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LINEAR DIFFERENTIAL GAMES WITH VECTOR-VALUED CRITERIA*

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Abstract

This paper deals with a problem of linear differential games with several quadratic objective criteria (with vector-objective). In this case the notion of Pareto min-max is used as optimum point of the differential game. We mention that the notion of Pareto min-max was introduced for the first time in [5]. Existence conditions (Theorem 1), necessary conditions (Theorem 2) and sufficient conditions (Theorem 3) are given.

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§ 1. Notations and Definitions

Let $\mathcal{X}$ and $\mathcal{Y}$ be real Banach spaces, $\emptyset \neq \mathcal{U} \subset \mathcal{X}$, $\emptyset \neq \mathcal{V} \subset \mathcal{Y}$ and $J : \mathcal{U} \times \mathcal{V} \to \mathbb{R}^m$, $m > 1$.

Definition 1. *Let $\mathcal{U}$ and $\mathcal{V}$ be convex sets. The function J is called convex with respect to $u \in \mathcal{U}$ and concave with respect to $v \in \mathcal{V}$ if and only if $J(\cdot, v) : \mathcal{U} \to \mathbb{R}^m$ is a convex function, $\forall v \in \mathcal{V}$ and $J(u, \cdot) : \mathcal{V} \to \mathbb{R}^m$ is a concave function, $\forall u \in \mathcal{U}$ (see [4]).*

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Definition 2. *The function J is called (weakly) lower semicontinuous with respect to $u \in \mathcal{U}$ at the point $(u^\circ, v^\circ) \in \mathcal{U} \times \mathcal{V}$ if $J(\cdot, v^\circ) : \mathcal{U} \to \mathbb{R}^m$ is (weakly) lower semicontinuous at the point $u^\circ \in \mathcal{U}$. The function J is called (weakly) lower semicontinuous with respect to $u \in \mathcal{U}$ on $\mathcal{U} \times \mathcal{V}$ if $J(\cdot, v) : \mathcal{U} \to \mathbb{R}^m$ is (weakly) lower semicontinuous with respect to u, $\forall v \in \mathcal{V}$.*

The function J is called (weakly) upper semicontinuous with respect to $v \in \mathcal{V}$ at the point $(u^\circ, v^\circ) \in \mathcal{U} \times \mathcal{V}$ if $J(u^\circ, \cdot) : \mathcal{V} \to \mathbb{R}^m$ is (weakly) upper semicontinuous at the point $v^\circ \in \mathcal{V}$. The function J is called (weakly) upper semicontinuous with respect to $v \in \mathcal{V}$ on $\mathcal{U} \times \mathcal{V}$ if $J(u, \cdot) : \mathcal{V} \to \mathbb{R}^m$ is (weakly) upper semicontinuous with respect to v, $\forall u \in \mathcal{U}$.

Definition 3. *An element $(u^\circ, v^\circ) \in \mathcal{U} \times \mathcal{V}$ is called Pareto local min-max point for the function $J : \mathcal{U} \times \mathcal{V} \to \mathbb{R}^m$ if $\left[\exists U_0 \in \mathcal{V}(u^\circ) \text{ and } \exists V_0 \in \mathcal{V}(v^\circ)\right]$ with the property that $\nexists (u, v) \in \left(\mathcal{U} \cap U_0\right) \times \left(\mathcal{V} \cap V_0\right)$ such that*

$$J(u, v^\circ) \le J(u^\circ, v^\circ) \le J(u^\circ, v), \tag{i}$$

and

$$\begin{cases} \text{either} & \|J(u^\circ, v^\circ) - J(u, v^\circ)\|_2 > 0, \\ \text{or} & \|J(u^\circ, v^\circ) - J(u^\circ, v)\|_2 > 0. \end{cases} \tag{ii}$$

An element $(u^\circ, v^\circ) \in \mathcal{U} \times \mathcal{V}$ is called Pareto global min-max point for the function $J : \mathcal{U} \times \mathcal{V} \to \mathbb{R}^m$ if $\nexists (u, v) \in \mathcal{U} \times \mathcal{V}$ such that

$$J(u, v^\circ) \le J(u^\circ, v^\circ) \le J(u^\circ, v), \tag{i'}$$

and

$$\begin{cases} \text{either} & \|J(u^\circ, v^\circ) - J(u, v^\circ)\|_2 > 0, \\ \text{or} & \|J(u^\circ, v^\circ) - J(u^\circ, v)\|_2 > 0. \end{cases} \tag{ii'}$$

§ 2. Differential Games with Vector-valued Criterion

We consider the following problem of a linear differential game with several quadratic criteria:

$\Omega_1 \subset \mathbb{R}^p$, $\Omega_2 \subset \mathbb{R}^r$, $p \ge 1 \le r$ convex and compact sets,
$\mathcal{U} := \{u(\cdot) |\, u(\cdot) \in L_2([0,T]; \mathbb{R}^p),\ u(t) \in \Omega_1,\, t \in [0,T]\}$,
$\mathcal{V} := \{v(\cdot) |\, v(\cdot) \in L_2([0,T]; \mathbb{R}^r),\ v(t) \in \Omega_2,\, t \in [0,T]\}$,
the system of linear differential equations

$$\begin{cases} \dot{x}^*(t) = A \cdot x^*(t) + B_1 \cdot u^*(t) + B_2 \cdot v^*(t), \\ x^*(0) = x_0^*, \end{cases} \tag{1}$$

where $A \in \mathcal{M}_{n\times n}(\mathbb{R})$, $B_1 \in \mathcal{M}_{n\times p}(\mathbb{R})$, $B_2 \in \mathcal{M}_{n\times r}(\mathbb{R})$ are constant matrix,

$$J = (J_1, ..., J_m) : \mathcal{U} \times \mathcal{V} \to \mathbb{R}^m$$

$$J_k\big(u(\cdot), v(\cdot)\big) := x_{uv}(T) \cdot \mathbf{P}_k \cdot x_{uv}^*(T) + \int_0^T \big[u(t) \cdot \mathbf{Q}_k \cdot u^*(t) + v(t) \cdot \mathbf{R}_k \cdot v^*(t)\big] dt, \tag{2}$$

where

$\mathbf{P}_k \in \mathcal{M}_{n\times n}(\mathbb{R})$ is a constant, symmetrical and positive semi-definite matrix,

$\mathbf{Q}_k \in \mathcal{M}_{p\times p}(\mathbb{R})$ is a constant, symmetrical and positive definite matrix,

$\mathbf{R}_k \in \mathcal{M}_{r\times r}(\mathbb{R})$ is a constant, symmetrical and negative definite matrix, $k \in \{1, ..., m\}$,

$x_{uv}(T) \in \mathbb{R}^n$ is the point where the system (1) trajectory reaches, according to the pair (u, v) at the final moment T,

$x_{uv}^*(T) \in \mathcal{M}_{1\times n}\big(\mathbb{R}\big)$ is the transpose of the vector $x_{uv}(T) \in \mathbb{R}^n$.

$\big($**P.O.**$\big)$ **The problem of optimum**:

$$\min\text{-}\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} J\big(u(\cdot), v(\cdot)\big).$$

From system (1) we deduce

$$x_{uv}^*(T) = e^{AT}\Big[x_0^* + \int_0^T e^{-As}\big(B_1 \cdot u^*(s) + B_2 \cdot v^*(s)\big) ds\Big] \tag{3}$$

and hence

$$J_k(u, v) = x_0 \cdot \widetilde{\mathbf{P}}_k \cdot x_0^* + 2x_0\widetilde{\mathbf{P}}_k \int_0^T e^{-As}\big(B_1 \cdot u^*(s) + B_2 \cdot v^*(s)\big) ds +$$

$$+ \int_0^T\int_0^T \big(u(\tau) \cdot B_1^* + v(\tau) \cdot B_2^*\big) \cdot \mathbf{H_k}(\tau, s) \cdot \big(B_1 \cdot u^*(s) + B_2 \cdot v^*(s)\big) d\tau ds +$$

$$+ \int_0^T \big(u(s) \cdot \mathbf{Q}_k \cdot u^*(s) + v(s) \cdot \mathbf{R}_k \cdot v^*(s)\big) ds, \tag{4}$$

where

$$\widetilde{\mathbf{P}}_k = e^{A^*T}\mathbf{P}_k e^{AT} \text{ and } \mathbf{H}_k(\tau,s) = e^{A^*(T-s)}\mathbf{P}_k e^{A(T-\tau)}, \ \ k \in \{1,...,m\}. \quad (5)$$

Lemma 1. *Let $C \in \mathcal{M}_{n\times n}(\mathbb{R})$ be a constant, symmetrical and positive semi-definite matrix. The quadratic form $\varphi : L_2\big([0,T],\mathbb{R}^n\big) \to \mathbb{R}$,*

$$\varphi(y) = \int_0^T\int_0^T y(\tau)\cdot C\cdot y^*(s)d\tau ds,$$

is positive semi-definite.

Proof. Since $C \geq 0$, $\exists C_1 \geq 0,\ C_1^* = C_1$ such that $C = C_1^2$. Hence

$$\varphi(y) = \int_0^T\int_0^T \big(y(\tau)\cdot C_1^*\big)\cdot\big(C_1\cdot y^*(s)\big)d\tau ds = \Big(\int_0^T y(\tau)\cdot C_1^* d\tau\Big)\Big(\int_0^T C_1\cdot y^*(s)ds\Big) \geq 0.$$

■

Remark 1. *The sets $\mathcal{U}$ and $\mathcal{V}$ are weak-sequentially compact* (*see* [12], *Lemma* 1*A*) *and convex.*

Remark 2. *From the above hypotheses it results that the function $J : \mathcal{U}\times\mathcal{V}\to\mathbb{R}^m$ is convex with respect to $u\in\mathcal{U}$* (*see* [7], *Lemma* 2).

Since J is continuous, it results that J is weakly lower semicontinuous with respect to $u\in\mathcal{U}$ (*see* [16], *th.* 8.2).

Assumption 1. The matrices A, B_2, $\mathbf{P}_k$ and $\mathbf{R}_k$ satisfy the condition

$$\int_0^T\Big[\int_0^T v(\tau)\cdot B_2^*\cdot\mathbf{H}_k(\tau,s)\cdot B_2 d\tau + v(s)\cdot\mathbf{R}_k\Big]\cdot v^*(s)ds \leq 0, \quad (6)$$

$\forall v(\cdot)\in L_2\big([0,T],\mathbb{R}^r\big)$, $\forall k\in\{1,...,m\}$.

Remark 3. *If $\mathbf{P}_k = 0$, $k\in\{1,...,m\}$ then Assumption 1 is true.*

Proposition 1. *If Assumption 1 is fulfilled, then the function $J : \mathcal{U}\times\mathcal{V}\to\mathbb{R}^m$ is concave with respect to $v(\cdot)\in\mathcal{V}$. In addition is weakly upper semicontinuous with respect to $v(\cdot)\in\mathcal{V}$.*

⌈The proof follows from the fact that for $\psi_k : [0,1] \to \mathbb{R}$, it should be $\psi_k(t) := \quad = J_k\Big(u(\cdot), t\cdot v'(\cdot) + (1-t)\cdot v''(\cdot)\Big)$, we get

$$\psi_k''(t) = \int_0^T\Big[\int_0^T \big(v'(\tau) - v''(\tau)\big)B_2^* \cdot \mathbf{H}_k(\tau,s)\cdot B_2 d\tau +$$

$$+\big(v'(s) - v''(s)\mathbf{R}_k\Big]\cdot \big(v'(s) - v''(s)\big)^* ds \le 0,$$

$\forall v'(\cdot), v''(\cdot) \in \mathcal{V}$ and $\forall k \in \{1,...,m\}$.⌋

Theorem 1. *If Assumption 1 is fulfilled for the problem* $\big(P.O.\big)$, *then there* $\exists\big(u^\circ, v^\circ\big) \in \mathcal{U}\times\mathcal{V}$ *Pareto min-max point for the function* J *on* $\mathcal{U}\times\mathcal{V}$.

Proof. Function $F_\lambda : \mathcal{U}\times\mathcal{V} \to \mathbb{R}$,

$$F_\lambda(u,v) := \big\langle \lambda, J(u,v)\big\rangle = \sum_{k=1}^m \lambda_k \cdot J_k(u,v), \tag{7}$$

where $\lambda \in \overset{\circ}{K}{}^m = int(\mathbb{R}_+^n)$, is convex and weakly lower semicontinuous with respect to u and concave and weakly upper semicontinuous with respect to v. Then there $\exists\big(u^\circ, v^\circ\big) \in \mathcal{U}\times\mathcal{V}$ a saddle point for F_λ on $\mathcal{U}\times\mathcal{V}$ $\big($see [7], Lemmas 4 and 6$\big)$. Hence $\big(u^\circ, v^\circ\big)$ is a Pareto min-max point for J on $\mathcal{U}\times\mathcal{V}$ $\big($see [5], [6]$\big)$. ∎

Theorem 2. *The necessary condition so that* $\big(u^\circ, v^\circ\big) \in \mathcal{U}\times\mathcal{V}$ *should be a Pareto min-max point for* J *on* $\mathcal{U}\times\mathcal{V}$ *is that*

$$\begin{gathered} W := \Big\{(h_1,h_2) \in L_2\big([0,T],\mathbb{R}^p\big)\times L_2\big([0,T],\mathbb{R}^r\big) \mid \exists(t_1 > 0 < t_2) \text{ such that} \\ \big(u^\circ + t_1 h_1, v^\circ\big) \in \mathcal{U}\times\mathcal{V},\ \big(u^\circ, v^\circ + t_2 h_2\big) \in \mathcal{U}\times\mathcal{V}, \\ \int_0^T\Big[x_0\widetilde{P}_k e^{-As}B_1 + u^\circ(s)\mathbf{Q}_k + \int_0^T\big(u^\circ(\tau)B_1^* + v^\circ(\tau)B_2^*\big)\mathbf{H}_k(\tau,s)B_1 d\tau\Big]h_1^*(s)ds < 0, \\ \int_0^T\Big[x_0\widetilde{P}_k e^{-As}B_2 + v^\circ(s)\mathbf{R}_k + \int_0^T\big(u^\circ(\tau)B_2^* + v^\circ(\tau)B_2^*\big)\mathbf{H}_k(\tau,s)B_2 d\tau\Big]h_2^*(s)ds > 0, \\ \forall k \in \{1,...,m\}\Big\} \end{gathered} \tag{8}$$

should be empty set $\big(W = \emptyset\big)$.

Proof. Let us assume, by reduction ad absurdum, that $W \neq \emptyset$. Let $(h_1, h_2) \in W$. There exist $t_1 > 0 < t_2$ such that $(u^\circ + t_1 h_1, v^\circ) \in \mathcal{U} \times \mathcal{V}$ and $(u^\circ, v^\circ + t_2 h_2) \in \mathcal{U} \times \mathcal{V}$. But $\mathcal{U}$ and $\mathcal{V}$ are convex, hence $\forall t \in]0, \min\{t_1, t_2\}]$ $\Rightarrow$

$$(u^\circ + th_1, v^\circ) \in \mathcal{U} \times \mathcal{V} \ \text{ and } \ (u^\circ, v^\circ + th_2) \in \mathcal{U} \times \mathcal{V}.$$

For $k \in \{1, ..., m\}$ and $t \in]0, \min\{t_1, t_2\}]$ we get

$$J_k\big(u^\circ + th_1, v^\circ\big) - J_k\big(u^\circ, v^\circ\big) = t \int_0^T \Big[2x_0 \widetilde{\mathbf{P}}_k e^{-As} B_1 + 2u^\circ(s)\mathbf{Q}_k +$$

$$+2\int_0^T \big(u^\circ(\tau)B_1^* + v^\circ(\tau)B_2^*\big) \cdot \mathbf{H}_k(\tau, s) \cdot B_1 d\tau\Big] h_1^*(s) ds +$$

$$+t^2 \int_0^T \Big(h_1(s)\mathbf{Q}_k + \int_0^T h_1(\tau)B_1^* \cdot \mathbf{H}_k(\tau, s) \cdot B_1 d\tau\Big) h_1^*(s) ds.$$

For a sufficiently small $t > 0$, it follows that

$$J_k\big(u^\circ + th_1, v^\circ\big) - J_k\big(u^\circ, v^\circ\big) < 0, \ \forall k \in \{1, ..., m\},$$

which contradicts definition 3 and the theorem is proved. ■

Theorem 3. *Consider that Assumption 1 is fulfilled. If for* $(u^\circ, v^\circ) \in \mathcal{U} \times \mathcal{V}$ *we get*

$$\begin{gathered}
W^* := \Big\{(h_1, h_2) \in L_2\big([0,T], \mathbb{R}^p\big) \times L_2\big([0,T], \mathbb{R}^r\big) \,|\, \exists (t_1 > 0 < t_2) \text{ such that} \\
\big(u^\circ + t_1 h_1, v^\circ\big) \in \mathcal{U} \times \mathcal{V}, \ \big(u^\circ, v^\circ + t_2 h_2\big) \in \mathcal{U} \times \mathcal{V}, \\
\int_0^T \Big[x_0 \widetilde{P}_k e^{-As} B_1 + u^\circ(s)\mathbf{Q}_k + \int_0^T \big(u^\circ(\tau)B_1^* + v^\circ(\tau)B_2^*\big)\mathbf{H}_k(\tau, s) B_1 d\tau\Big] h_1^*(s) ds \leq 0, \\
\int_0^T \Big[x_0 \widetilde{P}_k e^{-As} B_2 + v^\circ(s)\mathbf{R}_k + \int_0^T \big(u^\circ(\tau)B_2^* + v^\circ(\tau)B_2^*\big)\mathbf{H}_k(\tau, s) B_2 d\tau\Big] h_2^*(s) ds \geq 0, \\
\forall k \in \{1, ..., m\}\Big\} = \big\{(0,0)\big\},
\end{gathered} \tag{9}$$

then (u°, v°) *is a Pareto min-max point for* J *on* $\mathcal{U} \times \mathcal{V}$.

Proof. We suppose that (u°, v°) is not a Pareto min-max point for J on $\mathcal{U} \times \mathcal{V}$. Then there exists $(\overline{u}, \overline{v}) \in \mathcal{U} \times \mathcal{V}$ such that

$$J(\overline{u}, v^\circ) \leq J(u^\circ, v^\circ) \leq J(u^\circ, \overline{v}) \tag{10}$$

and there exist

either $i_0 \in \{1, ..., m\}$ for which

$$J_{i_0}(\overline{u}, v^\circ) < J_{i_0}(u^\circ, v^\circ) \leq J_{i_0}(u^\circ, \overline{v}), \tag{11}$$

or $k_0 \in \{1, ..., m\}$ for which

$$J_{k_0}(\overline{u}, v^\circ) \leq J_{k_0}(u^\circ, v^\circ) < J_{k_0}(u^\circ, \overline{v}) \tag{12}$$

$\big($hence $(u^\circ, v^\circ) \neq (\overline{u}, \overline{v})\big)$.

Since $\mathcal{U}$ and $\mathcal{V}$ are convex sets and J is convex with respect to u and concave with respect to v (Prop. 1), then for any $t \in]0, 1[$ we get

$$(\widehat{u}, \widehat{v}) = t(\overline{u}, \overline{v}) + (1 - t)(u^\circ, v^\circ) \in \mathcal{U} \times \mathcal{V}, \tag{13}$$

$$J(\widehat{u}, v^\circ) \leq J(u^\circ, v^\circ) \leq J(u^\circ, \widehat{v}),$$

and

$$\begin{cases} \text{either} & J_{i_0}(\widehat{u}, v^\circ) < J_{i_0}(u^\circ, v^\circ) \leq J_{i_0}(u^\circ, \widehat{v}) \\ \text{or} & J_{k_0}(\widehat{u}, v^\circ) \leq J_{k_0}(u^\circ, v^\circ) < J_{k_0}(u^\circ, \widehat{v}). \end{cases} \tag{14}$$

Let $(\overline{h}_1, \overline{h}_2) := (\overline{u}, \overline{v}) - (u^\circ, v^\circ)$.

For $t \in]0, 1[$, from relation (13) we deduce

$$\begin{gathered} J_k(u^\circ + t\overline{h}_1, v^\circ) - J_k(u^\circ, v^\circ) = 2t \int_0^T \Big[x_0 \widetilde{\mathbf{P}}_k e^{-As} B_1 + u^\circ(s) \mathbf{Q}_k + \\ + \int_0^T \big(u^\circ(\tau) B_1^* + v^\circ(\tau) B_2^* \big) \cdot \mathbf{H}_k(\tau, s) \cdot B_1 d\tau \Big] \overline{h}_1^*(s) ds + \\ + t^2 \int_0^T \Big[\overline{h}_1(s) \mathbf{Q}_k + \int_0^T \overline{h}_1(\tau) B_1^* \cdot \mathbf{H}_k(\tau, s) \cdot B_1 d\tau \Big] \overline{h}_1^*(s) ds \leq 0 \end{gathered} \tag{15}$$

and

$$J_k(u^\circ, v^\circ + t\overline{h}_2) - J_k(u^\circ, v^\circ) = 2t \int_0^T \Big[x_0 \widetilde{\mathbf{P}}_k e^{-As} B_2 + u^\circ(s)\mathbf{R}_k +$$
$$+ \int_0^T \big(u^\circ(\tau)B_1^* + v^\circ(\tau)B_2^*\big) \cdot \mathbf{H}_k(\tau, s) \cdot B_2 d\tau\Big]\overline{h}_2^*(s)ds + \tag{16}$$
$$+t^2 \int_0^T \Big[\overline{h}_2(s)\mathbf{R}_k + \int_0^T \overline{h}_2(\tau)B_2^* \cdot \mathbf{H}_k(\tau, s) \cdot B_2 d\tau\Big]\overline{h}_2^*(s)ds \geq 0$$
$$k \in \{1, ..., m\}.$$

Taking into account Assumption 1 and the relations (15) and (16) it results that

$$\int_0^T\Big[x_0\widetilde{\mathbf{P}}_k e^{-As}B_1 + u^\circ(s)\mathbf{Q}_k + \int_0^T\big(u^\circ(\tau)B_1^* + v^\circ(\tau)B_2^*\big)\cdot\mathbf{H}_k(\tau, s)\cdot B_1 d\tau\Big]\overline{h}_1^*(s)ds \leq 0, \tag{17}$$

and

$$\int_0^T\Big[x_0\widetilde{\mathbf{P}}_k e^{-As}B_2 + u^\circ(s)\mathbf{R}_k + \int_0^T\big(u^\circ(\tau)B_1^* + v^\circ(\tau)B_2^*\big)\cdot\mathbf{H}_k(\tau, s)\cdot B_2 d\tau\Big]\overline{h}_2^*(s)ds \geq 0, \tag{18}$$

$\forall k \in \{1, ..., m\}$, that is $(\overline{h}_1, \overline{h}_2) \in W^* \setminus \{(0, 0)\}$ which contradicts the hypothesis and the theorem is proved. ■

Example. We consider

$\Omega_1 = [0, 1], \qquad \Omega_2 = [0, 1], \qquad T = 1,$

$\mathcal{U} = \big\{u(\cdot) \,|\, u(\cdot) \in L_2\big([0, 1]; \mathbb{R}\big)\big\}, \quad u(t) \in [0, 1],\ t \in [0, 1],$

$\mathcal{V} = \big\{v(\cdot) \,|\, v(\cdot) \in L_2\big([0, 1]; \mathbb{R}\big)\big\}, \quad v(t) \in [0, 1],\ t \in [0, 1],$

and the motion equation

$$\begin{cases} \dot{x}(t) + x(t) = u(t) - v(t),\ t \in [0, 1], \\ x(0) = 1. \end{cases} \tag{$*$}$$

Let $J = \big(J_1, J_2\big) : \mathcal{U} \times \mathcal{V} \to \mathbb{R}^2$, where

$$J_1\big(u(\cdot), v(\cdot)\big) = x^2(1) + \int_0^1 \big[2u^2(t) - v^2(t)\big]\,dt,$$

$$J_2\big(u(\cdot), v(\cdot)\big) = 2x^2(1) + \int_0^1 \big[u^2(t) - 2v^2(t)\big]\,dt.$$

The solution of system $(*)$ is

$$x(t) = e^{-t}\Big[1 + \int_0^t \big(u(s) - v(s)\big)e^s ds\Big]$$

and

$$x(1) = e^{-1}\Big[1 + \int_0^1 \big(u(t) - v(t)\big)e^t dt\Big]$$

For finding a solution of the problem, we attach the functional

$$\widetilde{J}\big((u\cdot), v(\cdot)\big) = \frac{1}{3}\big[J_1\big((u\cdot), v(\cdot)\big) + J_2\big((u\cdot), v(\cdot)\big)\big] = x^2(1) + \int_0^1 \big(u^2(t) - v^2(t)\big)\,dt.$$

We get

$$\widetilde{J}\big((u\cdot), v(\cdot)\big) = \int_0^1 \big[u^2(t) - v^2(t)\big]\,dt + e^{-2}\Big[1 + \int_0^1 \big(u(t) - v(t)\big)e^t dt\Big]^2.$$

The functional $\widetilde{J}\big((u\cdot), v(\cdot)\big)$ is convex with respect to $u(\cdot) \in \mathcal{U}$ and one obtains

$$\varphi(v(\cdot)) = \min_{u(\cdot)\in\mathcal{U}} \widetilde{J}\big(u(\cdot), v(\cdot)\big) = \widetilde{J}\big(0, v(\cdot)\big) = -\int_0^1 v^2(t)\,dt + e^{-2}\Big[1 - \int_0^1 v(t)e^t dt\Big]^2.$$

In order to show that the functional φ is concave we define

$$\psi(t) := \varphi\big(tv_1(\cdot) + (1-t)v_2(\cdot)\big) =$$

$$= -\int_0^1 \big(tv_1(s) + (1-t)v_2(s)\big)^2 ds + e^{-2}\Big[1 - \int_0^1 \big(tv_1(s) + (1-t)v_2(s)\big)e^s ds\Big]^2,$$

from which, we deduce

$$\psi''(t) = -2\int_0^1 \big(v_1(s) - v_2(s)\big)^2 ds + \Big[\int_0^1 \big(v_1(s) - v_2(s)\big)e^{s-1}ds\Big]^2 ds.$$

Because

$$\Big[\int_0^1 \big(v_1(s) - v_2(s)\big)e^{s-1}ds\Big]^2 ds \le \int_0^1 \big(v_1(s) - v_2(s)\big)^2 ds \cdot \int_0^1 e^{2(s-1)}ds,$$

it follows

$$\psi''(t) \le 0,$$

, hence $\varphi(\cdot)$ is concave. We get

$$\max_{v(\cdot)\in\mathcal{V}} \min_{u(\cdot)\in\mathcal{U}} \widetilde{J}\big(u(\cdot), v(\cdot)\big) = \max_{v(\cdot)\in\mathcal{V}} \varphi\big(u(\cdot)\big) = \varphi(0) = \widetilde{J}\big(0,0\big) = e^{-2}.$$

On the other hand, because the functional $\widetilde{J}\big(u(\cdot), v(\cdot)\big)$ is convex with respect to $u(\cdot) \in \mathcal{U}$, we get

$$\varphi_1\big(u(\cdot)\big) := \max_{v(\cdot)\in\mathcal{V}} \widetilde{J}\big(u(\cdot), v(\cdot)\big) = \widetilde{J}\big(u(\cdot), 0\big) = \int_0^1 u^2(s)ds + e^{-2}\Big[1 + \int_0^1 u(s)e^s ds\Big]^2,$$

which is a convex functional.

Now

$$\min_{u(\cdot)\in\mathcal{U}} \max_{v(\cdot)\in\mathcal{V}} \widetilde{J}\big(u(\cdot), v(\cdot)\big) = \min_{u(\cdot)\in\mathcal{U}} \varphi_1\big(u(\cdot)\big) = \varphi_1(0) = \widetilde{J}\big(0,0\big) = e^{-2},$$

hence

$$\min_{u(\cdot)\in\mathcal{U}} \max_{v(\cdot)\in\mathcal{V}} \widetilde{J}\big(u(\cdot), v(\cdot)\big) = e^{-2} = \max_{v(\cdot)\in\mathcal{V}} \min_{u(\cdot)\in\mathcal{U}} \widetilde{J}\big(u(\cdot), v(\cdot)\big) = \widetilde{J}\big(0,0\big).$$

Therefore, a solution of the problem is $\big(u^\circ(\cdot), v^\circ(\cdot)\big) = \big(0,0\big)$.

Remark 4. *(i) The functionals $J_1\big(u(\cdot), v(\cdot)\big)$ and $J_2\big(u(\cdot), v(\cdot)\big)$ are convex with respect to $u(\cdot) \in \mathcal{U}$ and concave with respect to $v(\cdot) \in \mathcal{V}$. As above, we deduce*

$$\min_{u(\cdot)\in\mathcal{U}} \max_{v(\cdot)\in\mathcal{V}} J_1\big(u(\cdot), v(\cdot)\big) = e^{-2} = \max_{v(\cdot)\in\mathcal{V}} \min_{u(\cdot)\in\mathcal{U}} J_1\big(u(\cdot), v(\cdot)\big) = J_1(0,0),$$

and

$$\min_{u(\cdot)\in\mathcal{U}}\max_{v(\cdot)\in\mathcal{V}} J_2\big(u(\cdot),v(\cdot)\big) = 2e^{-2} = \max_{v(\cdot)\in\mathcal{V}}\min_{u(\cdot)\in\mathcal{U}} J_2\big(u(\cdot),v(\cdot)\big) = J_2(0,0),$$

hence

$$\min_{u(\cdot)\in\mathcal{U}}\max_{v(\cdot)\in\mathcal{V}} J\big(u(\cdot),v(\cdot)\big) = \big(e^{-2},2e^{-2}\big) = \max_{v(\cdot)\in\mathcal{V}}\min_{u(\cdot)\in\mathcal{U}} J\big(u(\cdot),v(\cdot)\big) = J(0,0).$$

In this case, we obtain the same solution $\big(u^\circ(\cdot),v^\circ(\cdot)\big) = \big(0,0\big)$.

(ii) *Let* $\alpha \in]0,1[$. *Consider the functional*

$$J_\alpha\big(u(\cdot),v(\cdot)\big) = \alpha J_1\big(u(\cdot),v(\cdot)\big) + (1-\alpha)J_2\big(u(\cdot),v(\cdot)\big) =$$

$$= \big[\alpha+2(1-\alpha)\big]x^2(1) + \int_0^1 \Big\{\big[2\alpha+(1-\alpha)\big]u^2(t) - \big[\alpha+2(1-\alpha)\big]v^2(t)\Big\}dt =$$

$$= \big(2-\alpha\big)\Big\{e^{-1}\Big[1+\int_0^1 \big(u(t)-v(t)\big)e^t dt\Big]\Big\}^2 + \int_0^1 \big[\big(1+\alpha\big)u^2(t) - \big(2-\alpha\big)v^2(t)\big]dt.$$

The functional $J_\alpha\big(u(\cdot),v(\cdot)\big)$ *is convex with respect to* $u(\cdot)\in\mathcal{U}$ *and concave with respect to* $v(\cdot)\in\mathcal{V}$ *and we deduce*

$$\min_{u(\cdot)\in\mathcal{U}}\max_{v(\cdot)\in\mathcal{V}} J_\alpha\big(u(\cdot),v(\cdot)\big) = \max_{v(\cdot)\in\mathcal{V}}\min_{u(\cdot)\in\mathcal{U}} J_\alpha\big(u(\cdot),v(\cdot)\big) = J_\alpha(0,0).$$

We get $\big(u^\circ(\cdot),v^\circ(\cdot)\big) = \big(0,0\big)$.

REFERENCES

1. Balakrishnan A. V., *Introduction to Optimization Theory in a Hilbert Space*, Springer Verlag, Berlin, 1971.

2. Barbu V., Precupanu Th., *Convexitate şi optimizare în spaţii Banach*, Editura Academiei RSR, Bucureşti, 1975.

3. Demianov V.F., Rubinov A.M., P *Priblijennâe metodâ reşenia extremalinâh zadaci*, Izdatelistvo Leningradscovo Universiteta, 1968.

4. Drăguşin C., *On Pareto optimization* (in Romanian), Studii şi Cercetări Matematice, 10, tom. 26, 1974, Ed. Academiei RSR.

5. Drăguşin C., *Minmax Pareto* (in Romanian), Inst. Politehnic Bucureşti, Seminarul de Matematici, 1975.

6. Drăguşin C., *Min-max pour des critéres multiple*, R.A.I.R.O., Recherche opérationelle, Vol. 12, nr. 2, 1978.

7. Drăguşin C., *Optimization Problems with Several Objective Functions* (in Romanian), Doctoral Thesis, Universitatea Bucureşti, 1978.

8. Drăguşin C., *On differential linear games with quadratic performance criteria*, (in Romanian) Cercetări operaţionale - curs ţinut la masteratul Logistica Transporturilor.

9. Duffin R.J., Peterson E.L., Zener C., *Geometric Programming Theory and Application*, John Wiley & Sons, Inc. New York-London-Sydney, 1967

10. Dunford N., Schwartz J., *Linear Operators. General Theory*, Interscience, New York, London, 1958.

11. Golstein E. G., *Convex Programming. Elementary Theory* (in Russian), Moscow, 1970.

12. Isaacs R., *Differential Games*, John Wiley & Sons, Inc., New York-London-Sydney, 1965.

13. Krassovski N., Soubbotine A., *Jeux Différentiels*, Éditions Mir, Moscou, 1977.

14. Lee E.B., Markus L., *Foundations of Optimal Control Theory*, New York-London-Sydney, John Wiley & Sons, Inc., 1967.

15. Mehlmann A., *Applied Differential Games*, Plenum Press - New York and London, 1988.

16. Pascali D., *Operatori neliniari*, Editura Academiei RSR, Bucureşti, 1974.

17. Penescu C., Ionescu V., Rosinger E., *Optimal Processes* (in Romanian), Editura Academiei RSR, Bucureşti, 1970

18. Rockafellar R.T., *Convex Analysis*, Princeton, New Jersey, Princeton University Press, 1970.

19. Vainberg M.M., *Variational Method and Method Monotonous Operators in the Theory of Nonlinear Equations* (in Russian), Moscow, 1972.