Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066 - 6594 Vol. 5, No. 1-2 / 2013

A DUALITY ALGORITHM FOR THE OBSTACLE PROBLEM*

Diana Merluşcă†‡

Abstract

We consider the obstacle problem in Sobolev spaces, of order strictly greater then the dimension of the domain. The aim is to propose an algorithm to find the solution of the obstacle problem, based on the solution of the dual approximating problem, which is, in fact, a finite dimensional quadratic minimization problem.
MSC: 65K10, 65K15, 90C59, 49N15.

keywords: obstacle problem, dual problem

1 Introduction

The obstacle problem has been studied by many authors due to its applicability in many fields, such as the study of fluid filtration in porous media, constrained heating, elasto-plasticity, optimal control, and financial mathematics (C. Baiocchi. [3] and G. Duvaut, J.-L. Lions [6]).

We find the obstacle problem in recent works as well, for example in M. Burger, N. Matevosyan, M.T Wolfram, [5], in which an obstacle problem is

*Accepted for publication in revised form on January 15, 2013

†dianam1985@yahoo.com Institute of Mathematics of the Romanian Academy, Bucharest, Romania.

‡This paper is supported by the Sectorial Operational Programme Human Resources Development (SOP HRD), financed from the European Social Fund and by the Romanian Government under contract number SOP HRD/107/1.5/S/82514

formulated as a shape optimization problem. Other references are R. Griesse, K. Kunisch, [7], C. M. Murea, D. Tiba [9]. Moreover, certain authors test their algorithms by applying them to the obstacle problem, for instance the work of L. Badea, [2], in which the one- and two-level domain decomposition methods are tested on a two obstacle problem.

In his book, R. Glowinski, [8], analyzes the obstacle problem on $H_0^1(\Omega)$. He treats this problem from the numerical point of view, by finite element methods, and gives some theoretical results of the existence and uniqueness of the solution, subject to the properties of the obstacle and the input data.

In their book, V. Barbu and Th. Precupanu, [4], studied the obstacle problem in $H_0^1(\Omega)$ from the duality point of view. They apply the Fenchel duality theorem for the following problem

$$\min\left\{\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega fu : \ u \in K\right\} \tag{1}$$

where $f \in L^2(\Omega)$ and $K = \{u \in H_0^1(\Omega) : u \geq 0 \text{ a.e. on } \Omega\}$. They end up formulating the dual problem associated to (1) as follows

$$\max\left\{-\frac{1}{2}\|p^* + h\|^2_{H^{-1}(\Omega)} : p^* \in H^{-1}(\Omega), p^* \geq 0\right\}$$

Interpreting this problem, using Theorem 2.4, page 188, [4], they restate the (1) as boundary value problem of unilateral type.

Keeping in mind this argument, we have started our study considering an approximating problem for the obstacle problem in $W^{1,p}(\Omega)$ for $p > dim\,\Omega$. Using the dual of the approximating problem we came upon a finite dimensional problem which is, in fact, a quadratic minimization problem, and thus, its solution can be computed much easier then the solution of an obstacle problem. Thus using the duality mapping we can construct the solution of the obstacle problem solving only a finite dimensional quadratic minimization problem.

The algorithm presented here was successfully tested from the numeric point of view.

2 Statement of the direct and approximating problem

We consider the following obstacle problem

$$\min_{y \in W_0^{1,p}(\Omega)_+} \left\{ \frac{1}{2}\|y\|^2_{W_0^{1,p}(\Omega)} - \int_\Omega fy \right\} \tag{2}$$

where $f \in L^1(\Omega)$, $p > d = dim\ \Omega$, and $W_0^{1,p}(\Omega)_+ = \{y \in W_0^{1,p}(\Omega) : y \geq 0\}$. We consider that Ω is a bounded open set with a strong local Lipschitz property.

It can be easily proved that (2) has a unique solution $\bar{y} \in W^{1,p}(\Omega)$, by using the compact imbedding $W^{1,p}(\Omega) \to L^\infty(\Omega)$, which follows from the Rellich-Kondrachov Theorem (R. Adams [1], Theorem 6.2, Part II, page 144).

Also, knowing that, by Sobolev Imbedding Theorem, we have $W^{1,p}(\Omega) \to C(\overline{\Omega})$, it makes sense to consider the following problem

$$\min \left\{ \frac{1}{2}\|y\|^2_{W_0^{1,p}(\Omega)} - \int_\Omega fy \ \ : \ y \in W_0^{1,p}(\Omega); y(x_i) \geq 0, i = 1, 2, \ldots, k \right\} \tag{3}$$

where $\{x_i\}_{i\in\mathbb{N}} \subseteq \Omega$ is a dense set in $\overline{\Omega}$. For each $k \in \mathbb{N}$, we denote

$$C_k = \{y \in W_0^{1,p}(\Omega) : y(x_i) \geq 0, i = 1, 2, \ldots, k\}$$

the closed convex cone.

We can prove that (3) has also an unique solution $\bar{y}_k \in C_k$ by using the same argument as in the proof of the existence and uniqueness for the solution of problem (2).

Moreover, we can prove the following result

Theorem 1 *The sequence $\{\bar{y}_k\}_k$ constructed from the solutions of problems* (3)*, for $k \in \mathbb{N}$, is a strongly convergent sequence in $W^{1,p}(\Omega)$ to the unique solution $\bar{y}$ of the problem* (2)*.*

As a consequence of Proposition 1, we can state that problem (3) is an approximating problem for (2).

In the following section we shall use the dual of problem (3) to solve problem (2).

3 The dual problem and the analysis of its solution

We will use Fenchel duality Theorem to state the dual problems associated to problems (2) and (3). For this purpose we consider the functional

$$F(y) = \frac{1}{2}\|y\|^2_{W_0^{1,p}(\Omega)} - \int_\Omega fy, \quad y \in W_0^{1,p}(\Omega)$$

Using the definition of the convex conjugate and the fact that the duality mapping $J : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$ is single-valued and bijective operator, we get that the convex conjugate of F is

$$F^*(y^*) = \frac{1}{2}\|f + y^*\|^2_{W^{-1,q}(\Omega)}$$

Considering now the functional $g = -I_{W_0^{1,p}(\Omega)_+}$ and using the concave conjugate definition we get that

$$g^\bullet(y^*) = \begin{cases} 0, & y^* \in (W_0^{1,p}(\Omega)_+)^* \\ -\infty, & y^* \notin (W_0^{1,p}(\Omega)_+)^* \end{cases}$$

with $(W_0^{1,p}(\Omega)_+)^* = \{y^* \in W^{-1,q}(\Omega) : (y, y^*) \geq 0, \forall y \in W_0^{1,p}(\Omega)_+\} = W^{-1,q}(\Omega)_+$.

Since F şi $-g$ are convex and proper functionals on $W^{1,p}(\Omega)$, the domain of g is $D(g) = W_0^{1,p}(\Omega)_+$, and F is continous everywhere on $W_0^{1,p}(\Omega)_+$ we are able to apply Fenchel duality Theorem (V. Barbu, Th. Precupanu, [4], Theorem 2.5, page 189) and obtain

$$\min\left\{\frac{1}{2}\|y\|^2_{W_0^{1,p}(\Omega)} - \int_\Omega fy : y \in W_0^{1,p}(\Omega)_+\right\}$$
$$= \max\left\{-\frac{1}{2}\|f + y^*\|^2_{W^{-1,q}(\Omega)} : y^* \in W^{-1,q}(\Omega)_+\right\}$$

So the dual problem associated to problem (2) is

$$\max\left\{-\frac{1}{2}\|f + y^*\|^2_{W^{-1,q}(\Omega)} : y^* \in W^{-1,q}(\Omega)_+\right\}$$

For the approximating problem (3) we only need the concave conjugated of $g_k = -I_{C_k}$ due to the fact that we minimize the same functional F over another cone. Thus, the concave conjugate is

$$g_k^\bullet(y^*) = \inf\{(y, y^*) - g(y) : y \in C_k\} = \begin{cases} 0, & y^* \in C_k^* \\ -\infty, & y^* \notin C_k^* \end{cases}$$

where $C_k^* = \{y^* \in W^{-1,q}(\Omega) : (y^*, y) \geq 0, \forall y \in C_k\}$.

Lemma 1 *The polar cone of* C_k *is*

$$C_k^* = \left\{ u = \sum_{i=1}^{k} \alpha_i \delta_{x_i} : \alpha_i \geq 0 \right\}$$

where δ_{x_i} *are the Dirac distributions concentrated in* $x_i \in \Omega$, *i.e.* $\delta_{x_i}(y) = y(x_i)$, $y \in W_0^{1,p}(\Omega)$.

Since the domain of g_k is $D(g_k) = C_k$ and the functional F is still continous on the closed convex cone C_k the hypothesis of Fenchel duality Theorem are satisfied once again. This implies that

$$\begin{aligned} &\min \left\{ \frac{1}{2} \|y\|^2_{W_0^{1,p}(\Omega)} - \int_\Omega fy \quad : \; y \in C_k \right\} \\ &= \max \left\{ -\frac{1}{2} \|y^* + f\|^2_{W^{-1,q}(\Omega)} : y^* \in C_k^* \right\} \end{aligned}$$

So we obtain the dual approximating problem associated to problem (3)

$$\max \left\{ -\frac{1}{2} \|y^* + f\|^2_{W^{-1,q}(\Omega)} : y^* \in C_k^* \right\} \tag{4}$$

Denoting y_k and y_k^* as the solution of problems (3) and its dual (4), we apply Theorem 2.4 (page 188, V. Barbu, Th. Precupanu, [4]) and obtain the system

$$y_k^* \in \partial F(y_k), \quad -y_k^* \in \partial I_{C_k}(y_k)$$

From $y_k^* \in \partial F(y_k)$ yields that $y_k^* + f \in J(y_k)$, where $J : W^{1,p}(\Omega) \to W^{-1,q}(\Omega)$ is the duality mapping. So, we conclude that

$$y_k^* = J(y_k) - f \tag{5}$$

From $-y_k^* \in \partial I_{C_k}(y_k)$ we obtain

$$\sum_{i=1}^{k} \alpha_i^* y_k(x_i) = 0$$

which means that

$$\alpha_i^* y_k(x_i) = 0, \quad \forall i = \overline{1, k}$$

Then, the Lagrange multipliers α_i^* are zero if $y_k(x_i) > 0$ and they are non-zero only if the constraint is active, i.e. $y_k(x_i) = 0$.

With the above arguments, we can state the main result as follows:

Theorem 2 *To compute the solution y_k^* of the dual approximating problem it is sufficient to compute the coefficients α_i^*, due to the formula*

$$y_k^* = \sum_{i=1}^{k} \alpha_i^* \delta_{x_i}$$

. Moreover, the solution of the approximating problem y_k is computed using $y_k = J^{-1}(y_k^* + f)$ and $\alpha_i^* y_k(x_i) = 0, \quad \forall i = \overline{1,k}$.

Example 1 *Let $\Omega \subset \mathbb{R}$ and $p = 2$. Then the duality mapping $J : H_0^1(\Omega) \to H^{-1}(\Omega)$ is, in this case, a linear operator and is define as $J(y) = -y''$. Let us denote $J^{-1}(\delta_{x_i}) = d_i$ and $J^{-1}(f) = y_f$.*

We obtain that the dual approximating problem formulated for dimension 1

$$\min_{y^* \in C_k^*} \{\frac{1}{2}\|y^* + f\|_{H^{-1}(\Omega)}^2\}$$

is, in fact, equivalent to the problem

$$\min_{\alpha \in \mathbb{R}_+^k} \left\{ \frac{1}{2}\alpha^T A \alpha + b^T \alpha \right\} \tag{6}$$

where A is the matrix of elements $a_{ij} = \int_\Omega d_i' d_j' dx$ for all $i, j = 1, 2, \ldots, k$, and the elements of b are $b_i = \int_\Omega d_i' y_f' dx$, for all $i = 1, 2, \ldots, k$.

Thus, solving problem (6) *we find α_i^*, for $i = 1, 2, \ldots, k$, we compute the solution of the approximating problem using the formula*

$$y_k = \sum_{i=1}^{k} \alpha_i^* d_i + y_f$$

taking into account the complementarity condition that $\alpha_i y_k(x_i) = 0$ for all $i = 1, 2, \ldots, k$.

References

[1] R. Adams. *Sobolev spaces*, Acad. Press, 1975.

[2] L. Badea. One- and Two-Level Domain Decomposition Methods for Nonlinear Problems. In *First International Conference on Parallel, Distributed and Grid Computing for Engineering* Stirlingshire, Scotland, 2009.

[3] Baiocchi, C., Su un problema di frontiera libera connesso a questioni di idraulica. *Ann. Mat. Pura Appl.*, 92, 107 127, 1972.

[4] V. Barbu, T. Precupanu. *Convexity and optimization in Banach spaces*, Noordhoff, 1978.

[5] Burger, M., Matevosyan, N., Wolfram, M.T. A level set based shape optimization method for an elliptic obstacle problem. Math. Models Methods Appl. Sci. 21, N. 4, 619649, 2011.

[6] Duvaut, G., Lions, J.-L. Les inéquations en mécanique et en physique. Travaux et Recherches Mathematiques, N. 21. Dunod, Paris, 1972.

[7] Griesse, R., Kunisch, K. A semi-smooth Newton method for solving elliptic equations with gradient constraints. M2AN Math. Model. Numer. Anal. 43, N. 2, 209238, 2009.

[8] R. Glowinski. *Numerical Methods for Nonlinear Variational Problems*, Springer-Verlag, New York, 1984.

[9] Murea, C.M., Tiba, D. A direct algorithm in some free boundary problems, *BCAM Publications*, 2012.