

# OPTIMAL THICKNESS OF A CYLINDRICAL SHELL – AN OPTIMAL CONTROL PROBLEM IN LINEAR ELASTICITY THEORY\*

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## Abstract

In this paper we discuss optimization problems for cylindrical tubes which are loaded by an applied force. This is a problem of optimal control in linear elasticity theory (shape optimization). We are looking for an optimal thickness minimizing the deflection (deformation) of the tube under the influence of an external force. From basic equations of mechanics, we derive the equation of deformation. We apply the displacement approach from shell theory and make use of the hypotheses of Mindlin and Reissner. A corresponding optimal control problem is formulated and first order necessary conditions for the optimal solution (optimal thickness) are derived. We present numerical examples which were solved by the finite element method.

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# 1 Introduction

In this paper, we discuss a particular class of optimal shape design for cylindrical shells. As a problem of shape optimization, it belongs to a very active field of research with extensive literature. We refer only to the books by Sokolowski and Zolesio [1], Pironneau [2], Haslinger and Mäkinen [3] and Neittaanmäki et al. [4], Delfour and Zolesio [5], or Masmoudi et al. [6] and to the references therein. Our problem is, in some sense, easier to handle, because it can be transformed to an optimal control problem of coefficients in a 4th order elliptic equation. We have been inspired by the papers [7,8,9,10] on this subject. First investigations and modelings of these problems can be found in the books of Ciarlet [11] and Timoshenko [12]. Lepik and Lepikult. [7], Lepikult et al. [7,8], and Lellep [9,10] contributed to this topic. Lepikult et al. [8] discuss related problems and solve them with the software GESOP. We should also mention Olenev [14], who examined the plastic deformation of cylindrical shells due to an external force. Lellep [10] developed optimization procedures for cylindrical shells with piecewise linear geometry. The work by Neittaanmäki et al. [4] and Sprekels and Tiba [15] is most close to ours. They deal with a similar problem in elliptic equations of fourth order.

The present paper considers the effect of an external force on a cylindrical shell (specific rotationally symmetric force). As a result of this force the cylinder tube is deformed. Our objective is to determine the thickness of the tube, which minimizes the deformation. The underlying physical process is described by a 4th order ordinary differential equation with boundary conditions, which results from the balance of power. As an additional condition, we require that the volume of the tube remains constant. To obtain practical solutions we also require the thickness to vary only within specified limits. We seek to find an optimal thickness numerically and we derive first order necessary conditions for the optimal solution. The particular way of numerical treatment is one of our main issues. First-order conditions for optimality are tested numerically to evaluate the precision of the computed optimal shape. This is another novelty of this work. In this paper, we treat the stationary case, which is formulated in the next section. In a forthcoming paper we will deal with the transient case, which results from the law of conservation of momentum.

## 2 Modeling of the Problem

Many practical problems deal with deformations of bodies caused by the influence of forces. Examples are the deflection of floors, vibration characteristics of bridges, deformations during processing of metals, and crash tests in car industry. These practical problems are analyzed in elasticity theory. By using the basic equations of mechanics (balance of power, stress reaction of the material) and taking into account the geometric properties of the body the aforementioned problems can be modeled fairly simple. As a result we obtain equations for the solution (deformation) of the problem.

Let  $\bar{\Omega}_{3D} \subset \mathbf{R}^3$  be the reference configuration for a body in the stress free state. The state is expressed by a map  $\phi : \bar{\Omega}_{3D} \rightarrow \mathbf{R}^3$ . This map includes the identity mapping and small displacements  $\mathbf{y}$ . The deviation from the identity mapping is expressed by the strain. The strain-tensor  $\varepsilon$  has the following components:

$$\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial y_i}{\partial x_j} + \frac{\partial y_j}{\partial x_i} \right) \quad i, j = 1, 2, 3.$$

The displacements depend on material parameters by Hooke's law

$$\sigma(\mathbf{y}) = 2\mu \varepsilon(\mathbf{y}) + \lambda(\text{trace}(\varepsilon(\mathbf{y}))) \cdot I,$$

with the Lamé-constants  $\lambda, \mu$  (material parameters), the identical tensor  $I$ , and the stress tensor  $\sigma$ . In linear elasticity theory, the goal is to minimize the energy functional

$$\Pi(\mathbf{y}) := \int_{\Omega_{3D}} \left[ \frac{1}{2} \sigma(\mathbf{y}) : \varepsilon(\mathbf{y}) - \mathbf{f} \cdot \mathbf{y} \right] d\mathbf{x} - \int_{\partial\Omega_{3D}} \mathbf{g} \cdot \mathbf{y} dS$$

for all admissible  $\mathbf{y}$ . The term  $\sigma : \varepsilon$  denotes the second order tensor product of  $\sigma$  and  $\varepsilon$ . The function  $\mathbf{f}$  represents the force exerted on the body and  $\mathbf{g}$  formulates possible boundary conditions derived from the specific problem. For modeling the cylindrical shell we use the hypotheses of Mindlin and Reissner [16,17]. This allows to reduce our 3-dimensional problem to a 1-dimensional. The deformation of the body under the force  $\mathbf{f}$  is modeled by the balance of power

$$-\text{div} \sigma(\mathbf{y}) = \mathbf{f}. \tag{1}$$

Let us introduce the cylindrical shell to be optimized. A cylindrical shell is most conveniently described in cylindrical coordinates. The surface of the cylinder  $\Omega_{2D}^C := [0, 1] \times [0, 2\pi]$  with radius  $R$  is given by

$$\mathbf{z}(x, \varphi) = \begin{bmatrix} x \\ R \cos \varphi \\ R \sin \varphi \end{bmatrix}, \quad x \in \bar{\Omega} := [0, 1], \varphi \in [0, 2\pi].$$

The cylindrical shell  $\mathcal{S}$  with center plane  $\mathbf{z}(x, \varphi)$  and thickness  $u$  is given as

$$\mathcal{S} = \left\{ \mathbf{z}(x, \varphi) + h \begin{bmatrix} 0 \\ \cos \varphi \\ \sin \varphi \end{bmatrix} \mid h \in \left[-\frac{u}{2}, \frac{u}{2}\right], (x, \varphi) \in \Omega_{2D}^C \right\},$$

in the natural coordinate system  $\mathbf{e}_i$ , ( $i = 1, 2, 3$ ), specified by the cylindrical shell,

$$\mathbf{e}_1 = \frac{\partial \mathcal{S}}{\partial x} \quad \mathbf{e}_2 = \frac{\partial \mathcal{S}}{\partial \varphi} \quad \mathbf{e}_3 = \frac{\partial \mathcal{S}}{\partial h}.$$

We mention that  $\Omega_{3D} = \mathcal{S}$ , to close the gap the setting above.

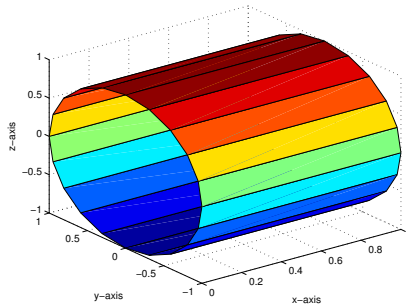


Figure 1: cylindrical shell

The Mindlin and Reissner hypotheses lead to the displacement law

$$\mathbf{y} = y_1(x, \varphi)\mathbf{e}_1 + y_2(x, \varphi)\mathbf{e}_2 + y_3(x, \varphi)\mathbf{e}_3 - h[\theta_1(x, \varphi)\mathbf{e}_1 + \theta_2(x, \varphi)\mathbf{e}_2]$$

with displacements  $y_i : \mathcal{S} \rightarrow \mathbf{R}$  with respect to all basis-directions and torsions  $\theta_i : \mathcal{S} \rightarrow \mathbf{R}$ . We assume a rotationally symmetric force. Let the

tube be fixed at its ends and let the Kirchhoff-Love hypothesis for a thin shell be fulfilled. We use soft clamped boundary conditions ( $y_i = 0, \nabla\theta_i = 0$  on the considered boundary) which are often considered in practice. Under these conditions, we have  $y_1 = y_2 = \theta_2 = 0$  and  $\partial_x y_1 = \theta_1$ . We insert the displacement law into the balance of power (1). With  $w := y_3, f_z := \mathbf{f} \cdot \mathbf{e}_3$ , there follows the equation of the stationary problem in the weak formulation: Find a solution  $w \in V := H^2(\Omega) \cap H_0^1(\Omega)$  such that

$$\int_{\Omega} \left\{ (2\mu + \lambda) \left[ \frac{Ru^3}{12} d_x^2 w d_x^2 \tilde{w} + \left( \frac{u^3}{12R^3} + \frac{u}{R} \right) w \tilde{w} \right] \right\} dx = R \int_{\Omega} f_z \tilde{w} dx \quad (2)$$

for each  $\tilde{w} \in V$ . We recall that  $\Omega = (0, 1)$ . Here and what follows,  $d_x^2 w$  stands for  $\frac{d^2 w}{dx^2}$ . The space  $H^k(\Omega)$  denotes the standard well know Sobolev space of order  $k$ . The existence of a solution of equation (2) can be shown by the Lemma of Lax and Milgram. This variational formulation has the advantage that we look for the solution in the weak sense (weak solution)  $w \in V$ . Later we need the higher regularity  $w \in V \cap H^k(\Omega)$  with  $k > 2$ . To achieve this higher regularity, we require that the force  $f_z$  belongs to  $H^{k-1}(\Omega)$  and that the coefficients of the differential equation are sufficiently smooth, for more information see [18].

Under sufficient smoothness,  $w$  is a classical solution of the equation (2). We allow the function  $f_z$  to be only square integrable, i.e.  $f_z \in L^2(\Omega)$ . This generalization fits better to the practical situation.

To cover the non-linearities with respect to the control  $u$ , we define Nemytskij operators  $\Phi, \Psi : L^\infty(\Omega) \rightarrow L^\infty(\Omega)$ :

$$\Phi(u) := (2\mu + \lambda) \frac{Ru^3}{12} \quad \Psi(u) := (2\mu + \lambda) \left( \frac{u^3}{12R^3} + \frac{u}{R} \right).$$

These operators are continuously differentiable. We need them for the derivatives of optimality criteria for the optimal control problem. These derivatives can be expressed for a direction  $h \in L^\infty(\Omega)$  by

$$\Phi'(u)h = (2\mu + \lambda_{ESZ}) \frac{Ru^2}{4} h \quad \Psi'(u)h = (2\mu + \lambda_{ESZ}) \left( \frac{u^2}{4R^3} + \frac{1}{R} \right) h.$$

### 3 The Optimal Control Problem

For the formulation of the problem and its solvability, we assume the existence of an optimal control  $\bar{u}$ . The goal is to determine a thickness  $\bar{u}$ , which

minimizes the deformation of the cylinder tube. Additionally, we require the cylindrical shell to have a constant volume:

$$\min_{u \in U_{ad}} J(w) := \min_{u \in U_{ad}} \int_{\Omega} f_z(x) w(x) dx$$

subject to

$$\begin{aligned} d_x^2 \left( \Phi(u) d_x^2 w \right) + \Psi(u) w &= R f_z \\ w(0) = w(1) = d_x^2 w(0) = d_x^2 w(1) &= 0 \end{aligned}$$

where

$$U_{ad} = \left\{ u \in L^\infty(\Omega), u_a \leq u(x) \leq u_b \text{ a.e.}, \int_{\Omega} u(x) dx = C \right\},$$

and  $0 < u_a < u_b$  are given. Additionally, the objective function is weighted by the applied force  $f_z$ . In regular situations, we can assume that the resulting deformation  $w$  has the same direction as the applied force. Then the objective function is positive. The thickness  $u = u(x)$  is the control function that influences the displacement (deflection)  $w = w(x)$  for a given force  $f_z = f_z(x)$ . The constant  $C := \frac{V_Z}{2\pi R}$  considers the constant volume of the shell  $V_Z$ . In this formulation of the optimal control problem we have used the strong formulation to highlight the type of equation and the soft clamped boundary conditions. In the following and in the numerical calculations we use only the weak formulation (2).

Let us assume for convenience that a (globally) optimal control exists that we denote in the following by  $\bar{u}$ . In general, this problem of existence is fairly delicate in the theory of shape optimization. We refer to the preface in Sokolowski and Zolesio [1], who underline the intrinsic difficulties of this issue. The existence of optimal controls can be proved in certain classes of functions that are compact in some sense. We refer also to recent discussions on bounded perimeter sets in shape optimization discussed in [5,6,18,19]. In the case of optimal shaping of some thin elastic structures such as arches or curved rods, another method was presented by Sprekels and Tiba [15], cf. also Neittaanmäki et al. [4]. In our case, this method is not applicable, because we have a volume constraint that cannot be handled this way. We also might work in a set  $U_{ad}$  that is compact in  $L^\infty$ . This is not useful for our application. However, in the numerical discretization, the existence of an optimal control follows by standard compactness arguments. Moreover, as

in classical calculus of variations, we might assume the existence of a locally optimal control. The whole theory of our paper remains true without any change for any locally optimal control  $\bar{u}$ .

Next, we transform this problem to a nonlinear optimization problem in a Banach space. For this, we define the control-to-state operator  $G : u \mapsto w$ ,  $G : L^\infty(\Omega) \rightarrow V$  where  $w \in V$  is the solution of the state equation. This allows us to eliminate the state  $w$  in the objective functional and to formulate the so-called reduced optimal control problem:

$$\min_{u \in U_{ad}} J(G(u)) = \min_{u \in U_{ad}} \int_{\Omega} [f_z G(u)](x) dx. \tag{3}$$

Let us define the reduced functional  $f$  by

$$f(u) := \int_{\Omega} [f_z G(u)](x) dx.$$

Next, we formulate the first order necessary conditions of this problem. Notice that  $f$  is continuously Fréchet-differentiability.

**Lemma 1** *Let  $\bar{u} \in U_{ad}$  be a solution of the problem (3). Then the variational inequality*

$$f'(\bar{u})(u - \bar{u}) \geq 0 \quad \forall u \in U_{ad}$$

*is fulfilled.*

We refer, for instance, to [19] for the proof of this standard result. By the chain rule, we find for any  $h \in L^\infty(\Omega)$

$$f'(\bar{u})h = \int_{\Omega} [f_z G'(\bar{u})h](x) dx$$

as the derivative of the objective functional. The derivative of the control-to-state operator  $G$  is given by the following theorem as a solution of a boundary value problem.

**Theorem 1** *Let  $\Omega$  be a bounded Lipschitz domain and  $\Phi, \Psi$  be differentiable Nemytskij operators in  $L^\infty(\Omega)$ . Then the control-to-state operator  $G$  is continuously Fréchet-differentiable. The derivative at  $\bar{u}$  in direction  $h$  given by*

$$G'(\bar{u})h = y$$

with  $y$  being the weak solution of the boundary value problem

$$d_x^2(\Phi(\bar{u})d_x^2y) + \Psi(\bar{u})y = -d_x^2(\Phi'(\bar{u})hd_x^2\bar{w}) - \Psi'(\bar{u})h\bar{w}$$

with boundary conditions  $y(0) = d_x^2y(0) = y(1) = d_x^2y(1) = 0$ . Here,  $\bar{u}$  and  $\bar{w} \in V$  denote the optimal control and the associated optimal state, respectively.

**Proof.** Let  $\bar{w} = G(\bar{u})$  be the weak solution of the boundary value problem

$$\begin{aligned} d_x^2(\Phi(\bar{u})d_x^2\bar{w}) + \Psi(\bar{u})\bar{w} &= Rf_z \\ \bar{w}(0) = \bar{w}(1) = d_x^2\bar{w}(0) &= d_x^2\bar{w}(1) = 0 \end{aligned}$$

and let  $w_u = G(\bar{u} + h)$ ,  $h \in L^\infty(\Omega)$ , be the weak solution of the boundary value problem

$$\begin{aligned} d_x^2(\Phi(\bar{u} + h)d_x^2w_u) + \Psi(\bar{u} + h)w_u &= Rf_z \\ w_u(0) = w_u(1) = d_x^2w_u(0) &= d_x^2w_u(1) = 0. \end{aligned}$$

We consider the difference  $G(\bar{u} + h) - G(\bar{u})$  and use the Fréchet-differentiability of the Nemytskij operators:

$$\begin{aligned} d_x^2([\Phi'(\bar{u})h + r_\Phi(\bar{u}, h)]d_x^2\bar{w} + \Phi(\bar{u} + h)d_x^2(w_u - \bar{w})) \\ + [\Psi'(\bar{u})h + r_\Psi(\bar{u}, h)]\bar{w} + \Psi(\bar{u} + h)(w_u - \bar{w}) = 0. \end{aligned}$$

For the boundary conditions, it follows

$$\begin{aligned} w_u(0) - \bar{w}(0) = 0 \quad d_x^2(w_u(0) - \bar{w}(0)) = 0 \\ w_u(1) - \bar{w}(1) = 0 \quad d_x^2(w_u(1) - \bar{w}(1)) = 0. \end{aligned}$$

We define  $w_u - \bar{w} = y + y_r$  with  $y \in V$  being the weak solution of the equation

$$d_x^2(\Phi(\bar{u})d_x^2y) + \Psi(\bar{u})y = -d_x^2(\Phi'(\bar{u})hd_x^2\bar{w}) - \Psi'(\bar{u})h\bar{w}$$

with boundary conditions  $y(0) = d_x^2y(0) = y(1) = d_x^2y(1) = 0$ . Now we consider the difference in order to derive an equation for the function  $y_r \in V$ :

$$\begin{aligned} d_x^2(\Phi(\bar{u} + h)d_x^2y_r) + \Psi(\bar{u} + h)y_r + d_x^2([\Phi'(\bar{u})h + r_\Phi(\bar{u}, h)]d_x^2y) \\ + [\Psi'(\bar{u})h + r_\Psi(\bar{u}, h)]y + d_x^2(r_\Phi(\bar{u}, h)d_x^2\bar{w}) + r_\Psi(\bar{u}, h)\bar{w} = 0. \end{aligned}$$



For the boundary conditions it follows

$$y_r(0) = d_x^2 y_r(0) = y_r(1) = d_x^2 y_r(1) = 0.$$

In order to prove the existence of a solution by the Lemma of Lax and Milgram, we consider the variational formulation. To this aim, we define the bilinear forms

$$\begin{aligned} a_y(y, v) &:= \int_{\Omega} \{ \Phi(\bar{u}) d_x^2 y d_x^2 v + \Psi(\bar{u}) y v \} dx \\ a_{y_r}(y_r, v) &:= \int_{\Omega} \{ \Phi(\bar{u} + h) d_x^2 y_r d_x^2 v + \Psi(\bar{u} + h) y_r v \} dx \end{aligned}$$

for each  $v \in V$ . Moreover, we introduce linear functionals  $f_y : V \rightarrow \mathbf{R}$  and  $f_{y_r} : V \rightarrow \mathbf{R}$  by

$$\begin{aligned} f_y(v) &:= \int_{\Omega} \{ -\Phi'(\bar{u}) h d_x^2 \bar{w} d_x^2 v - \Psi'(\bar{u}) h \bar{w} v \} dx \\ f_{y_r}(v) &:= \int_{\Omega} \left\{ -r_{\Phi}(\bar{u}, h) d_x^2 \bar{w} d_x^2 v - r_{\Psi}(\bar{u}, h) \bar{w} v \right. \\ &\quad \left. - [\Phi'(\bar{u}) h + r_{\Phi}(\bar{u}, h)] d_x^2 y d_x^2 v - [\Psi'(\bar{u}) h + r_{\Psi}(\bar{u}, h)] y v \right\} dx \end{aligned}$$

for fixed  $h \in L^{\infty}(\Omega)$  sufficiently small. Notice that  $d_x^2 \bar{w} \in L^2(\Omega)$  is satisfied, hence the expressions on the right-hand side are well-posed. This leads to the following problems: Find functions  $y$  and  $y_r$  that satisfy the equations

$$a_y(y, v) = f_y(v) \text{ and } a_{y_r}(y_r, v) = f_{y_r}(v)$$

for all  $v \in V$ . Obviously, the bilinear forms satisfy the conditions of the Lemma of Lax and Milgram. In order to estimate the linear form  $f_y(v)$ , we define  $\Theta_1(\bar{u})(x) := \max\{|\Phi'(\bar{u})(x)|, |\Psi'(\bar{u})(x)|\}$ ,  $x \in \Omega$ , and it holds

$$|f_y(v)| \leq \int_{\Omega} |\Theta_1(\bar{u}) h| |d_x^2 \bar{w} d_x^2 v + \bar{w} v| dx \leq \|\Theta_1\|_{L^{\infty}} \|h\|_{L^{\infty}} \|\bar{w}\|_{H^2} \|v\|_{H^2}.$$

For the norm of the functional it follows

$$\|f_y\|_{V^*} = \sup_{v \in V} \frac{|f_y(v)|}{\|v\|_V} \leq \|\Theta_1\|_{L^{\infty}} \|h\|_{L^{\infty}} \|\bar{w}\|_{H^2}.$$

The Lemma of Lax and Milgram yields

$$\|y\|_{H^2} \leq \frac{\|\Theta_1\|_{L^{\infty}} \|h\|_{L^{\infty}} \|\bar{w}\|_{H^2}}{\beta_0}$$

with a constant  $\beta_0 > 0$ . The solution  $y$  depends linearly on  $h$ . In order to estimate the linear form  $f_{y_r}(v)$  we define

$$r_{y_r}(\bar{u}, h)(x) := \max\{|r_\Phi(\bar{u}, h)(x)|, |r_\Psi(\bar{u}, h)(x)|\}, \quad x \in \Omega.$$

We have

$$\begin{aligned} |f_{y_r}(v)| &\leq \int_{\Omega} |r_{y_r}(\bar{u}, h)| |d_x^2(\bar{w} + y)d_x^2v + (\bar{w} + y)v| + |\Theta_1| |h| |d_x^2y d_x^2v + yv| \, dx \\ &\leq \|r_{y_r}(\bar{u}, h)\|_{L^\infty} \|\bar{w} + y\|_{H^2} \|v\|_{H^2} + \|\Theta_1\|_{L^\infty} \|h\|_{L^\infty} \|y\|_{H^2} \|v\|_{H^2}, \end{aligned}$$

hence

$$\|f_{y_r}\|_{V^*} \leq \|r_{y_r}(\bar{u}, h)\|_{L^\infty} \|\bar{w} + y\|_{H^2} + \|\Theta_1\|_{L^\infty} \|h\|_{L^\infty} \|y\|_{H^2}.$$

Therefore it holds

$$\|y_r\|_{H^2} \leq c_\alpha \|f_{y_r}\|_{V^*}$$

with a constant  $c_\alpha > 0$ . It remains to show that the remainder term  $y_r$  satisfies the required property. We divide by  $\|h\|_{L^\infty} > 0$ ,

$$\frac{\|y_r\|_{H^2}}{\|h\|_{L^\infty}} \leq c_\alpha \frac{\|f_{y_r}\|_{V^*}}{\|h\|_{L^\infty}},$$

and consider the limit  $\|h\|_{L^\infty} \rightarrow 0$ . In the following we analyze each term separately. First, we invoke the remainder property of Nemytskij operators. For the second term, we use our estimate of the solution  $y$ ,

$$\begin{aligned} \frac{\|r_{y_r}(\bar{u}, h)\|_{L^\infty} \|y\|_{H^2}}{\|h\|_{L^\infty}} &\leq \frac{\|r_{y_r}(\bar{u}, h)\|_{L^\infty} \|\Theta_1\|_{L^\infty} \|h\|_{L^\infty} \|\bar{w}\|_{H^2}}{\beta_0 \|h\|_{L^\infty}} \\ &= \frac{\|r_{y_r}(\bar{u}, h)\|_{L^\infty} \|\Theta_1\|_{L^\infty} \|\bar{w}\|_{H^2}}{\beta_0} \rightarrow 0, \end{aligned}$$

as  $\|h\|_{L^\infty} \rightarrow 0$ . The last term is handled by

$$\begin{aligned} \frac{\|\Theta_1\|_{L^\infty} \|h\|_{L^\infty} \|y\|_{H^2}}{\|h\|_{L^\infty}} &\leq \frac{\|\Theta_1\|_{L^\infty}^2 \|h\|_{L^\infty}^2 \|\bar{w}\|_{H^2}}{\beta_0 \|h\|_{L^\infty}} \\ &= \frac{\|\Theta_1\|_{L^\infty}^2 \|h\|_{L^\infty} \|\bar{w}\|_{H^2}}{\beta_0} \rightarrow 0 \end{aligned}$$

for  $\|h\|_{L^\infty} \rightarrow 0$ . In view of the remainder property, we conclude

$$\frac{\|y_r\|_{H^2}}{\|h\|_{L^\infty}} \leq c_\alpha \frac{\|f_{y_r}\|_{V^*}}{\|h\|_{L^\infty}} \leq o(\|h\|_{L^\infty}).$$

Thus we showed the Fréchet-differentiability of the operator  $G$ :

$$G(\bar{u} + h) - G(\bar{u}) = G'(\bar{u})h + r_G(\bar{u}, h)$$

with  $G'(\bar{u})h = y$  and  $r_G(\bar{u}, h) = y_r$ .

□

By an adjoint state, we are able to formulate this derivative more useful.

**Definition 1** *The adjoint state  $p \in V$  associated with  $\bar{u}$  is the weak solution of the boundary value problem*

$$d_x^2 (\Phi(\bar{u})d_x^2 p) + \Psi(\bar{u})p = f_z$$

with boundary conditions  $p(0) = d_x^2 p(0) = p(1) = d_x^2 p(1) = 0$ .

We use the function  $p$  in order to express the first order necessary condition more conveniently. The function  $p$  can be interpreted as Lagrange multiplier associated with the state equation.

**Lemma 2** *Let functions  $\bar{u}, h \in L^\infty(\Omega)$  be given. Furthermore, let  $y$  and  $p$  be the weak solutions of*

$$\begin{aligned} d_x^2 (\Phi(\bar{u})d_x^2 y) + \Psi(\bar{u})y &= -d_x^2 (\Phi'(\bar{u})hd_x^2 w - \Psi'(\bar{u})hw) \\ y(0) = y(1) = d_x^2 y(0) &= d_x^2 y(1) = 0 \end{aligned}$$

and

$$\begin{aligned} d_x^2 \Phi(\bar{u})d_x^2 p + \Psi(\bar{u})p &= f_z \\ p(0) = p(1) = d_x^2 p(0) &= d_x^2 p(1) = 0. \end{aligned}$$

Then it holds

$$\int_{\Omega} f_z y \, dx = \int_{\Omega} [-\Phi'(\bar{u})d_x^2 w d_x^2 p - \Psi'(\bar{u})wp]h \, dx.$$

Using this lemma, it follows for  $h = u - \bar{u}$  that

$$\begin{aligned} f'(\bar{u})(u - \bar{u}) &= \int_{\Omega} [f_z G'(\bar{u})h](x) \, dx = \int_{\Omega} f_z y(x) \, dx \\ &= \int_{\Omega} [-\Phi'(\bar{u})d_x^2 \bar{w} d_x^2 p - \Psi'(\bar{u})\bar{w}p](u - \bar{u}) \, dx. \end{aligned} \quad (4)$$

**Corollary 1 (Necessary condition)** *Any optimal control  $\bar{u}$  and the corresponding optimal state  $\bar{w} = G(\bar{u})$  must fulfill the optimality system*

$$\int_{\Omega} \{ \Phi(\bar{u}) d_x^2 p d_x^2 v + \Psi(\bar{u}) p v \} dx = \int_{\Omega} f_z v dx, \quad \forall v \in V$$

$$\int_{\Omega} \{ -\Phi'(\bar{u}) d_x^2 \bar{w} d_x^2 p - \Psi'(\bar{u}) \bar{w} p \} (u - \bar{u}) dx \geq 0, \quad \forall u \in U_{ad},$$

where the Lagrange multiplier  $p \in V$  is the weak solution of the adjoint equation.

This formulation implicitly contains the constraints in terms of the control-to-state operator  $G$ .

## 4 Numerical Implementation and Solution of the Problem

Our equality is a fourth order ordinary differential equation. We solve this equation by the finite element method. For the resulting optimal control problem we use the optimization solver *fmincon* that is part of the software-package MATLAB. Subsequently, we give some details to the finite element method (FEM) [16,20] that is used to determine approximate solutions of the state equations and the associated adjoint equation. The starting point of this method is the variational formulation (2).

In order to approximate the solution space  $V$ , we use Hermite interpolation of the functions  $w$  and  $p$  with step size parameter  $h$ :

$$w_h = \sum_{j=0}^n w_j^0 p_j + w_j^1 q_j$$

with basis functions  $p_j, q_j \in \mathcal{P}_3(\Omega)$  for  $j = 0, 1, \dots, n$  (cubic polynomials), where  $n$  is the number of grid points, and the  $w_j^0, w_j^1$  are certain real node parameters. For the node parameters, we have in mind  $w_j^0 \approx w_h(x_j)$  and  $w_j^1 \approx d_x w_h(x_j)$ . First, we define the discrete solution space

$$V_h = \left\{ w_h(\cdot) : w_h(\cdot) = \sum_{j=0}^n w_j^0 p_j(\cdot) + w_j^1 q_j(\cdot) \right\} \subset V.$$

In this definition of the solution space, we use conformal finite elements. Furthermore, we define a discrete bilinear form and a linear form

$$\begin{aligned}
 a_h(w_h, v_h) &:= \int_{\Omega} \Phi(u_h) d_x^2 w_h d_x^2 v_h + \Psi(u_h) w_h v_h \, dx \\
 F_h(v_h) &:= R \int_{\Omega} f_z v_h \, dx.
 \end{aligned}$$

For the weak formulation of our state equation or adjoint equation (with small modifications in the linear form), we define the following discrete problem: Find  $w_h \in V_h$  solving the equation

$$a_h(w_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h.$$

For this discrete problem and our approach, we can use the standard techniques of linear algebra to calculate an approximate solution for the state  $w_h$  and the associated adjoint state  $p_h$ . For the control function  $u$ , we use a piecewise linear interpolation,

$$u_h = \sum_{j=0}^n u_j l_j$$

with linear continuous basis functions  $l_j$ ; then it holds  $u_h \in \mathcal{C}(\bar{\Omega})$ . With the linear interpolation of the control  $u$  it follows for the Nemytskij-operators  $\Phi(u)$  and  $\Psi(u)$  on each interval  $E^{(i)} := [x_{i-1}, x_i]$ :

$$\begin{aligned}
 \Phi(u_h)|_{E^{(i)}} &:= \frac{(2\mu + \lambda_{ESZ})R}{12} \left( \sum_{k=i-1}^i u_k l_k(x) \right)^3 \\
 \Psi(u_h)|_{E^{(i)}} &:= \frac{(2\mu + \lambda_{ESZ})}{12R^3} \left\{ 12R^2 \left( \sum_{k=i-1}^i u_k l_k(x) \right) + \left( \sum_{k=i-1}^i u_k l_k(x) \right)^3 \right\}.
 \end{aligned} \tag{5}$$

The coefficients  $u_{i-1}$  and  $u_i$  are equal to  $u_h(x_{i-1})$  and  $u_h(x_i)$  within the considered element  $E^{(i)}$ . Therefore, we have established an isomorphism

$$u_h \iff \vec{u}_h = [u_0, \dots, u_n] \in \mathbf{R}^{n+1}$$

and the objective functional  $f_h(u_h) = \int_{\Omega} f_z^h w_h \, dx$  can be expressed by a mapping  $\varphi : \mathbf{R}^{n+1} \rightarrow \mathbf{R}$ :

$$\varphi_h : \vec{u}_h \mapsto u_h \mapsto f_h(u_h).$$

This function will be used in the numerical implementation. The optimal control problem is finally approximated by

$$\min_{\vec{u} \in U_{ad}^h} \varphi_h(\vec{u}_h),$$

where the set  $U_{ad}^h$  will be defined later.

#### 4.1 Implementation of the Derivative of the Objective Functional

In order to express the derivative of the objective functional  $f$ , we invoke the adjoint state  $p$ . We get from (4)

$$f'(u)z = \int_{\Omega} [-\Phi'(u)d_x^2 w d_x^2 p - \Psi'(u)wp]z \, dx \quad \forall z \in L^\infty(\Omega).$$

The optimization solver *fmincon* calculates the discrete derivative  $\nabla\varphi_h(\vec{u}_h)$  by finite differences. This means that, in principle, our adjoint calculus is not needed by *fmincon*. However, proceeding in this way, the computing times will be very long. The tool *fmincon* can be accelerated by providing information on the derivative. During the analytical treatment of the optimal control problem, we determined the derivative  $f'(u)$  by (4). Numerically, we implement this derivative by the finite element approximation to compute  $\nabla\varphi_h(\vec{u}_h)$ . During the numerical optimization process, this gradient is passed to the solver *fmincon*. This reduces the running time of the optimization algorithm considerably. On the other hand, this approach might be problematic, because we approximate  $\nabla\varphi_h$  by means of a discretized continuous adjoint equation. This is not necessarily equal to the exact discrete gradient  $\nabla\varphi_h$ . The solver *fmincon* is testing the quality of the transmitted gradient  $\nabla\varphi_h$  by finite differences. During our numerical experiments, it turned out that the difference between our gradient  $\nabla\varphi_h$  (computed via the adjoint equation) and the "exact" discrete gradient was marginal of the order  $10^{-6}$  for sufficiently small discretization parameters  $h$ .

We should mention that the computation of the gradient via the adjoint equation was numerically more stable than the use of the one generated by *fmincon*. Of course, the solution of the optimization problem with finite element gradients is identical with the one obtained from finite difference gradients. We used a discretized version of this gradient, where we consider

arbitrary directions  $z_h \in \mathcal{C}(\bar{\Omega})$  (the analog approach to  $u$ ), and it holds

$$\begin{aligned} \nabla f_h(u_h)z_h &= \int_{\Omega} [-\Phi'(u_h)d_x^2 w_h d_x^2 p_h - \Psi'(u_h)w_h p_h]z_h \, dx \\ &= \sum_{i=1}^n \int_{E^{(i)}} [-\Phi'(u_h)d_x^2 w_h d_x^2 p_h - \Psi'(u_h)w_h p_h]z_h \, dx \end{aligned}$$

for each  $z_h$ . The derivatives of the Nemytskij operators  $\Phi'(u_h)$ ,  $\Psi'(u_h)$  are discretized analogously to (5). For any direction  $z_h$ , we can choose the nodal basis functions  $l_j$  successively for  $j = 0, \dots, n$ . It follows for the numerical implementation of the discrete gradient vector that

$$[\nabla \varphi_h]_j = \sum_{i=1}^n \int_{E^{(i)}} [-\Phi'(u_h)d_x^2 w_h d_x^2 p_h - \Psi'(u_h)w_h p_h]l_j \, dx \quad j = 0, \dots, n.$$

The same implementation is used for checking the first order necessary optimality conditions, see Corollary 1.

## 4.2 Numerical Solution of the Optimization Problem

The reduced problem (3) is our starting point for the direct solution of the optimal control problem. It is a finite dimensional optimization problem:

$$\min_{\vec{u}_h \in U_h^{ad}} \varphi_h(\vec{u}_h) \quad (6)$$

subject to

$$A_h \vec{u}_h = C, \quad (7)$$

where

$$U_h^{ad} = \{\vec{u}_h \in \mathbf{R}^{n+1} \mid \vec{u}_a \leq \vec{u}_h \leq \vec{u}_b\}$$

and

$$A_h = \begin{bmatrix} \frac{h}{2} & h & \dots & h & \frac{h}{2} \end{bmatrix} \in \mathbf{R}^{n+1}.$$

The volume condition (7) is formulated as an additional constraint. It was derived by the trapezoidal rule. The constant  $C$  (volume) depends on the particular problem. The restrictions on  $\vec{u}_h$  are defined componentwise.

We use the *Optimization Toolbox* of MATLAB, in particular the tool *fmincon*, for obtaining numerical solutions. It is designed for solving optimization

problems with linear (or nonlinear) objective functions and linear (or nonlinear) constraints, both in form of equations as well as inequalities. The routine *fmincon* requires the following inputs: the values of the discrete objective functional  $\varphi_h(\vec{u}_h)$ , the volume condition (7) by input of  $A_h$  and  $C$ , the vectors  $\vec{u}_a, \vec{u}_b$ , and the discrete gradient vector  $\nabla\varphi_h$  that leads to a reduction of running time. The program stops, if changes of the objective functional are smaller than a prescribed threshold, the violations of the constraint be located within the tolerances, and the necessary conditions for the optimality of the solution are fulfilled. The output is the optimal solution vector  $\vec{u}_h$ , the discrete Lagrange multiplier  $q_h$  for the volume condition, and the Lagrange multipliers  $\vec{\mu}_a, \vec{\mu}_b$  for the control restrictions, where they are active. Whether the solution  $\vec{u}_h$  of the discrete optimal control problem really be a candidate for the optimal solution is verified by checking the first order necessary condition. Let us call this "optimality test".

The variational inequality

$$\int_{\Omega} [-\Phi'(\bar{u})d_x^2\bar{w}d_x^2p - \Psi'(\bar{u})\bar{w}p](z - \bar{u}) dx \geq 0, \quad \forall z \in U_{ad}, \quad (8)$$

is the starting point for evaluating the optimality of the numerically computed optimal solution  $\vec{u}_h$ . Pointwise evaluation of the first order necessary condition, as done in [10] for optimal control problems with box constraints, is not applicable in our case due to the non-constant ansatz for the control function  $u$  and the additional volume condition. Let us define the linear form

$$b(z) := \int_{\Omega} [\Phi'(\bar{u})d_x^2\bar{w}d_x^2p + \Psi'(\bar{u})\bar{w}p]z dx.$$

The inequality (8) is equivalent to

$$\max_{z \in U_{ad}} b(z) = b(\bar{u}),$$

hence, to be optimal,  $\bar{u}$  must solve a linear continuous optimization problem. As done for the control  $u$  we linearly interpolate the function  $z$ . We have also an isomorphism

$$\psi_h : \vec{z}_h \mapsto z_h \mapsto b_h(z_h),$$

with

$$b_h(z_h) = \int_{\Omega} [\Phi'(\bar{u}_h)d_x^2\bar{w}_h d_x^2p_h + \Psi'(\bar{u}_h)\bar{w}_h p_h]z_h dx.$$



Hence we obtain the discrete problem:

$$\max_{\bar{z}_h \in U_h^{ad}} \psi_h(\bar{z}_h) \quad (9)$$

subject to

$$A_h \bar{z}_h = C.$$

A first test is performed as follows: After computing  $\bar{u}_h$ , we compute the state  $\bar{w}_h$  and the adjoint state  $p_h$ . Then we solve the optimal problem (9). The solution is  $\bar{z}_h$ . If  $\bar{u}_h$  is optimal for the discretized problem, the equation  $\bar{u}_h = \bar{z}_h$  should hold. In general, there is an error and the difference  $\|\bar{u}_h - \bar{z}_h\|$  indicates the precision of  $\bar{u}_h$ . This procedure shows, how good  $\bar{u}_h$  solves the discretized optimization problem. Another test is used to estimate, how well  $\bar{u}_h$  solves the reduced problem.

Now, we give some implementation details. The problem (9) is also solved by the optimization solver *fmincon*. The gradient  $\nabla \psi_h$  is calculated by finite elements and is passed to *fmincon*. We used  $\bar{z}_0 = \bar{u}_h + \epsilon \cdot \mathbf{1}$  as starting approximation for our examples below. The parameter  $\epsilon \in \mathbf{R}$  generates a perturbation in all components of optimal solution  $\bar{u}_h$  of (6). The output is the optimal solution  $\bar{z}_h$  of (8).

## 5 Examples

Let us discuss some simple test examples for our optimal control problem. They are used to evaluate the quality of the necessary optimality conditions and the advantage of using finite elements gradients. In the examples, we use the material parameters

$$E = 2.1 \cdot 10^2, \nu = 0.3, R = 1,$$

that is the elastic modulus  $E$ , the Poisson number  $\nu$  and the radius  $R$ . The first two parameters depend on the material and the last parameter depends on the particular geometry. To discretize the stationary problem, we choose an equidistant grid. For this grid, we compute the discrete optimal solution  $\bar{u}_h$  and the corresponding optimal state  $\bar{w}_h$ .

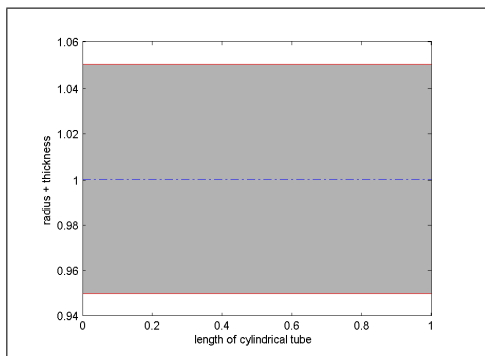


Figure 2: Starting configuration

Figure 2 shows a longitudinal cut through the cylindrical shell, where we only plot the upper part due to symmetry. The dash-dotted-line represents the central plane of the cylindrical shell. This is the starting configuration for all examples. We define the vector  $\mathbf{1} \in \mathbf{R}^{n+1}$  containing the integer number in all entries. As restrictions to the control we set the constants  $u_a = 0.05 \cdot \mathbf{1}$  and  $u_b = 0.2 \cdot \mathbf{1}$ . The fixed volume is prescribed by  $C = 0.6283$ , and  $u_0 = 0.1 \cdot \mathbf{1}$  is the initial value on the control. The figures below show the optimal solution  $\bar{u}_h$ , the solution  $\bar{z}_h$  of the variational inequality and the corresponding shape of the cylindrical shell for different choices of  $f_z$ .

**Example 1.** We take as force  $f_z = \sin(2\pi x)$ .

First, we justify the use of the finite element gradient via the adjoint equation.

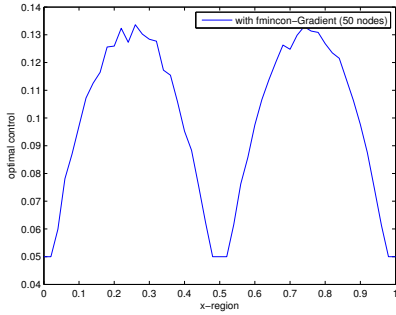


Figure 3: Optimal thickness  $\bar{u}_h$  on using the finite difference method of *fmincon* for gradients (fdm-gradient)

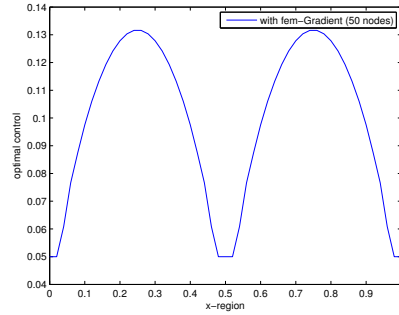


Figure 4: Optimal thickness  $\bar{u}_h$  by using the finite element method and the adjoint equation for the gradients (fem-gradient)

As can be easily seen, the use of the finite element gradient is of great advantage. The second major advantage is the acceleration of the optimization solver *fmincon*:

grid-points	<i>fmincon</i> with fdm-gradient	<i>fmincon</i> with fem-gradient
50	246.433	2.49704
100	1166.62	5.58509

The values of the 2nd and 3rd column are given in seconds. These values were calculated for Example 1.

The next pictures show the numerically calculated optimal control  $\bar{u}_h$  and the associated configuration of the cylindrical shell. Again, we only display its upper half.

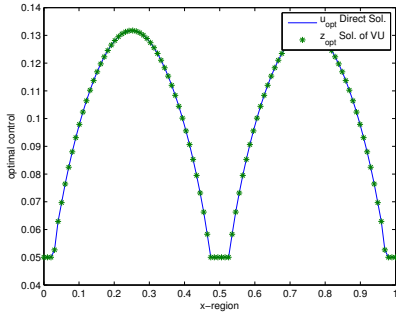


Figure 5: Optimal thickness  $\bar{u}_h$

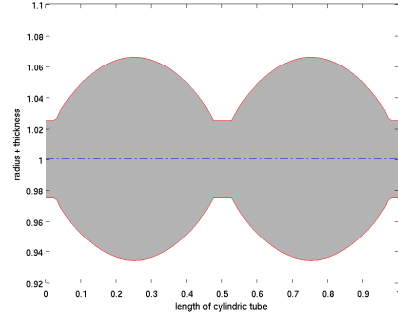


Figure 6: Configuration of the cylindrical shell

The following table shows how the error developed with consideration of the step size. For  $\bar{u}$ , we take the solution  $\bar{u}_h$  on a very fine grid with  $h = 1.25 \cdot 10^{-3}$ .

step-size $h$	$\ \bar{u} - \bar{u}_h\ _\infty$	$\ \bar{u} - \bar{u}_h\ _2$
$4.00 \cdot 10^{-2}$	$5.215633 \cdot 10^{-3}$	$1.322039 \cdot 10^{-3}$
$2.00 \cdot 10^{-2}$	$3.393303 \cdot 10^{-3}$	$5.254619 \cdot 10^{-4}$
$1.00 \cdot 10^{-2}$	$1.465212 \cdot 10^{-3}$	$2.395697 \cdot 10^{-4}$
$5.00 \cdot 10^{-3}$	$8.900138 \cdot 10^{-4}$	$7.009283 \cdot 10^{-5}$
$2.50 \cdot 10^{-3}$	$6.143035 \cdot 10^{-4}$	$5.524533 \cdot 10^{-5}$

The next table displays the discretization error of (9) in different norms.

step-size $h$	$\ \bar{u} - \bar{z}_h\ _\infty$	$\ \bar{u} - \bar{z}_h\ _2$
$2.00 \cdot 10^{-2}$	$1.398209 \cdot 10^{-1}$	$3.264695 \cdot 10^{-2}$
$1.00 \cdot 10^{-2}$	$1.476560 \cdot 10^{-1}$	$1.978807 \cdot 10^{-2}$
$5.00 \cdot 10^{-3}$	$1.105508 \cdot 10^{-4}$	$9.371137 \cdot 10^{-6}$
$2.50 \cdot 10^{-3}$	$6.110761 \cdot 10^{-5}$	$4.862581 \cdot 10^{-6}$
$1.25 \cdot 10^{-3}$	$1.024119 \cdot 10^{-5}$	$2.125253 \cdot 10^{-6}$

**Example 2.** We take as force  $f_z = x(1 - x)$ .

In the next example a "simple" symmetric force is acting on the cylindrical shell. We again show the optimal control  $\bar{u}_h$  and the cut through the cylindrical shell.

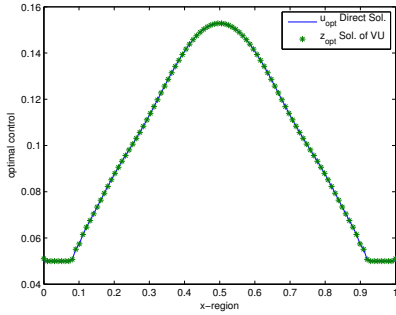


Figure 7: Optimal thickness  $\bar{u}_h$

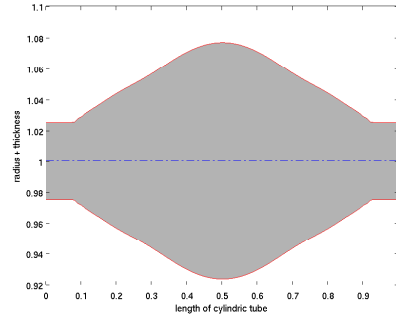


Figure 8: Configuration of the cylindrical shell

For the error, the following values are obtained:

step-size $h$	$\ \bar{u} - \bar{u}_h\ _\infty$	$\ \bar{u} - \bar{u}_h\ _2$
$4.00 \cdot 10^{-2}$	$4.279621 \cdot 10^{-3}$	$1.663178 \cdot 10^{-3}$
$2.00 \cdot 10^{-2}$	$3.652657 \cdot 10^{-3}$	$1.513568 \cdot 10^{-3}$
$1.00 \cdot 10^{-2}$	$3.312656 \cdot 10^{-3}$	$1.432902 \cdot 10^{-3}$
$5.00 \cdot 10^{-3}$	$2.552378 \cdot 10^{-3}$	$1.187955 \cdot 10^{-3}$
$2.50 \cdot 10^{-3}$	$1.194550 \cdot 10^{-3}$	$1.520591 \cdot 10^{-4}$

step-size $h$	$\ \bar{u} - \bar{z}_h\ _\infty$	$\ \bar{u} - \bar{z}_h\ _2$
$2.00 \cdot 10^{-2}$	$1.100842 \cdot 10^{-3}$	$3.894984 \cdot 10^{-5}$
$1.00 \cdot 10^{-2}$	$1.100871 \cdot 10^{-3}$	$3.895043 \cdot 10^{-5}$
$5.00 \cdot 10^{-3}$	$1.100853 \cdot 10^{-3}$	$3.894977 \cdot 10^{-5}$
$2.50 \cdot 10^{-3}$	$1.100852 \cdot 10^{-3}$	$3.895002 \cdot 10^{-5}$
$1.25 \cdot 10^{-3}$	$1.092301 \cdot 10^{-3}$	$3.864768 \cdot 10^{-5}$

Figure 7 indicates that, in the major parts of Intervall  $[0, 1]$  the solution is almost linear. This explains the very good approximation already for  $h = 0.02$ .

**Example 3.** The force is given by  $f_z = \exp(x)$ .

In this example, an exponential power is applied to the cylindrical shell.

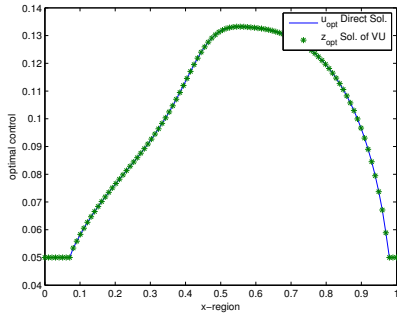


Figure 9: Optimal thickness  $\bar{u}_h$

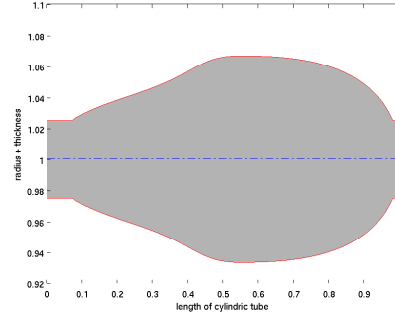


Figure 10: Configuration of the cylindrical shell

For this example, we calculated the following errors:

step-size $h$	$\ \bar{u} - \bar{u}_h\ _\infty$	$\ \bar{u} - \bar{u}_h\ _2$
$4.00 \cdot 10^{-2}$	$6.617402 \cdot 10^{-3}$	$1.663178 \cdot 10^{-4}$
$2.00 \cdot 10^{-2}$	$1.062276 \cdot 10^{-3}$	$1.513568 \cdot 10^{-4}$
$1.00 \cdot 10^{-2}$	$1.073535 \cdot 10^{-3}$	$1.432902 \cdot 10^{-5}$
$5.00 \cdot 10^{-3}$	$9.485243 \cdot 10^{-4}$	$1.187955 \cdot 10^{-5}$
$2.50 \cdot 10^{-3}$	$5.911537 \cdot 10^{-4}$	$1.520591 \cdot 10^{-5}$

step-size $h$	$\ \bar{u} - \bar{z}_h\ _\infty$	$\ \bar{u} - \bar{z}_h\ _2$
$2.00 \cdot 10^{-2}$	$1.479757 \cdot 10^{-1}$	$2.707877 \cdot 10^{-2}$
$1.00 \cdot 10^{-2}$	$3.221490 \cdot 10^{-3}$	$2.032824 \cdot 10^{-4}$
$5.00 \cdot 10^{-3}$	$1.617830 \cdot 10^{-4}$	$6.937561 \cdot 10^{-6}$
$2.50 \cdot 10^{-3}$	$8.919767 \cdot 10^{-5}$	$3.873755 \cdot 10^{-6}$
$1.25 \cdot 10^{-3}$	$1.159797 \cdot 10^{-5}$	$1.552477 \cdot 10^{-6}$

**Example 4.** We select as force

$$f_z(x) = \begin{cases} \exp(x) & x \in [0, 0.5] \\ \exp(-x) & x \in (0.5, 1] \end{cases} .$$

In this example, we considered also an exponential power influence on the cylindrical shell. But on half of the interval, we used a negative exponential power.

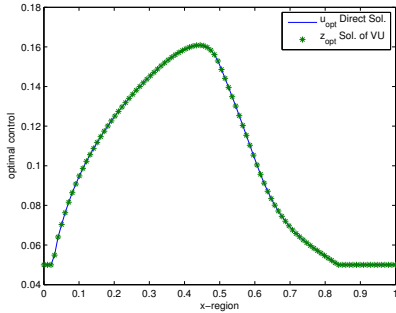


Figure 11: Optimal thickness  $\bar{u}_h$

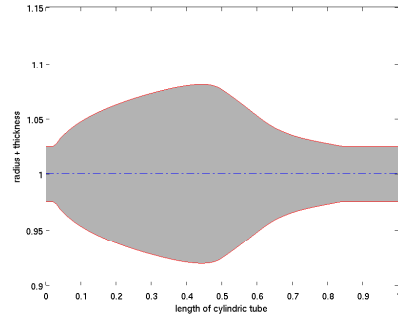


Figure 12: Configuration of the cylindrical shell

step-size $h$	$\ \bar{u} - \bar{u}_h\ _\infty$	$\ \bar{u} - \bar{u}_h\ _2$
$4.00 \cdot 10^{-2}$	$5.145951 \cdot 10^{-3}$	$8.806373 \cdot 10^{-4}$
$2.00 \cdot 10^{-2}$	$2.393243 \cdot 10^{-3}$	$2.094697 \cdot 10^{-4}$
$1.00 \cdot 10^{-2}$	$1.718650 \cdot 10^{-3}$	$1.538653 \cdot 10^{-4}$
$5.00 \cdot 10^{-3}$	$9.667670 \cdot 10^{-4}$	$9.869548 \cdot 10^{-5}$
$2.50 \cdot 10^{-3}$	$3.907290 \cdot 10^{-4}$	$6.034828 \cdot 10^{-5}$

step-size $h$	$\ \bar{u} - \bar{z}_h\ _\infty$	$\ \bar{u} - \bar{z}_h\ _2$
$2.00 \cdot 10^{-2}$	$1.363838 \cdot 10^{-1}$	$1.535772 \cdot 10^{-2}$
$1.00 \cdot 10^{-2}$	$2.291533 \cdot 10^{-3}$	$1.604229 \cdot 10^{-4}$
$5.00 \cdot 10^{-3}$	$1.537531 \cdot 10^{-4}$	$8.267414 \cdot 10^{-6}$
$2.50 \cdot 10^{-3}$	$1.023066 \cdot 10^{-4}$	$4.271060 \cdot 10^{-6}$
$1.25 \cdot 10^{-3}$	$1.004273 \cdot 10^{-5}$	$1.403717 \cdot 10^{-6}$

## 6 Concluding Remarks

We discussed a problem of optimal shape design in linear elasticity theory. The optimal thickness of a cylindrical tube is determined that minimizes the displacement of the tube under the influence of given external force. Necessary optimality conditions for the optimal solution are formulated and proved. In contrast to previous work on this subject, we selected a direct method for the optimization for a finite element discretized model. We also use finite element method to generate gradients and to test necessary optimality conditions. We considered only small deformations. The case of large deformations that might lead to effects of plasticity is not considered here.

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