A_2) -invariant subspace which includes \mathcal{B} is*

$$\min I(A_1, A_2; \mathcal{B}) = S_{i_0, j_0} = \sum_{i=0}^{i_0} \sum_{j=0}^{j_0} A_1^i A_2^j \mathcal{B}. \quad (16)$$

3 The state space representation of the 2D continuous-discrete systems

In this section we consider the linear spaces $X = \mathbf{R}^n$, $U = \mathbf{R}^m$ and $Y = \mathbf{R}^p$, called respectively the *state*, *input* and *output spaces*. The time set is $T = \mathbf{R} \times \mathbf{Z}$. By $(s, l) < (t, k)$ for $(s, l), (t, k) \in T$ we mean $s \leq t, l \leq k$ and $(s, l) \neq (t, k)$ and $(s, l) \leq (t, k)$ means $s \leq t, l \leq k$.

Definition 2 *A two-dimensional continuous-discrete linear system (2Dcd) is a quintuplet $\Sigma = (A_1(t, k), A_2(t, k), B(t, k), C(t, k), D(t, k)) \in \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times m} \times \mathbf{R}^{p \times n} \times \mathbf{R}^{p \times m}$ with $A_1(t, k)A_2(t, k) = A_2(t, k)A_1(t, k) \forall (t, k) \in T$, where all matrices are continuous with respect to $t \in \mathbf{R}$ for any $k \in \mathbf{Z}$; the state space representation of Σ is given by the following state and output equations (where $\dot{x}(t, k) = \frac{\partial x}{\partial t}(t, k)$).*

$$\begin{aligned} \dot{x}(t, k+1) &= A_1(t, k+1)x(t, k+1) + A_2(t+1, k)\dot{x}(t, k) \\ &\quad - A_1(t, k)A_2(t, k)x(t, k) + B(t, k)u(t, k) \end{aligned} \quad (17)$$

$$y(t, k) = C(t, k)x(t, k) + D(t, k)u(t, k). \quad (18)$$

We denote by $\Phi(t, t_0; k)$ or $\Phi_{A_1}(t, t_0; k)$ the (continuous) fundamental matrix of $A_1(t, k)$ with respect to $t \in \mathbf{R}$, for any fixed $k \in \mathbf{Z}$. $\Phi(t, t_0; k)$ has the following properties, for any $t, t_0, t_1 \in \mathbf{R}$:

- i) $\frac{d}{dt}\Phi(t, t_0; k) = A_1(t, k)\Phi(t, t_0; k),$
- ii) $\Phi(t_0, t_0; k) = I_n,$
- iii) $\Phi(t, t_1; k)\Phi(t_1, t_0; k) = \Phi(t, t_0; k),$
- iv) $\Phi(t, t_0; k)^{-1} = \Phi(t_0, t; k),$

$$v) \Phi(t, t_0; k) = I + \sum_{l=1}^{\infty} \int_{t_0}^t A_1(s_1, k) \int_{t_0}^{s_1} A_1(s_2, k) \cdots \int_{t_0}^{s_{l-1}} A_1(s_l, k) ds_l ds_{l-1} \cdots ds_2 ds_1.$$

If A_1 is a constant matrix, then $\Phi(t, t_0; k) = \sum_{l=0}^{\infty} \frac{A_1^l (t-t_0)^l}{l!} = e^{A_1(t-t_0)}.$

The *discrete fundamental matrix* $F(t; k, k_0)$ of the matrix $A_2(t, k)$ is defined by

$$F(t; k, k_0) = \begin{cases} A_2(t, k-1)A_2(t, k-2) \cdots A_2(t, k_0) & \text{for } k > k_0 \\ I_n & \text{for } k = k_0 \end{cases}$$

for any fixed $t \in \mathbf{R}$.

If A_2 is a constant matrix, then $F(t; k, k_0) = A_2^{k-k_0}.$

$\Phi(t, t_0; k)$ and $F(s; l, l_0)$ are commutative matrices for any $t, t_0, s \in \mathbf{R}$ and $k, l, l_0 \in \mathbf{Z}$ since $A_1(t, k)$ and $A_2(t, k)$ are commutative matrices.

Definition 3 *A vector $x_0 \in X$ is called the initial state of Σ at the moment $(t_0, k_0) \in T$ if, for any $(t, k) \in T$ with $(t, k) \geq (t_0, k_0)$ the following conditions hold:*

$$x(t, k_0) = \Phi(t, t_0; k_0)x_0, \quad x(t_0, k) = F(t_0; k, k_0)x_0. \tag{19}$$

For $(t_0, k_0) \leq (t, k)$ we denote by $[t_0, t; k_0, k]$ the set $[t_0, t; k_0, k] = [t_0, t] \times \{k_0, k_0 + 1, \dots, k\}.$

From [11], Proposition 2.3 we obtain:

Proposition 4 *The state of the system Σ at the moment $(t, k) \in T$ determined by the control $u(\cdot, \cdot) : [t_0, t; k_0, k] \rightarrow U$ and by the initial state $x_0 \in X$ is*

$$x(t, k) = \Phi(t, t_0; k)F(t_0; k, k_0)x_0 + \int_{t_0}^t \sum_{l=k_0}^{k-1} \Phi(t, s; k)F(s; k, l+1)B(s, l)u(s, l)ds. \tag{20}$$

By replacing the state $x(t, k)$ given by (5) in the output equation (2) we get

Proposition 5 *The input-output map of the system Σ is given by the formula*

$$y(t, k) = C(t, k)\Phi(t, t_0; k)F(t_0; k, k_0)x_0 + \int_{t_0}^t \sum_{l=k_0}^{k-1} C(t, k)\Phi(t, s; k)F(s; k, l+1)B(s, l)u(s, l)ds + D(t, k)u(t, k). \tag{21}$$

4 Reachability of time-variable 2D continuous-discrete systems

For the concept of reachability we need only the state equation (17), hence a 2Dcd system can be reduced to the triplet $\Sigma = (A_1(t, k), A_2(t, k), B(t, k))$.

A triplet $(t, k, x) \in \mathbf{R} \times \mathbf{Z} \times X$ is said to be a *phase* of Σ if x is the state of Σ at the moment (t, k) (i.e. $x = x(t, k)$, where $x(t, k)$ is given by (20)).

Definition 4 *A phase (t, k, x) of Σ is said to be reachable on $[t_0, t; k_0, k]$ if there exists a control $u(\cdot, \cdot)$ which transfers the phase $(t_0, k_0, 0)$ to (t, k, x) .*

A phase (t_0, k_0, x) is said to be controllable on $[t_0, t; k_0, k]$ if there exists a control $u(\cdot, \cdot)$ which transfers the phase (t_0, k_0, x) to $(t, k, 0)$.

If every phase is reachable (controllable) on $[t_0, t; k_0, k]$, the system Σ is said to be completely reachable (completely controllable) on $[t_0, t; k_0, k]$.

Definition 5 *For $(t_0, k_0) \leq (t, k)$ the matrix*

$$\mathcal{R}_\Sigma(t_0, t; k_0, k) = \int_{t_0}^t \sum_{l=k_0}^{k-1} \Phi(t, s; k)F(s; k, l+1)B(s, l)B(s, l)^T F(s; k, l+1)^T \Phi(t, s; k)^T ds, \tag{22}$$

is called the reachability Gramian of Σ .

We have proved in [11]:

Proposition 6 *The set of the states of the system Σ which are reachable on $[t_0, t; k_0, k]$ is the subspace $X_r = \text{Im}\mathcal{R}_\Sigma(t_0, t; k_0, k)$.*

It follows that Σ is completely reachable on $[t_0, t; k_0, k]$ iff $\text{Im}\mathcal{R}_\Sigma(t_0, t; k_0, k) = X_r = X = \mathbf{R}^n$, i.e. iff $\text{rank}\mathcal{R}_\Sigma(t_0, t; k_0, k) = \dim \text{Im}\mathcal{R}_\Sigma(t_0, t; k_0, k) = n$. One obtains

Theorem 1 *The system Σ is completely reachable on $[t_0, t; k_0, k]$ if and only if*

$$\text{rank}\mathcal{R}_\Sigma(t_0, t; k_0, k) = n.$$

5 Reachability of LTI 2D continuous-discrete systems

Let us consider an LTI system $\Sigma = (A_1, A_2, B) \in \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times m}$, i.e. a system with A_1, A_2 and B constant matrices. In this case we can consider the initial moment $(t_0, k_0) = (0, 0)$ and the time set $T = \mathbf{R}^+ \times \mathbf{Z}^+$. Then the state formula (20) and the input-output map (21) become

$$x(t, k) = e^{A_1 t} A_2^k x_0 + \int_0^t \sum_{l=0}^{k-1} e^{A_1(t-s)} A_2^{k-l-1} B u(s, l) ds, \tag{23}$$

$$y(t, k) = C e^{A_1 t} A_2^k x_0 + \int_0^t \sum_{l=0}^{k-1} C e^{A_1(t-s)} A_2^{k-l-1} B u(s, l) ds + D u(t, k). \tag{24}$$

Definition 6 *The system Σ is called completely reachable if for any state $x \in X$ there exists $(t, k) \in T$ such that the phase (t, k, x) is reachable on $[0, t; 0, k]$.*

We associate to Σ the *reachability matrix* $C_\Sigma = [B \ A_1 B \ \dots \ A_1^{n-1} B \ A_2 B \ A_1 A_2 B \ \dots \ A_1^{n-1} A_2 B \ \dots \ A_2^{n-1} B \ A_1 A_2^{n-1} B \ \dots \ A_1^{n-1} A_2^{n-1} B.]$
Theorem 1 gives (see [11], Theorem 4.2)

Theorem 2 $\Sigma = (A_1, A_2, B)$ *is completely reachable if and only if*

$$\text{rank}C_\Sigma = n. \tag{25}$$

We can prove by Proposition 6:

Proposition 7 *The set of all reachable states of Σ is $X_r = \text{Im}C_\Sigma$.*

It follows from Proposition 7 that a state $x \in X$ is reachable iff there exists $v \in \mathbf{R}^{nm}$ such that $x = C_\Sigma v$. Taking into account the definition of the controllability matrix C_Σ , this is equivalent to the equality

$x = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=1}^m \alpha_{ijk} A_1^i A_2^j b_k$, where $\alpha_{ijk} \in \mathbf{R}$ are the corresponding components of the vector v and $b_k, k \in \{1, 2, \dots, m\}$ are the columns of the matrix B . We denote by \mathcal{B} the subspace $\mathcal{B} = \text{Im}B$. Then $A_1^i A_2^j b_k \in A_1^i A_2^j \mathcal{B}, \forall i, j \geq 0, k \in \{1, 2, \dots, m\}$ and we get

Proposition 8 *The set of all reachable states of Σ is the subspace*

$$X_r = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=1}^m A_1^i A_2^j \mathcal{B}.$$

By Proposition 1 one obtains:

Proposition 9 *The set of all reachable states of Σ is the minimal subspace of X which is (A_1, A_2) -invariant and includes \mathcal{B} .*

An immediate consequence of Proposition 9 is the following

Theorem 3 *$\Sigma = (A_1, A_2, B)$ is completely reachable if and only if X is the minimal subspace which is (A_1, A_2) -invariant and includes \mathcal{B} .*

Definition 7 *Two systems $\Sigma = (A_1, A_2, B, C)$ and $\tilde{\Sigma} = (\tilde{A}_1, \tilde{A}_2, \tilde{B}, \tilde{C})$ are said to be isomorphic if there exists a nonsingular matrix $T \in \mathbf{R}^{n \times n}$ such that*

$$\tilde{A}_i = T^{-1} A_i T, \quad i = 1, 2; \quad \tilde{B} = T^{-1} B, \quad \tilde{C} = CT. \tag{26}$$

Theorem 4 *A system $\Sigma = (A_1, A_2, B)$ is not completely reachable if and only if Σ is isomorphic to a system $\tilde{\Sigma} = (\tilde{A}_1, \tilde{A}_2, \tilde{B})$ of the form*

$$\begin{aligned} \tilde{A}_1 &= \begin{bmatrix} A_{111} & A_{121} \\ 0 & A_{221} \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} A_{112} & A_{122} \\ 0 & A_{222} \end{bmatrix}, \\ \tilde{B} &= \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \end{aligned} \tag{27}$$

with $A_{111}, A_{112} \in \mathbf{R}^{q \times q}, B_1 \in \mathbf{R}^{q \times m}, q < n$. The triplet $\Sigma_1 = (A_{111}, A_{112}, B_1)$ is completely reachable.

Proof: We consider the direct sum decomposition of the state space $X = \mathbf{R}^n$ as $X = X_1 \oplus X_2$ where $X_r = X_1$. The partitions of the matrices in (27) are obtained with respect to this decomposition, since by Proposition 8 X_r is (A_1, A_2) -invariant and contains the columns of B ; q is the dimension of the subspace X_r .

We can derive other criteria of reachability.

Theorem 5 $\Sigma = (A_1, A_2, B)$ is completely reachable if and only if there is no common left eigenvector of matrices A_1 and A_2 , orthogonal on the columns of B .

Proof: Let us assume that there exists $v \in \mathbf{C}^n \setminus \{0\}$ such that $\exists \lambda, \mu \in \mathbf{C}$ with $v^T A_1 = \lambda v^T, v^T A_2 = \mu v^T$ and $v^T B = 0$. Then $v^T A_1^i A_2^j B = \lambda^i \mu^j v^T B = 0 \quad \forall i, j \geq 0$, hence $v^T C_\Sigma = 0$, i.e. $\Sigma = (A_1, A_2, B)$ is not completely reachable.

Conversely, if Σ is not completely reachable, then there exists $v \in \mathbf{C}^n \setminus \{0\}$ such that $v^T C_\Sigma = 0$, hence the subspace $S_1 = \{x \in \mathbf{C}^n | x^T C_\Sigma = 0\}$ contains a vector $v \neq 0$. If $x \in S_1$, then $x^T A_1^i A_2^j B = 0$ for any $i, j = \overline{0, n-1}$ and by Hamilton-Cayley Theorem this equality is true for any $i, j \geq 0$. Then, for any $x \in S_1$, $(A_1^T x)^T A_1^i A_2^j B = x^T A_1^{i+1} A_2^j B = 0, \forall i, j \geq 0$, hence $A_1^T x \in S_1$, i.e. S_1 is A_1^T -invariant; analogously, S_1 is A_2^T -invariant. It follows that S_1 contains an eigenvector x of A_1^T ; let λ be the corresponding eigenvalue. Let us consider the subspace $S_2 = \{x \in X | A_1^T x = \lambda x\}$. If $x \in S_2$ then $A_1^T (A_2^T x) = A_2^T A_1^T x = \lambda A_2^T x$, hence $A_2^T x \in S_2$, that is S_2 is A_2^T -invariant and so is $S_3 = S_1 \cap S_2$. Then S_3 contains an eigenvector w of A_2^T and since $S_3 \subset S_2$, w is an eigenvector of A_1^T too. Moreover, since $S_3 \subset S_1$, we have $w^T C_\Sigma = 0$ and particularly $w^T B = 0$, hence w is a common left eigenvector of A_1 and A_2 orthogonal on the columns of B .

The following theorem is an extension to 2Dcd systems of the Popov-Hautus-Belevitch criterion of reachability.

Theorem 6 $\Sigma = (A_1, A_2, B)$ is completely reachable if and only if for any $\lambda_1, \lambda_2 \in \mathbf{C}$

$$\text{rank} \begin{bmatrix} B & \lambda_1 I - A_1 & \lambda_2 I - A_2 \end{bmatrix} = n. \quad (28)$$

Proof: Obviously, the existence of $\lambda_1, \lambda_2 \in \mathbf{C}$ such that $\text{rank} \begin{bmatrix} B & \lambda_1 I - A_1 & \lambda_2 I - A_2 \end{bmatrix} < n$ is equivalent to the existence of $v \in \mathbf{R}^n \setminus \{0\}$ such that $v^T \begin{bmatrix} B & \lambda_1 I - A_1 & \lambda_2 I - A_2 \end{bmatrix} = 0$ which means $v^T B =$

$0, v^T A_1 = \lambda_1 v^T, v^T A_2 = \lambda_2 v^T$ that is, by Theorem 5, to the fact that Σ is not completely reachable.

Since $\text{rank}[\lambda I - A] = n \forall \lambda \in \mathbf{C} \setminus \sigma(A)$ for any $n \times n$ matrix A (where $\sigma(A)$ is the spectrum of A), one obtains by Theorem 6

Corollary 1 $\Sigma = (A_1, A_2, B)$ is completely reachable if and only if (28) holds $\forall \lambda_1 \in \sigma(A_1)$ and $\lambda_2 \in \sigma(A_2)$.

6 The determination of reachable states subspace

Let us consider an LTI system $\Sigma = (A_1, A_2, B) \in \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times m}$ with A_1, A_2 commutative matrices. We will adapt Algorithm 1 to determine the reachable states subspace X_r of the system Σ .

Algorithm 2

Stage 1. Determine the controllability matrix C_Σ .

Stage 2. Compute $\text{rank} C_\Sigma$. If $\text{rank} C_\Sigma = n$, then $X_r = \mathbf{R}^n$. STOP

Stage 3. Construct the sequence of subspaces $(S_{0,j})_{0 \leq j \leq n-1}$ of the space $X = \mathbf{R}^n$:

$$S_{0,0} = \text{Im}B; \quad (29)$$

$$S_{0,j} = \text{Im}B + A_2 S_{0,j-1}, j = 1, \dots, n. \quad (30)$$

Stage 4. Determine j_0 , the first index in $\{0, 1, \dots, n-1\}$ which verifies

$$S_{0,j_0+1} = S_{0,j_0}. \quad (31)$$

If $j_0 = n-1$, then $X_r = \min I(A_1, A_2; \text{Im}B) = \mathbf{R}^n$. STOP

If $j_0 < n-1$, GO TO Stage 5.

Stage 5. Construct the sequence of subspaces $(S_{i,j_0})_{0 \leq i \leq n}$ of the space $X = \mathbf{R}^n$:

$$S_{i,j_0} = S_{i-1,j_0} + A_1 S_{i-1,j_0}. \quad (32)$$

Stage 6. Determine i_0 , the first index in $\{0, 1, \dots, n-1\}$ which verifies

$$S_{i_0+1,j_0} = S_{i_0,j_0}. \quad (33)$$

Then $X_r = S_{i_0, j_0}$. STOP

Proof. By Proposition 9, $X_r = \min I(A_1, A_2; \mathcal{B})$ where $\mathcal{B} = \text{Im}B$.

If $\text{rank}C_\Sigma = n$, then Σ is completely reachable (by Theorem 2), hence $X_r = \mathbf{R}^n$. Otherwise, $\min I(A_1, A_2; \mathcal{B}) = S_{i_0, j_0}$, hence $X_r = S_{i_0, j_0}$.

The *Matlab* program presented below and based upon the algorithm above calculates the dimension and an orthonormal basis of the reachable states subspace for the bi-dimensional case.

The instructions make use of the m-functions *ima* and *sums* included in the Geometric Approach toolbox published by G. Marro and G. Basile at <http://www3.deis.unibo.it/Staff/FullProf/GiovanniMarro/geometric.htm>; this GA toolbox works with with Matlab 5, Matlab 6 and Matlab 7 and the Control System Toolbox.

More precisely, given the matrices A_1, A_2 that commute and the matrix B , the next commands will compute and display the dimension of a basis and an orthonormal one in the space $S = I(A_1, A_2; B)$.

```
% begin m-file
S = ima(B, 0); [n, dimInv] = size(S);
for j= 0:n-2      % first loop
    S = sums(S, A2*S); [n, m1] = size(S);
    if m1 == dimInv break;    else dimInv = m1;    end
end
for i= 0:n-2      % second loop
    S = sums(S, A1*S); [n, m1] = size(S);
    if dimInv == m1 break;    else dimInv = m1;    end
end
disp(['The reachable states subspace has the dimension ', ...
num2str(dimInv)])
disp('An orthonormal basis for the reachable states subspace
is:')
disp(S)
% end m-file
```

For example, given the matrices

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \\ 0 & 0 \end{bmatrix},$$

the above Matlab program gives the answers:

The reachable states subspace has the dimension 3

An orthonormal basis for the reachable states subspace is:

$$\begin{array}{ccc} 0.7071 & 0.2357 & - 0.6667 \\ 0 & 0.9428 & 0.3333 \\ 0.7071 & -0.2357 & 0.6667 \\ 0 & 0 & 0 \end{array} .$$

7 Conclusion

The minimal subspace which is invariant with respect to some commutative matrices and which includes a given subspace is determined by a suitable algorithm. This algorithm is applied to determine the subspace of the reachable states of a hybrid 2D system. The state space representation of these systems is studied and reachability criteria are obtained. Necessary and sufficient conditions of reachability are derived for LTI 2D systems as well as the characterization of the reachable states subspace.

These results and the proposed algorithms can be extended to n D systems with $n > 2$.

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