ISSN 2066 - 6594

Ann. Acad. Rom. Sci.
Ser. Math. Appl.
Vol. 4, No. 2 / 2012

CONVERGENCE ESTIMATES FOR ABSTRACT SECOND ORDER SINGULARLY PERTURBED CAUCHY PROBLEMS WITH MONOTONE NONLINEARITIES*

Andrei PERJAN † Galina RUSU ‡

Abstract

We study the behavior of solutions to the problem

$$\begin{cases} \varepsilon\Big(u''_\varepsilon(t) + A_1 u_\varepsilon(t)\Big) + u'_\varepsilon(t) + A_0 u_\varepsilon(t) + B\big(u_\varepsilon(t)\big) = f_\varepsilon(t), \quad t \in (0,T), \\ u_\varepsilon(0) = u_{0\varepsilon}, \quad u'_\varepsilon(0) = u_{1\varepsilon}, \end{cases}$$

in the Hilbert space H as $\varepsilon \to 0$, where A_1, A_0 are two linear self-adjoint operators and B is a locally Lipschitz and monotone operator.

*Accepted for publication in revised form on June 15, 2012.

†Department of Mathematics and Informatics, Moldova State University, A. Mateevici str. 60, MD 2009, Chisinau, Fax and tel. no: +(373)22577627, Email: perjan@usm.md

‡Department of Mathematics and Informatics, Moldova State University, A. Mateevici str. 60, MD 2009, Chisinau, Fax and tel. no: +(373)22577627, Email: rusugalina@mail.md

‡ Researches supported by the State Program of Republic of Moldova under Contract 12.839.08.06F and by the Independent Project for Young Researchers under Contract 11.819.08.02F

MSC: 35B25, 35K15, 35L15, 34G10

Keywords: singular perturbation; abstract second order Cauchy problem; boundary layer function; a priori estimate.

1 Introduction

Let H be a real Hilbert space endowed with the scalar product $(\cdot,\cdot)$ and the norm $|\cdot|$. Let $A_i : D(A_i) \subset H \to H$, $i = 0,1$, be two linear self-adjoint operators and $B : D(B) \subset H \to H$ a locally Lipschitz and monotone operator. Consider the following Cauchy problem:

$$\begin{cases} \varepsilon\Big(u_\varepsilon''(t) + A_1 u_\varepsilon(t)\Big) + u_\varepsilon'(t) + A_0 u_\varepsilon(t) + B\big(u_\varepsilon(t)\big) = f_\varepsilon(t), \quad t \in (0,T), \\ u_\varepsilon(0) = u_{0\varepsilon}, \quad u_\varepsilon'(0) = u_{1\varepsilon}, \end{cases} \tag{P_ε}$$

where $\varepsilon > 0$ is a small parameter($\varepsilon \ll 1$), $u_\varepsilon, f_\varepsilon : [0,T) \to H$.

We investigate the behavior of solutions u_ε to the problems (P_ε) when $u_{0\varepsilon} \to u_0$, $f_\varepsilon \to f$ as $\varepsilon \to 0$. We establish a relationship between solutions to the problems (P_ε) and the corresponding solution to the following unperturbed problem:

$$\begin{cases} v'(t) + A_0 v(t) + B\big(v(t)\big) = f(t), \quad t \in (0,T), \\ v(0) = u_0. \end{cases} \tag{P_0}$$

If in some topology, the solutions u_ε to the perturbed problems (P_ε) tend to the corresponding solution v to the unperturbed problem (P_0) as $\varepsilon \to 0$, then the problem (P_0) is called *regularly perturbed.* In the opposite case, the problem (P_0) is called *singularly perturbed.* In the last case, a subset of $[0,\infty)$ in which solutions u_ε have a singular behavior relative to ε arises. This subset is called *the boundary layer.* The function which defines the singular behavior of solution u_ε within the boundary layer is called *the boundary layer function.*

The problem (P_ε) is the abstract model of singularly perturbed problems of hyperbolic-parabolic type. Such kind of problems arises in the mathematical modeling of elasto-plasticity phenomena. These abstract results are new and can be applied to singularly perturbed problems of hyperbolic-parabolic type with stationary part defined by strongly elliptic operators of high order.

A large class of works is dedicated to the study of singularly perturbed Cauchy problems for differential equations of second order. Without pretending to do a complete analysis of these works, we will mention some of them, which contain a rich bibliography. In [15], [17], [28], asymptotic expansions of solutions and theirs derivatives for linear wave equations have been obtained. In [3], [5], [8], [14], [22] the nonlinear problems of hyperbolic-parabolic type have been studied. In [4], [7], [9], [16], [21], [23], [25] the behavior of solutions u_ε to the abstract linear Cauchy problem (P_ε) has been established as $\varepsilon \to 0$, in the case when A_0 and A_1 are positive operators and $B = 0$. The nonlinear abstract problems of hyperbolic-parabolic type have been studied in [10], [11], [12], [13], [18]. Under some assumptions, closely related to those we use in this article, in [19] and [20] the author analyzed the behavior of solutions to the Cauchy problem for the semi-linear equation $\varepsilon u''(t) + Au'(t) + Bu(t) + f(u) = 0$ in a Hilbert space, as $\varepsilon \to 0$. The coefficients are supposed to be commuting self-adjoint operators and the function f is locally Lipschitz or monotone. The difference of the solution and its singular limit has been estimated. The convergence rate has been established in terms of the small parameter ε. Also the difference of solutions of nonhomogeneous equations with initial data $u(0) = u'(0) = 0$ has been evaluated. All results from these papers were obtained by using the theory of semigroups of linear operators.

Different to other methods, our approach is based on two key points. The first one is the relationship between solutions to the problems (P_ε) and (P_0) in the linear case. The second key point are *a priori* estimates of solutions to the unperturbed problem, which are uniform with respect to the small parameter ε. Moreover, the problem (P_ε) is studied for a larger class of functions f_ε, i. e. $f_\varepsilon \in W^{1,p}(0,T;H)$. We also obtain the convergence rate, as $\varepsilon \to 0$, which depends on p.

Similar results have been established in the work [24], under the same assumptions on the operators A_0 and A_1 and by assuming that the operator B is Lipschitz.

The organization of this paper is as follows. In the next section the theorems of existence and uniqueness of solutions to the problems (P_ε) and (P_0) are presented. In Section 3 we present some *a priori* estimates of these solutions. In Section 4 we present a relationship between solutions to the problem (P_ε) and the corresponding solution to the problem (P_0). The main result of this paper is established in the Section 5. More precisely, we prove the convergence estimates of the difference of solutions and theirs derivatives to the problems (P_ε) and (P_0). At last, an example is given to show the applications of our main result.

In what follows we will need some notations. Let $k \in N^*$, $1 \le p \le +\infty$, $(a,b) \subset (-\infty,+\infty)$ and X be a Banach space. By $W^{k,p}(a,b;X)$ denote the Banach space of vectorial distributions $u \in D'(a,b;X)$, $u^{(j)} \in L^p(a,b;X)$, $j = 0,1,\dots,k$, endowed with the norm

$$\|u\|_{W^{k,p}(a,b;X)} = \left(\sum_{j=0}^{k} \|u^{(j)}\|^p_{L^p(a,b;X)} \right)^{\frac{1}{p}} \quad \text{for} \quad p \in [1,\infty),$$

$$\|u\|_{W^{k,\infty}(a,b;X)} = \max_{0\le j\le k} \|u^{(j)}\|_{L^\infty(a,b;X)} \quad \text{for} \quad p = \infty.$$

In the particular case $p = 2$ we put $W^{k,2}(a,b;X) = H^k(a,b;X)$. If X is a

Hilbert space, then $H^k(a,b;X)$ is also a Hilbert space with the scalar product

$$(u,v)_{H^k(a,b;X)} = \sum_{j=0}^{k} \int_a^b \left(u^{(j)}(t), v^{(j)}(t)\right)_X dt.$$

For $s \in \mathbb{R}$, $k \in \mathbb{N}$ and $p \in [1,\infty]$ define the Banach spaces

$$W_s^{k,p}(a,b;H) = \{f : (a,b) \to H; f^{(l)}(\cdot)e^{-st} \in L^p(a,b;X),\, l = 0,\dots,k\},$$

with the norms

$$\|f\|_{W_s^{k,p}(a,b;X)} = ||fe^{-st}||_{W^{k,p}(a,b;X)}.$$

The framework of our study will be determined by the following conditions:

(H1) *The operator $A_0 : D(A_0) \subseteq H \to H$ is linear, self-adjoint and positive definite, i. e. there exists $\omega_0 > 0$ such that*

$$(A_0 u, u) \geq \omega_0 \, |u|^2, \quad \forall u \in D(A_0);$$

(H2) *The operator $A_1 : D(A_1) \subseteq H \to H$ is linear, self-adjoint, $D(A_0) \subseteq D(A_1)$ and there exists $\omega_1 > 0$ such that*

$$|(A_1 u, u)| \leq \omega_1 \, (A_0 u, u)\,, \quad \forall u \in D(A_0).$$

(HB1) *The operator $B : D(B) \subseteq H \to H$ is $A_0^{1/2}$ locally Lipschitz, i.e. $D(A_0^{1/2}) \subset D(B)$ and for every $R > 0$ there exists $L(R) \geq 0$ such that*

$$|B(u_1) - B(u_2)| \leq L(R)|A_0^{1/2}(u_1 - u_2)|,\, \forall u_i \in D(A_0^{1/2}),\; |A_0^{1/2} u_i| \leq R,\; i = 1,2;$$

(HB2) *The operator B is the Fréchet derivative of some convex and positive functional $\mathcal{B}$ with $D(A_0^{1/2}) \subset D(\mathcal{B})$.*

The hypothesis **(HB2)** implies, in particular, that the operator B is monotone and verifies the condition

$$\frac{d}{dt}\mathcal{B}(u(t)) = \big(B(u(t)), u'(t)\big), \quad \forall t \in [a,b] \subset \mathbb{R}$$

in the case when $u \in C([a,b], D(A_0^{1/2})) \cap C^1([a,b], H)$ (see, for example [26], p. 29).

(HB3) *The operator B possesses the Fréchet derivative B' in $D(A_0^{1/2})$ and there exists $L_1(R) \geq 0$ such that*

$$\big|\big(B'(u_1) - B'(u_2)\big)v\big| \leq L_1(R)\,\big|A_0^{1/2}(u_1 - u_2)\big|\,|A_0^{1/2}v|, \ \ \forall u_1,\, u_2,\, v \in D(A_0^{1/2}),$$

$$|A_0^{1/2} u_i| \leq R, \quad i = 1, 2.$$

In what follows, we present an inequality of Gronwall-Bellman type, which will be used to prove the main results of this work.

Lemma 1.1. *Suppose that $v,\, z,\, h : [a,b] \subset \mathbb{R} \to \mathbb{R}$, $v \in C([a,b])$, $z \in L^2(a,b)$, $h \in L^1(a,b)$, $v(t) \geq 0$ for $t \in [a,b]$ and $z(t) \geq 0$, $h(t) \geq 0$, a. e. $t \in (a,b)$. If*

$$v(t) + \Big(\int_{t_0}^t z^2(s)\,ds\Big)^{1/2}$$

$$\leq c_0 \Big(v(t_0) + \int_{t_0}^t h(s)\,ds\Big) + c_1 \int_{t_0}^t z(s)\,ds, \quad \forall t_0,\, t \in [a,b], \quad t > t_0 \tag{1.1}$$

with $c_0 > 0$, $c_1 > 0$, then

$$v(t) + \Big(\int_a^t z^2(s)\,ds\Big)^{1/2}$$

$$\leq \max\Big\{(2c_0)^{4\,c_1^2\,(t-a)+1},\ (2c_0)^{-4\,c_1^2\,(t-a)+1}\Big\}\Big(v(a) + \int_a^t h(s)\,ds\Big), \quad \forall t \in [a,b]. \tag{1.2}$$

Proof. The inequality (1.1) implies

$$\Big(\int_{t_0}^t z^2(s)\,ds\Big)^{1/2}$$

$$\leq c_0 v(t_0) + c_0 \int_{t_0}^t h(s)\,ds + c_1\,(t - t_0)^{1/2}\,\Big(\int_{t_0}^t z^2(s)\,ds\Big)^{1/2}, t,\, t_0 \in [a,b], t > t_0.$$

If $0 \leq t - t_0 \leq (2\, c_1)^{-2}$, $t,\, t_0 \in [a, b]$, then from this inequality, it follows that

$$\Big(\int_{t_0}^{t} z^2(s)\, d\tau \Big)^{1/2} \leq 2c_0\, v(t_0) + 2c_0 \int_{t_0}^{t} h(s)\, ds.$$

From the last inequality and (1.1), it follows that

$$v(t) + \Big(\int_{t_0}^{t} z^2(s)\, ds \Big)^{1/2}$$

$$\leq 2c_0\, v(t_0) + 2c_0 \int_{t_0}^{t} h(s)\, ds, \quad \forall t,\, t_0 \in [a, b], \quad 0 \leq t - t_0 \leq (2\, c_1)^{-2}. \quad (1.3)$$

Let

$$t_k = a + \frac{k}{(2\, c_1)^2}, \quad k = 0, 1 \ldots, n, \quad t_k \in [a, b].$$

Denote by

$$y(t) = v(t) + \Big(\int_{a}^{t} z^2(s)\, ds \Big)^{1/2}, \quad g(t, t_k) = \int_{t_k}^{t} h(s)\, ds.$$

Then, from (1.3), we get

$$v(t) + \Big(\int_{t_k}^{t} z^2(s)\, ds \Big)^{1/2} \leq 2c_0 \Big(v(t_k) + g(t, t_k) \Big), \quad t \in [t_k, t_{k+1}] \subset [a, b]. \quad (1.4)$$

In particular, from (1.4), it follows that

$$v(t_k) + \Big(\int_{t_{k-1}}^{t_k} z^2(s)\, ds \Big)^{1/2} \leq 2c_0 \Big(v(t_{k-1}) + g(t_k, t_{k-1}) \Big), \quad [t_{k-1}, t_k] \subset [a, b]. \quad (1.5)$$

Using (1.5), we deduce the inequalities

$$y(t_k) \leq c_0 y(t_{k-1}) + c_0 v(t_{k-1}) + 2c_0\, g(t_k, t_{k-1}) \leq \cdots$$

$$\leq c_0^k v(a) + \sum_{j=0}^{k-1} c_0^{k-j} v(t_j) + 2 \sum_{j=0}^{k-1} c_0^{k-j} g(t_{j+1}, t_j), \quad t_k \in [a, b], \quad (1.6)$$

$$v(t_k) \leq 2\, c_0 \Big(v(t_{k-1}) + g(t_k, t_{k-1}) \Big) \leq \cdots$$

$$\leq (2c_0)^k\, v(a) + \sum_{j=0}^{k-1} (2c_0)^{k-j}\, g(t_{j+1}, t_j), \quad t_k \in [a,b]. \tag{1.7}$$

Inequalities (1.6) and (1.7) imply

$$v(t_k) + \Big(\int_a^{t_k} z^2(s)\, ds \Big)^{1/2} \leq (2c_0)^k\, v(a) + \sum_{j=0}^{k-1} (2c_0)^{k-j}\, g(t_{j+1}, t_j)$$

$$\leq (2c_0)^k \Big(v(a) + \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (2c_0)^{-j}\, h(s)\, ds \Big)$$

$$\leq (\max\{2c_0,\, (2c_0)^{-1})^k \Big(v(a) + \int_a^{t_k} h(s)\, ds \Big). \tag{1.8}$$

For each $t \in [a,b]$ there exists $t_k \in (a,b]$ such that $t \in [t_k, t_{k+1}]$ or $t \in (t_{k+1}, b]$ and $b - t_{k+1} < 1/4c_1^2$. Therefore, using (1.3) and (1.8), we obtain

$$v(t) + \Big(\int_a^t z^2(s)\, ds \Big)^{1/2} \leq v(t) + \Big(\int_a^{t_k} z^2(s)\, ds \Big)^{1/2} + \Big(\int_{t_k}^t z^2(s)\, ds \Big)^{1/2}$$

$$\leq (2c_0) v(t_k) + (2c_0) \int_{t_k}^t h(s)\, ds + \Big(\int_a^{t_k} z^2(s)\, ds \Big)^{1/2}$$

$$\leq \max\big\{ (2c_0)^{k+1}, (2c_0)^{-k+1} \big\} \Big(v(a) + \int_a^t h(s)\, ds \Big), \quad t \in [t_k, t_{k+1}].$$

As $k \leq 4\, c_1^2\, (t-a)$ for $t \in [t_k, t_{k+1}]$, from the last inequality, we get (1.2). □

2 Existence of solutions to problems (P_ε) and (P_0)

In this section we will present the results about the solvability of problems (P_ε) and (P_0) and also on the regularity of their solutions. They are not new (see, for example, [1], p. 127) but we formulate and prove them in terms of conditions **(HB1)** - **(HB3)** to specify the properties of smoothness of solutions.

Definition 2.1. *Let $T > 0$ and $f \in L^2(0,T;H)$, $A : D(A) \subseteq H \to H$, $B : D(B) \subseteq H \to H$. The function $u \in L^2(0,T;D(A) \cap D(B))$ with $u' \in L^2(0,T;H)$ and $u'' \in L^2(0,T;H)$ is called strong solution to the Cauchy problem*

$$u''(t) + u'(t) + Au(t) + B\big(u(t)\big) = f(t), \quad \forall t \in (0,T), \tag{2.1}$$

$$u(0) = u_0, \quad u'(0) = u_1, \tag{2.2}$$

if u satisfies the equality (2.1) *in the sense of distributions* a. e. $t \in (0,T)$ *and the initial conditions* (2.2) .

Definition 2.2. *Let $T > 0$ and $f \in L^2(0,T;H)$, $A : D(A) \subseteq H \to H$, $B : D(B) \subseteq H \to H$. The function $v \in L^2(0,T;D(A) \cap D(B))$ with $v' \in L^2(0,T;H)$ is called strong solution to the Cauchy problem*

$$v'(t) + Av(t) + B\big(v(t)\big) = f(t), \quad \forall t \in (0,T), \tag{2.3}$$

$$v(0) = u_0. \tag{2.4}$$

if v verifies the equality (2.3) *in the sense of distributions* a. e. $t \in (0,T)$ *and the initial condition* (2.4).

Theorem 2.1. *Let $T > 0$. Let us assume that the operator $A : D(A) \subset H \to H$ is linear, self-adjoint and positive definite, i. e. there exists $\omega > 0$ such that*

$$(Au, u) \geq \omega |u|^2, \quad \forall u \in D(A), \tag{2.5}$$

and the operator $B : D(B) \subset H \to H$ satisfies **(HB1)** *and* **(HB2)**.

If $u_0 \in D(A)$, $\quad u_1 \in D(A^{1/2})$ and $f \in W^{1,1}(0,T;H)$, then there exists a unique strong solution u to problem (2.1), (2.2), *such that $u \in C^2([0,T];H)$, $A^{1/2}u \in C^1([0,T];H)$, $Au \in C([0,T];H)$.*

If, in addition, $u_1 \in D(A)$, $f(0) - B(u_0) - Au_0 - u_1 \in D(A^{1/2})$, $f \in W^{2,1}(0,T;H)$ and (**HB3**) *is fulfilled, then $A^{1/2}u \in W^{2,\infty}(0,T;H)$ and $u \in W^{3,\infty}(0,T;H)$.*

Proof. Let $\mathcal{H} = D(A^{1/2}) \times H$ be the real Hilbert space endowed with the scalar product

$$(U_1, U_2)_{\mathcal{H}} = \left(A^{1/2}u_1, A^{1/2}u_2\right) + (v_1, v_2), \quad U_i = (u_i; v_i), \quad i = 1, 2. \tag{2.6}$$

Let $\mathcal{L} : \mathcal{V} = D(A) \times D(A^{1/2}) \to \mathcal{H}$ be the operator which is defined by

$$\mathcal{L}U = (-v;\ Au + v), \quad U = (u; v) \in \mathcal{V}. \tag{2.7}$$

Let $\mathcal{F} : D(\mathcal{F}) = \mathbb{R} \times \mathcal{H}$,

$$\mathcal{F}(t, U) = (0;\ -B(u) + \tilde{f}(t)), \quad t \in \mathbb{R}, \quad U = (u; v) \in \mathcal{H},$$

where $\tilde{f} : \mathbb{R} \to H$ is the extension of function f such that $\tilde{f} \in W^{1,1}(\mathbb{R}; H)$ and
$||\tilde{f}||_{W^{1,1}(\mathbb{R};H)} \leq C(T)\, ||f||_{W^{1,1}(0,+\infty;H)}$. We examine the following Cauchy problem in $\mathcal{H}$

$$\begin{cases} U'(t) + \mathcal{L}\, U(t) = \mathcal{F}(t, U), & t \in \mathbb{R}, \\ U(0) = U_0, \end{cases} \tag{2.8}$$

where $U(t) = (u(t); v(t))$, $U_0 = (u_0; u_1)$. Since

$$(\mathcal{L}\, U, U)_{\mathcal{H}} = |v|^2 \geq 0, \quad \forall U = (u; v) \in \mathcal{V}, \tag{2.9}$$

it follows that the operator $\mathcal{L}$ is monotone. We will show that $R(I + \mathcal{L}) \supseteq \mathcal{H}$, from which it will follow that $\mathcal{L}$ is even maximal monotone. Let $G = (g; h) \in \mathcal{H}$ be arbitrary. The equation

$$U + \mathcal{L}\, U = G$$

is equivalent to the system

$$\begin{cases} v = u - g, \\ A\, u + 2\, u = 2\, g + h. \end{cases} \tag{2.10}$$

If $g \in D(A^{1/2})$ and $h \in H$, then the second equation from (2.10) has a unique solution $u \in D(A) \subset D(A^{1/2})$. From the first equation of the system (2.10), it follows that $v \in D(A^{1/2})$. Hence $R(I+\mathcal{L}) \supseteq \mathcal{H}$. Therefore, the operator $\mathcal{L}$ is maximal monotone in $\mathcal{H}$ (see, for example, [1], p. 34). According to Lumer - Phillips's Theorem ([27], p. 58), the operator $-\mathcal{L}$ is an infinitesimal generator of a C_0- semigroup $\{S(t)\,;\, t \geq 0\}$ of contractions on $\mathcal{H}$.

From **(HB1)**, it follows that

$$||\mathcal{F}(t,U_1)-\mathcal{F}(t,U_2)||_{\mathcal{H}} = |B(u_1)-B(u_2)| \leq L(R)\,||U_1-U_2||_{\mathcal{H}}$$

for $U_i=(u_i;v_i)$, $||U_i||_{\mathcal{H}} \leq C(R)$, $i=1,2$. Hence, the mapping $\mathcal{F}$ is locally Lipschitz in $\mathcal{H}$ with respect to the second variable. Then, there exists $a>0$ such that the problem (2.8) has a unique C^0-solution $U \in C([0,a);\mathcal{H})$ (see, for example, [27], p. 183). As $U_0 \in D(\mathcal{L})$ and $\tilde{f} \in W^{1,1}(\mathbb{R};H)$, it follows that this solution is also a classical solution in $[0,a)$. Indeed, let us examine the function

$$v(t) = \int_0^t S(t-s)\,\mathcal{F}(s,U(s))\,ds.$$

For $t \in (0,a)$ and $h>0$, $t+h \in (0,a)$, we have

$$v(t+h)-v(t) = \int_{-h}^0 S(t-s)\,\mathcal{F}(s+h,U(s+h))\,ds$$

$$+\int_0^t S(t-s)\Big(\mathcal{F}(s+h,U(s+h))-\mathcal{F}(s,U(s))\Big)\,ds. \tag{2.11}$$

We observe that the function $\mathcal{F}$ is continuous in $\mathbb{R}\times\mathcal{H}$ and it maps the bounded sets in $\mathbb{R}\times\mathcal{H}$ into bounded sets in $\mathcal{H}$, because

$$||\mathcal{F}(t,U)||_{\mathcal{H}} = |-B(u)+\tilde{f}(t)| \leq |B(0)|+L(R)\,|A^{1/2}u|+|f(0)|+||\tilde{f}||_{W^{1,1}(\mathbb{R};H)}$$

$$\leq C\big(R,||\tilde{f}||_{W^{1,1}(\mathbb{R};H)}\big), \quad U \in D(\mathcal{F}), \quad ||U||_{\mathcal{H}} \leq R, \quad t \in [0,a).$$

Therefore, from (2.11), it follows that

$$||v(t+h)-v(t)||_{\mathcal{H}} \leq \int_0^t \Big(|\tilde{f}(s+h)-\tilde{f}(s)|+|B(u(s+h))-B(u(s))|\Big)\,ds$$

$$+Mh \leq h\left(M+\int_0^{t+h}|\tilde{f}'(s)|ds\right)+L(R)\int_0^t ||U(s+h)-U(s)||_{\mathcal{H}}\,ds, \quad (2.12)$$

where $M=\max\limits_{t\in[0,a],||U(t)||_{\mathcal{H}}\leq R}|\mathcal{F}(t,U)|$. Since $U(t)=S(t)U_0+v(t)$ and

$$||S(t+h)U_0-S(t)U_0||_{\mathcal{H}} \leq ||S(h)U_0-U_0||_{\mathcal{H}} \leq ||\mathcal{L}U_0||_{\mathcal{H}}\,h,$$

from (2.12), we obtain

$$||U(t+h)-U(t)||_{\mathcal{H}} \leq h\left(M+\int_0^{\infty}|\tilde{f}'(s)|ds+||\mathcal{L}U_0||_{\mathcal{H}}\right)$$

$$+L(R)\int_0^t ||U(s+h)-U(s)||_{\mathcal{H}}\,ds.$$

From the last inequality, using Gronwall's Lemma (see, for example, [2], p. 156), we deduce that

$$||U(t+h)-U(t)||_{\mathcal{H}} \leq e^{L(R)\,t}\left(M+\int_0^{\infty}|\tilde{f}'(s)|ds+||\mathcal{L}U_0||_{\mathcal{H}}\right)h, \quad t,t+h\in[0,a).$$

From here, it follows that the function $t\in[0,a)\to U(t)\in\mathcal{H}$ is Lipschitz. As $\tilde{f}\in W^{1,1}(0,+\infty;H)$, then it follows that $\mathcal{F}\in W^{1,1}(0,a;\mathcal{H})$. Because $U_0\in D(\mathcal{L})$, from the equality

$$U(t)=S(t)U_0+\int_0^t S(t-s)\,\mathcal{F}(s,U(s))\,ds,$$

it follows that U is a classical solution to the problem (2.8) in $[0,a)$.

In addition, if for some $a>0$ U is the classical solution to problem (2.6) in $[0,a)$, then, due to **(HB2)**, U is bounded on $[0,a)$. Indeed, from the equality

$$||U(t)||^2_{\mathcal{H}}+2\int_0^t\Big(\mathcal{L}(U(s)),U(s)\Big)_{\mathcal{H}}ds+2\mathcal{B}(u(t))$$

$$=||U_0||^2_{\mathcal{H}}+2\mathcal{B}(u_0)+2\int_0^t\big(\tilde{f}(s),v(s)\big)\,ds, \quad t\in[0,a),$$

it follows that

$$||U(t)||^2_{\mathcal{H}} \leq ||U_0||^2_{\mathcal{H}}+2\mathcal{B}(u_0)+2\int_0^t\big(\tilde{f}(s),v(s)\big)\,ds \quad t\in[0,a).$$

Using Lemma of Brézis, we obtain

$$||U(t)||_{\mathcal{H}} \leq \big(||U_0||_{\mathcal{H}} + 2\big(\mathcal{B}(u_0)\big)^{1/2} + ||\tilde{f}||_{L^1(\mathbb{R};H)}, \quad t \in [0,a),$$

i. e. solution U is bounded on $[0,a)$. This solution is also a C^0-solution in $[0,a)$. Moreover, the function U is a global classical solution to the problem (2.8) (see, for example, [27], p. 183).

Now, we will show that U possesses the right derivative at $t=0$. Let $h>0$. Then we have that

$$\frac{d}{dh}||U(h)-U_0||^2_{\mathcal{H}}$$

$$= -2\big(\mathcal{L}(U(h)) - \mathcal{L}(U_0), U(h)-U_0\big)_{\mathcal{H}} + 2\big(\mathcal{F}(h,U(h)) - \mathcal{L}(U_0), U(h)-U_0\big)_{\mathcal{H}}.$$

From the last equality, using (2.9), we obtain the the inequality

$$||U(h)-U_0||^2_{\mathcal{H}} \leq 2\int_0^h ||\mathcal{F}(s,U(s)) - \mathcal{L}(U_0)||_{\mathcal{H}}\, ||U(s)-U_0||_{\mathcal{H}}\, ds,$$

from which, using Lemma of Brézis, it follows that

$$||U(h)-U_0||_{\mathcal{H}} \leq \int_0^h ||\mathcal{F}(s,U(s)) - \mathcal{L}(U_0)||_{\mathcal{H}}\, ds. \tag{2.13}$$

Since $\mathcal{F}(s,U(s)) \to \mathcal{F}(0,U_0)$ as $s\to 0$ în $\mathcal{H}$, we divide (2.13) on both sides by h and pass to the limit as $h\to 0$. We obtain

$$\limsup_{h\downarrow 0} \frac{1}{h}\, ||U(h)-U_0||_{\mathcal{H}} \leq ||\mathcal{F}(0,U_0) - \mathcal{L}(U_0)||_{\mathcal{H}}. \tag{2.14}$$

As U is the strong solution to the problem (2.8) and the operator $\mathcal{L}$ is monotone, then, for every $z \in D(\mathcal{L})$, we have

$$\frac{1}{2}\, ||U(t)-z||^2_{\mathcal{H}} \leq \frac{1}{2}\, ||U(s)-z||^2_{\mathcal{H}} + \int_s^t \big(\mathcal{F}(\tau,U(\tau)) - \mathcal{L}z, U(\tau)-z\big)_{\mathcal{H}}\, d\tau, \ 0\leq s\leq t,$$

from which it follows that

$$\big(U(h)-U_0, U_0-z\big)_{\mathcal{H}} \leq \frac{1}{2}\, ||U(h)-z||^2_{\mathcal{H}}$$

$$-\frac{1}{2}\,||U_0 - z||_{\mathcal{H}}^2 \le \int_0^h \big(\mathcal{F}(\tau, U(\tau)) - \mathcal{L}z, U(\tau) - z\big)_{\mathcal{H}}\, d\tau, \quad h > 0. \tag{2.15}$$

In virtue of (2.14), there exists a subsequence $h_k \downarrow 0$ such that

$$h_k^{-1}\big(U(h_k) - U_0)\big) \to q, \quad \text{weakly in} \quad \mathcal{H}.$$

Put $h = h_k$ in (2.15), then divide by h_k and, in the obtained inequality, pass to the limit as $h_k \downarrow 0$ to get the following inequality

$$\big(q - \mathcal{F}(0, U_0) + \mathcal{L}z, z - U_0\big)_{\mathcal{H}} \ge 0, \quad \forall z \in D(\mathcal{L}).$$

Since the operator $\mathcal{L}$ is maximal monotone in $\mathcal{H}$, then, from the last inequality, it follows that $q = \mathcal{F}(0, U_0) - \mathcal{L}U_0$ and q does not depend on the subsequence h_k. Since all subsequences $h_k^{-1}\big(U(h_k) - U_0)\big)$ converge in the weak sense to q and these subsequences, due to inequality (2.14), are bounded, it follows that q is a weak limit of the sequence $h^{-1}\big(U(h) - U_0)\big)$. It means that

$$h^{-1}\big(U(h) - U_0)\big) \to \mathcal{F}(0, U_0) - \mathcal{L}U_0, \quad \text{weakly in} \quad \mathcal{H}, \quad h \downarrow 0.$$

From the last relationship and (2.14), it follows that

$$\frac{d^+}{dt}U(0) = \lim_{h\downarrow 0} \frac{1}{h}\big(U(h) - U_0\big) = \mathcal{F}(0, U_0) - \mathcal{L}(U_0).$$

Consequently, we have that $U \in C^1([0,\infty); \mathcal{H})$. It follows that u is the unique strong solution to the problem (2.1), satisfying: $u \in C^2([0,\infty); H)$, $A^{1/2}u \in C^1([0,\infty); H)$ and $u(t) \in D(A)$ for each $t \in [0, +\infty)$. Since

$$\big|B\big(u(t+h)\big) - B\big(u(t)\big)\big| \le L(R)\,\big|A^{1/2}\big(u(t+h) - u(t)\big)\big| \to 0, \quad h \to 0,$$

where $R = \max_{\tau\in[t,t+1]} \big|A^{1/2}u(\tau)\big|$ and for each $t \in [0, +\infty)$

$$||u(t+h) - u(t)|| \le \omega^{-1/2}\,\big|A^{1/2}\big(u(t+h) - u(t)\big)\big| \to 0, \quad h \to 0,$$

then it follows that $B(u) \in C([0, +\infty; H)$. Therefore, from the equation (2.1) it follows that $Au \in C([0, \infty)\,; H)$. Consequently, we conclude that

$$u \in C^2([0,T]; H), \quad A^{1/2}u \in C^1([0,T]; H), \quad Au \in C([0,T]; H).$$

Let, now, $u_1 \in D(A)$, $f(0) - B(u_0) - Au_0 - u_1 \in D(A^{1/2})$, $f \in W^{2,1}(0,T;H)$ and the condition (**HB3**) be fulfilled. Then $\mathcal{F}(0) - \mathcal{L}U_0 \in D(\mathcal{L})$, $\mathcal{F} \in W^{1,1}(0,T;H)$ and

$$U'(t) = S(t)\big(\mathcal{F}(0) - \mathcal{L}U_0\big) + \int_0^t S(t-s)\mathcal{F}'(s)\,ds, \quad t \geq 0.$$

Therefore, for the function $U'_h(t) = U'(t+h) - U'(t)$ the equality

$$U'_h(t) = S(h)(S(t) - I)\big(\mathcal{F}(0) - \mathcal{L}U_0\big) + \int_0^t S(t-s)\mathcal{F}'_h(s)\,ds, \quad t \geq 0 \quad (2.16)$$

is valid and the estimate

$$||S(t)(S(h) - I)\big(\mathcal{F}(0) - \mathcal{L}U_0\big)||_{\mathcal{H}} \leq ||\mathcal{L}\big(\mathcal{F}(0) - \mathcal{L}U_0\big)||_{\mathcal{H}}\, h, \quad (2.17)$$

$$\Big|\Big| \int_0^t S(t-s)\mathcal{F}'_h(s)\,ds \Big|\Big|_{\mathcal{H}} \leq I_1(h) + I_2(t,h), \quad (2.18)$$

holds, where

$$I_1(h) = \int_{-h}^0 \big|\tilde{f}'(s+h) - B'\big(u(s+t)\big)u'u(s+h)\big|\,ds,$$

and

$$I_2(t,h) = \int_0^t \Big(\big|\tilde{f}'_h\big| + \big|\big(B(u(s))\big)'_h\big|\Big)\,ds.$$

Due to (**HB3**), for $I_1(h)$, we have that

$$I_1(h) \leq C_1(h)\,h, \quad (2.19)$$

where

$$C_1(h) = |f'(0)| + ||\tilde{f}''||_{L^1(0,h:H)} + \Big(L_1(R)\,R + ||B'(0)||\omega^{-1/2}\Big) \max_{s\in[0,h]} \big|A^{1/2}u'(s)\big|$$

and $R = \max_{s\in[0,h]} \left|A^{1/2}u(s)\right|$. For $I_2(t,h)$, we have that

$$I_2(t,h) \le ||\tilde{f}''||_{L^1(0,t+h;H)}\, h + C_2(T,h) \int_0^t \left|A^{1/2}u_h'(s)\right| ds$$

$$\le ||\tilde{f}''||_{L^1(0,t+h;H)}\, h + C_2(T,h) \int_0^t \left|\left|U_h'(s)\right|\right|_{\mathcal{H}} ds, \qquad (2.20)$$

where

$$C_2(T,h) = \Big(L_1(R_1)\, R_1 + ||B'(0)||\omega^{-1/2}\Big), \quad R_1 = \left|\left|A^{1/2}u\right|\right|_{C^1([0,T+h];H)}.$$

From (2.16), using the estimates (2.17), (2.18), (2.19) and (2.20), we deduce that

$$\left|\left|U_h'(t)\right|\right|_{\mathcal{H}} \le \Big(||\mathcal{L}\big(\mathcal{F}(0) - \mathcal{L}U_0\big)||_{\mathcal{H}} + C_1(h)$$

$$+||\tilde{f}''||_{L^1(0,t+h;H)}\Big)\, h + C_2(T,h) \int_0^t \left|\left|U_h'(s)\right|\right|_{\mathcal{H}} ds, \quad t \in [0,T].$$

Applying Lemma of Brézis to the last inequality, we get

$$\left|\left|U_h'(t)\right|\right|_{\mathcal{H}} \le \Big(||\mathcal{L}\big(\mathcal{F}(0) - \mathcal{L}U_0\big)||_{\mathcal{H}} + C_1(h) + ||\tilde{f}''||_{L^1(0,t+h;H)}\Big)\, h\, e^{C_2(T,h)\,t},$$

$$t \in [0,T].$$

It follows that the function $U' : [0,T] \to H$ is Lipschitz. Therefore, $U' \in W^{1,\infty}(0,T;\mathcal{H})$. It follows that $A^{1/2}u \in W^{2,\infty}(0,T;H)$ and $u \in W^{3,\infty}(0,T;H)$. □

Theorem 2.2. *Let $T > 0$. Let us assume that the operator $A : D(A) \subset H \to H$ is linear, self-adjoint, positive definite, satisfies condition* (2.5) *and the operator B verifies* **(HB1)** *and* **(HB2)***. If $u_0 \in D(A)$ and $f \in W^{1,1}(0,T;H)$, then there exists a unique strong solution to the problem* (2.3), (2.4), *such that $v \in C^1([0,T];H)$, $Av \in C([0,T];H)$. For this solution the following estimates*

$$\left|\left|v\right|\right|_{C([0,\,t];\,H)} + ||A^{1/2}v||_{L^2(0,\,t;H)} \le C\,\mathbf{M}_0(t), \quad \forall t \in [0,T], \qquad (2.21)$$

$$\big|\big|A^{1/2}v\big|\big|_{C([0,\,t];\,H)} + ||v\,'||_{C([0,\,t];H)} + \big|\big|A^{1/2}v\,'\big|\big|_{L^2(0,\,t;\,H)}$$
$$\leq C(\omega)\,\mathbf{M}_1(t), \quad \forall t \in [0,T], \tag{2.22}$$

are valid, where

$$\mathbf{M}_0(t) = \big|u_0\big| + \int_0^t \big(|f(s)| + |B(0)|\big)\,ds,$$

$$\mathbf{M}_1(t) = \big|Au_0\big| + ||f||_{W^{1,\,1}(0,\,t;\,H)} + |B(0)| + |f(0)|.$$

Proof. First of all we will show that every classical solution to the problem (2.3), (2.4) verifies the estimates (2.21), (2.22). To this end we multiply in H the equation (2.3) by $v(t)$ and then integrate the obtained equality. Taking into account that the operator B is monotone, we obtain

$$|v(t)|^2 + 2\int_0^t \big(Av(s), v(s)\big)\,ds \leq |u_0|^2 + 2\int_0^t \big(f(s) - B(0), v(s)\big)\,ds, \quad t \geq 0.$$

From the last inequality, using Lemma of Brézis, we obtain the estimate (2.21).

To prove the estimate (2.22) denote $v_h(t) = v(t+h) - v(t)$, $h > 0$. Then, as the operator B is monotone, for v_h, we obtain

$$|v_h(t)|^2 + 2\int_0^t \Big(Av_h(s), v_h(s)\Big)\,ds \leq |v_h(0)|^2 + 2\int_0^t \big(f_h(s), v_h(s)\big)\,ds, \quad t \geq 0,$$

from which, using Lemma of Brézis, it follows the inequality

$$|v_h(t)| + \int_0^t \Big(Av_h(s), v_h(s)\Big)\,ds \leq |v_h(0)| + \int_0^t |f_h(s)|\,ds, \quad t \geq 0.$$

Divide the last inequality by h and pass to the limit as $h \to 0$ in the obtained inequality, in addition, using Fatou's Lemma, we obtain

$$||v\,'||_{C([0,\,t];H)} + \big|\big|A^{1/2}v\,'\big|\big|_{L^2(0,\,t;\,H)} \leq \mathbf{M}_1(t), \quad t \in [0,T]. \tag{2.23}$$

Multiplying scalar in H the equation (2.3) by v, using (2.23) and the fact that the operator B is monotone, we get

$$\big|A^{1/2}v(t)\big|^2 \leq \big(f(t),v(t)\big) - \big(v'(t),v(t)\big) - \big(B(0),v(t)\big)$$

$$\leq |v(t)|\,\big(|f(t)|+|B(0)|+|v'(t)|\big) \leq C\,\omega^{-1/2}\big|A^{1/2}v(t)\big|\,\mathbf{M}_1(t), \quad t\in[0,T].$$

From the last estimate and (2.23) the estimate (2.22) follows.

Let us prove the solvability of the problem (2.3), (2.4). Let $\{S(t);t\geq 0\}$ be the C_0-semigroup of linear operators with the infinitesimal generator $-A$. Let $\tilde{f}$ be the extension of function f on $\mathbb{R}$, which is defined in Theorem 2.1, and $\mathcal{F}(t,v(t))=\tilde{f}(t)-B\big(v(t)\big)$. Similarly, as in Theorem 2.1 it is proved that $\mathcal{F}$ is a locally Lipschitz function in H with respect to the second variable, $\mathcal{F}$ is continuous on $\mathbb{R}\times H$ and maps the bounded sets in $\mathbb{R}\times H$ into bounded sets in H. Therefore, the proof of Theorem 2.2 follows the very same way as the proof of Theorem 2.1. □

3 A priori estimates for solutions to the problem (P_ε)

In what follows, we will give some *a priori* estimates of solutions to the problem

$$\varepsilon\, u_\varepsilon''(t)+u_\varepsilon'(t)+Au_\varepsilon(t)+B\big(u_\varepsilon(t)\big)=f(t), \quad t\in(0,T), \tag{3.1}$$

$$u_\varepsilon(0)=u_0, \quad u_\varepsilon'(0)=u_1, \tag{3.2}$$

in the case when the operator B is monotone. These estimates will be uniform with respect to the small ε and will be used to study the behavior of solutions to the problem (P_ε) when $\varepsilon\to 0$.

Lemma 3.1. *Let us assume that the operator $A:D(A)\subset H\to H$ is linear, self-adjoint, positive definite, satisfies* (2.5) *and the operator B verifies*

(HB1) *and* **(HB2)**. *If* $u_0 \in D(A), u_1 \in D(A^{1/2})$ *and* $f \in W^{1,1}(0,\infty;H)$, *then there exists* $C = C(\omega) > 0$ *such that for every strong solution* u_ε *to the problem* (3.1), (3.2), *the estimates*

$$||A^{1/2}u_\varepsilon||_{C([0,t];H)} + ||u_\varepsilon'||_{L^2(0,t;H)} + \Big(\mathcal{B}\big(u_\varepsilon(t)\big)\Big)^{1/2} \le \mathbf{m},\ \forall \varepsilon \in (0,1],\ \forall t \ge 0, \tag{3.3}$$

$$\varepsilon||u_\varepsilon''||_{C([0,\,t];H)} + ||u_\varepsilon'||_{C([0,\,t];H)} + \big|\big|A^{1/2}u_\varepsilon'\big|\big|_{L^2(0,\,t;\,H)}$$
$$\le C\, e^{12\,L^2(\mathbf{m})\,t}\,\mathbf{m_1}, \quad \forall \varepsilon \in (0,1], \quad \forall t \ge 0, \tag{3.4}$$

$$||Au_\varepsilon(t)||_{C([0,t];H)} \le C\,\mathbf{m_2}\, e^{\big(6\,L^2(\mathbf{m})+1\big)\,t}, \quad \forall \varepsilon \in (0,1/2], \quad \forall t \ge 0, \tag{3.5}$$

are valid, where

$$\mathbf{m} = |A^{1/2}u_0| + |u_1| + |\mathcal{B}(u_0)|^{1/2} + ||f||_{L^2(0,\infty;H)},$$

$$\mathbf{m_1} = |Au_0| + |A^{1/2}u_1| + |B(u_0)| + |\mathcal{B}(u_0)|^{1/2} + ||f||_{W^{1,1}(0,\,\infty;H)},$$

$$\mathbf{m_2} = \big(L(\mathbf{m}) + 1\big)\mathbf{m_1}.$$

If $B = 0$, *then, in* (3.3), (3.4) *and* (3.5), $L(\mathbf{m}) = 0$, $\mathbf{m_2} = \mathbf{m_1}$,

$\mathbf{m} = |A^{1/2}u_0| + |u_1| + ||f||_{L^2(0,\infty;H)}\big)$, $\mathbf{m_1} = |Au_0| + |A^{1/2}u_1| + ||f||_{W^{1,1}(0,\,\infty;H)}$.

Proof. Proof of the estimate (3.3). Denote by

$$E_0(u,t) = \varepsilon|u'(t)|^2 + \Big(Au(t), u(t)\Big) + 2\int_0^t |u'(\tau)|^2 d\tau + 2\mathcal{B}(u(t)).$$

Using Theorem 2.1, by direct computations, we obtain that, for every strong solution u_ε to the problem (3.2), the equality

$$\frac{d}{dt}E_0(u_\varepsilon, t) = 2\Big(f(t), u_\varepsilon'(t)\Big), \quad \forall t \ge 0$$

holds. Integrating this equality, we get

$$E_0(t, u_\varepsilon) \le E_0(u_\varepsilon, 0) + \int_0^t |f(s)|\,|u_\varepsilon'(s)|\,ds, \quad \forall t \ge 0. \tag{3.6}$$

If $f \in W^{1,1}(0,\infty;H)$, then $f \in L^p(0,\infty;H)$, $p \in [1,\infty]$ and

$$||f||_{L^p(0,\infty;H)} \leq C(p)\,||f||_{W^{1,1}(0,\infty;H)}. \tag{3.7}$$

Therefore, from (3.7), via to Hölder's inequality, it follows the estimate

$$||A^{1/2}u_\varepsilon||_{C([0\,,t;\,H)} + ||u_\varepsilon||_{L^2(0\,,t;\,H)} + \Big(\mathcal{B}\big(u_\varepsilon(t)\big)\Big)^{1/2}$$

$$\leq E_0^{1/2}(u_\varepsilon,0) + ||f||_{L^2(0\,,t;\,H)} + |\mathcal{B}(u_0)|^{1/2}, \quad \forall \varepsilon \in (0,1], \quad \forall t \geq 0,$$

from which we get the estimate (3.3).

Proof of the estimate (3.4). Denote by $u_{\varepsilon h}(t) = u_\varepsilon(t+h) - u_\varepsilon(t)$, $\forall h > 0$, $\forall t \geq 0$ and

$$E(u,t) = \varepsilon^2|u'(t)|^2 + \frac{1}{2}|u(t)|^2 + \varepsilon\Big(Au(t),u(t)\Big) + \varepsilon\int_0^t |u'(\tau)|^2 d\tau$$

$$+\varepsilon\Big(u(t),u'(t)\Big) + \int_0^t \Big(Au(\tau),u(\tau)\Big)d\tau. \tag{3.8}$$

For every strong solution u_ε to (3.2), the equality

$$\frac{d}{dt}E(u_{\varepsilon h},t) = \big(2\varepsilon u'_{\varepsilon h}(t) + u_{\varepsilon h}(t), f_h(t) - (B(u_\varepsilon(t)))_h\big), \quad \forall t > 0 \tag{3.9}$$

holds. According to **(HB1)** and (3.3), we have that

$$\big|(B(u_\varepsilon(t)))_h\big| = \big|B(u_\varepsilon(t+h)) - B(u_\varepsilon(t))\big| \leq L(\mathbf{m})\,\big|A^{1/2}u_{\varepsilon h}(t)\big|$$

and

$$|2\varepsilon u'_{\varepsilon h} + u_{\varepsilon h}(t)| \leq 2(E(u_{\varepsilon h},t))^{1/2}.$$

Integrating the equality (3.9) on (t_0,t), we obtain

$$E(u_{\varepsilon h},t)$$

$$\leq E(u_{\varepsilon h},t_0) + 2\int_{t_0}^t \Big(|f_h(\tau)| + L(\mathbf{m})\big|A^{1/2}u_{\varepsilon h}(\tau)\big|\Big)\, E^{1/2}(u_{\varepsilon h},\tau)\,d\tau,\ t > t_0 \geq 0.$$

From the last inequality, using Lemma of Brézis and Lemma 1.1, we get

$$|u_{\varepsilon h}(t)| + \Big(\int_0^t \big|A^{1/2}u_{\varepsilon h}(\tau)\big|^2 d\tau\Big)^{1/2}$$

$$\leq C\, e^{4\,L^2(\mathbf{m})\,t}\Big(E^{1/2}(u_{\varepsilon h},0) + \int_0^t |f_h(\tau)|\,d\tau\Big), \quad \forall t \geq 0. \tag{3.10}$$

To obtain the estimate (3.4), divide (3.10) by h, then pass to the limit as $h \downarrow 0$.

Proof of the estimate (3.5). Let A_λ be the Yosida approximation of operator A. Let us define

$$E_1(u,t) = \varepsilon\Big(A_\lambda u'(t), u'(t)\Big) + \Big(A_\lambda u(t), u(t)\Big)$$

$$+\Big(A_\lambda u(t), Au(t)\Big) + 2\varepsilon\Big(A_\lambda u(t), u'(t)\Big)$$

$$+2(1-\varepsilon)\int_0^t \Big(A_\lambda u'(\tau), u'(\tau)\Big)d\tau + 2\int_0^t \Big(A_\lambda u(\tau), Au(\tau)\Big)d\tau. \tag{3.11}$$

Due to Theorem 2.1, by direct computations, for every strong solution u_ε to the problem (3.2), we get

$$\frac{d}{dt}E_1(u_\varepsilon,t) = 2\Big(f(t) - B\big(u_\varepsilon(t)\big), A_\lambda u_\varepsilon(t) + A_\lambda u_\varepsilon'(t)\Big), \quad \forall t > 0.$$

Integrating this equality, we obtain

$$E_1(u_\varepsilon,t)$$

$$= E_1(u_\varepsilon,0) + 2\int_0^t \Big(f(\tau) - B\big(u_\varepsilon(\tau)\big), A_\lambda u_\varepsilon(\tau) + A_\lambda u_\varepsilon'(\tau)\Big)\,d\tau, \quad \forall t \geq 0. \tag{3.12}$$

Due to **(HB2)** and (3.4), for every $t > 0$, we have that $B(u_\varepsilon) \in W^{1,2}(0,t;H)$ and

$$\int_0^t \Big|\Big(B\big(u_\varepsilon(\tau)\big)\Big)'\Big|^2 d\tau \leq L^2(\mathbf{m}) \int_0^t \big|A^{1/2}u_\varepsilon'(\tau)\big|^2\,d\tau$$

$$\leq C\,L^2(\mathbf{m})\,e^{4\,L^2(\mathbf{m})\,t}\,\mathbf{m_1}^2, \quad \forall \varepsilon \in (0,1], \quad \forall t \geq 0. \tag{3.13}$$

Hence, $B(u_\varepsilon) \in W^{1,1}(0,t;H)$ for every $t > 0$ and the function $t \in [0,\infty) \to B\big(u_\varepsilon(t)\big) \in H$ is absolutely continuous. Then

$$\int_0^t \Big(B\big(u_\varepsilon(\tau)\big), A_\lambda u_\varepsilon'(\tau)\Big)\, d\tau = \Big(B\big(u_\varepsilon(t)\big), A_\lambda u_\varepsilon(t)\Big)$$

$$-\Big(B\big(u_0\big), A_\lambda u_0\Big) - \int_0^t \Big(\big(B\big(u_\varepsilon(\tau)\big)\big)', A_\lambda u_\varepsilon(\tau)\Big)\, d\tau$$

and the equality (3.12) will take the form

$$E_1(u_\varepsilon,t) = E_1(u_\varepsilon,0) + I_1(t,\varepsilon) + I_2(t,\varepsilon) + I_3(t,\varepsilon), \quad \forall t \geq 0, \tag{3.14}$$

where

$$I_1(t,\varepsilon) = 2\Big(f(t) - B\big(u_\varepsilon(t)\big), A_\lambda u_\varepsilon(t)\Big) - 2\Big(f(0) - B(u_0), A_\lambda u_0\Big),$$

$$I_2(t,\varepsilon) = 2\int_0^t \Big(f(\tau) - f'(\tau) - B\big(u_\varepsilon(\tau)\big), A_\lambda u_\varepsilon(\tau)\Big) d\tau,$$

$$I_3(t,\varepsilon) = 2\int_0^t \Big(\big(B\big(u_\varepsilon(\tau)\big)\big)', A_\lambda u_\varepsilon(\tau)\Big) d\tau.$$

Using **(HB1)**, (3.3) and proprieties of the Yosida approximation ([1], p. 99), for $I_1(t,\varepsilon)$, we obtain

$$\big|I_1(t,\varepsilon)\big| \leq \frac{1}{2}\big|A_\lambda u_\varepsilon(t)\big|^2 + L^2(\mathbf{m})\big(\big|A^{1/2}u_\varepsilon(t)\big|^2 + \big|A^{1/2}u_0\big|\big)$$

$$+C\left(|Au_0|^2 + |B(u_0)|^2 + ||f||^2_{W^{1,\,1}(0,\,\infty;\,H)}\right)$$

$$\leq \frac{1}{2}\Big(A_\lambda u_\varepsilon(t), Au_\varepsilon(t)\Big) + C\,\mathbf{m_2}^2, \quad \forall \varepsilon \in (0,1/2], \quad \forall t \geq 0. \tag{3.15}$$

Due to **(HB2)**, (3.3) and the properties of Yosida approximation, we have that

$$\Big|B\big(u_\varepsilon(\tau)\big)\Big| \leq \big|B(u_0)\big| + L(\mathbf{m})\left(\big|A^{1/2}u_\varepsilon(\tau)\big| + \big|A^{1/2}u_0\big|\right), \quad \forall \tau \geq 0,$$

and

$$E_1(u_\varepsilon,t) \geq 0, \quad \big|A_\lambda u_\varepsilon(t)\big| \leq E_1^{1/2}(u_\varepsilon,t), \quad \forall \varepsilon \in (0,1], \quad \forall t \geq 0.$$

Therefore, for $I_2(t,\varepsilon)$, we obtain

$$\big|I_2(t,\varepsilon)\big| \le \int_0^t k(\tau)E_1^{1/2}(u_\varepsilon,\tau)\,d\tau, \quad \forall\varepsilon\in(0,1], \quad \forall t\ge 0, \tag{3.16}$$

where

$$k(\tau)=|f(\tau)|+|f'(\tau)|+L(\mathbf{m})\left(\big|A^{1/2}u_\varepsilon(\tau)\big|+\big|A^{1/2}u_0\big|\right)+\big|B(u_0)\big|.$$

Using the estimate (3.3), for $k(\tau)$, we get

$$\int_0^t k(\tau)\,d\tau \le C\left(1+t\,L(\mathbf{m})\right)\mathbf{m_1}, \quad \forall\varepsilon\in(0,1/2], \quad \forall t\ge 0. \tag{3.17}$$

Using the estimate (3.13) and the properties of Yosida approximation, for $I_3(t,\varepsilon)$, we obtain

$$\begin{aligned}\big|I_3(t,\varepsilon)\big| \le \int_0^t &\left(A_\lambda u_\varepsilon(\tau),Au_\varepsilon(\tau)\right)d\tau\\ +C\,L^2(\mathbf{m})\,e^{4\,L^2(\mathbf{m})\,t}\,{\mathbf{m_1}}^2,& \quad \forall\varepsilon\in(0,1], \quad \forall t\ge 0.\end{aligned} \tag{3.18}$$

Using the properties of Yosida approximation, for $E_1(u_\varepsilon,0)$, we get

$$E_1(u_\varepsilon,0)\le C\left(\big|A^{1/2}u_1\big|^2+\big|Au_0\big|^2\right), \quad \forall\varepsilon\in(0,1]. \tag{3.19}$$

Hence, from (3.14), using the estimates (3.15), (3.16) and (3.18), we get

$$E_1(u_\varepsilon,t)\le C\left({\mathbf{m_2}}^2\,e^{4\,L^2(\mathbf{m})\,t}+\int_0^t k(\tau)E_1^{1/2}(u_\varepsilon,\tau)d\tau\right),\ \forall\varepsilon\in(0,1/2],\ \forall t\ge 0.$$

From this inequality, using Lemma of Brézis and the estimate (3.17), we obtain the inequality

$$E_1^{1/2}(u_\varepsilon,t)\le \mathbf{m_2}\,e^{\left(2L^2(\mathbf{m})+1\right)\,t}, \quad \forall\varepsilon\in(0,1/2], \quad \forall t\ge 0,$$

from which it follows that

$$\left(\mathcal{A}_\lambda u_\varepsilon(t),Au_\varepsilon(t)\right)\le C\,{\mathbf{m_2}}^2\,e^{2\left(2L^2(\mathbf{m})+1\right)\,t}, \quad \forall\varepsilon\in(0,1/2], \quad \forall t\ge 0.$$

Finally, passing to the limit in the last inequality as $\lambda\to 0$ and using the properties of Yosida approximation, we obtain the estimate (3.5). $\square$

Let u_ε be the strong solution to the problem (3.1), (3.2) and let us denote by

$$z_\varepsilon(t) = u_\varepsilon'(t) + he^{-t/\varepsilon}, \quad h = f(0) - u_1 - Au_0 - B(u_0). \tag{3.20}$$

Lemma 3.2. *Let us assume that the operator* $A : D(A) \subset H \to H$ *is linear, self-adjoint, positive definite, verifies* (2.5) *and the operator* B *verifies* **(HB1)**, **(HB2)** *and* **(HB3)**. *If* $u_0, u_1, h \in D(A)$ *and* $f \in W^{2,1}(0,\infty; H)$, *then for* z_ε, *defined by* (3.20), *the estimates*

$$||A^{1/2} z_\varepsilon||_{C([0,\,t];\,H)} + ||z_\varepsilon'||_{C([0,\,t];\,H)} + \big|\big|A^{1/2} z_\varepsilon'\big|\big|_{L^2(0,\,t;\,H)}$$

$$\leq C\, \mathbf{m_3}\, e^{\gamma t}, \quad \forall \varepsilon \in (0,1], \quad \forall t \geq 0, \tag{3.21}$$

are valid, where $\gamma = \gamma(\mathbf{m}) = 12\Big(L^2(\mathbf{m}) + \big[\mathbf{m}\, L_1(\mathbf{m}) + ||B'(0)||\, \omega^{-1/2}\big]^2\Big)$, $C = C(\omega, ||B'(0)||)$ *and*

$$\mathbf{m_3} = ||f||_{W^{2,1}(0,\infty;H)} + |Ah| + L_1(\mathbf{m})\, \mathbf{m_1}\big(1 + \big|A^{1/2}h\big| + \big|A^{1/2}u_0\big|\big).$$

If $B = 0$, *then* $h = f(0) - Au_0 - u_1$ *and*

$$||A^{1/2} z_\varepsilon||_{C([0,\,t];\,H)} + ||z_\varepsilon'||_{C([0,\,t];\,H)} + \big|\big|A^{1/2} z_\varepsilon'\big|\big|_{L^2(0,\,t;\,H)}$$

$$\leq C\left(\big|A(h+u_1)\big| + ||f||_{W^{2,1}(0,t;H)}\right), \quad \forall \varepsilon \in (0,1], \quad \forall t \geq 0.$$

Proof. Under the conditions of this lemma $\big(B(u_\varepsilon)\big)' \in W^{1,1}(0,T;H)$ for $\varepsilon \in (0,1]$, where u_ε is solution to the problem (3.2). Indeed, by Theorem 2.1, $u_\varepsilon \in W^{3,\infty}(0,T;H)$ and $A^{1/2}u_\varepsilon \in W^{2,\infty}(0,T;H)$. Therefore, using **(HB3)** and Lemma 3.1, we deduce

$$\big|\big(B(u_\varepsilon(t))\big)'\big| = \big|B'(u_\varepsilon(t))\, u_\varepsilon'(t)\big|$$

$$\leq \Big(L(\mathbf{m})\,\mathbf{m} + \omega^{-1/2}\, ||B'(0)||\Big)\big|A^{1/2}u_\varepsilon'(t)\big|, \quad \forall t \in [0,T]. \tag{3.22}$$

For $h > 0$ and $t, t+h \in [0,T]$, we have that

$$\Big|h^{-1}\left(\big(B(u_\varepsilon(t))\big)'\right)_h\Big|$$

$$\leq \Big|h^{-1}\Big(B'\big(u_\varepsilon(t+h)\big) - B'\big(u_\varepsilon(t)\big)\Big)\, u'_\varepsilon(t+h)\Big| + \Big|h^{-1}B'\big(u_\varepsilon(t)\big)\, u'_{\varepsilon h}(t)\Big|$$
$$\leq L_1(\mathbf{m})\,\big|h^{-1}\, A^{1/2}u_{\varepsilon h}(t)\big|\,\big|A^{1/2}u'_\varepsilon(t+h)\big|$$
$$+\Big(L_1(\mathbf{m})\,\mathbf{m} + \omega^{-1/2}\,||B'(0)||\Big)\,\big|h^{-1}\,A^{1/2}u'_{\varepsilon h}\big|, \quad \forall t \in [0, T-h], \qquad (3.23)$$

where

$$u_{\varepsilon h}(t) = u_\varepsilon(t+h) - u_\varepsilon(t), \quad \forall h > 0, \quad \forall t \in [0, T-h).$$

Then we can state that $\big(B(u_\varepsilon)\big)' \in W^{1,2}(0,T;H)$ (see, for example [1], p. 34). So $\big(B(u_\varepsilon)\big)' \in W^{1,1}(0,T;H)$ for every $T > 0$. Consequently, the functional $\mathcal{F}(t,\varepsilon) = f'(t) - \Big(B\big(u_\varepsilon(t)\big)\Big)' + e^{-t/\varepsilon}\, A\alpha$ belongs to $W^{1,1}(0,T;H)$ for each $T > 0$. Thus, according to Theorem 2.1, the function z_ε, defined by (3.20), is a strong solution to the problem

$$\begin{cases} \varepsilon z''_\varepsilon(t) + z'_\varepsilon(t) + Az_\varepsilon(t) = \mathcal{F}(t,\varepsilon), \quad \text{a. e.} \quad t \in (0,T), \\ z_\varepsilon(0) = v_1 + \alpha, \quad z'_\varepsilon(0) = 0, \end{cases} \qquad (3.24)$$

where

$$\mathcal{F}(t,\varepsilon) = f'(t) - \Big(B\big(u_\varepsilon(t)\big)\Big)' + e^{-t/\varepsilon}\, A\alpha \qquad (3.25)$$

and possesses the following regularity properties

$$z_\varepsilon \in C^2([0,\infty);H), \quad A^{1/2}z_\varepsilon \in C^1([0,\infty);H), \quad Az_\varepsilon \in C([0,\infty);H).$$

Let $h > 0$, $z_{\varepsilon h}(t) = z_\varepsilon(t+h) - z_\varepsilon(t)$ and let the functional $E(u,t)$ be defined by (3.8). By the direct computations, we obtain

$$\frac{d}{dt}E(z_{\varepsilon h}, t) = \Big(\mathcal{F}_h(t,\varepsilon), z_{\varepsilon h}(t) + 2\varepsilon z'_{\varepsilon h}(t)\Big), \quad \text{a. e.} \quad t \in (0, T-h). \quad (3.26)$$

Using **(HB1)**, **(HB3)** and (3.3), we get

$$\Big|\Big(\Big(\big(B\big(u_\varepsilon(t)\big)\big)'\Big)_h\Big| \leq \gamma_0\big|A^{1/2}z_{\varepsilon h}(t)\big| + k(t,h,\varepsilon),$$

where $\gamma_0 = \mathbf{m}\, L_1(\mathbf{m}) + ||B'(0)||\,\omega^{-1/2}$ and

$$k(t,h,\varepsilon) = L_1(\mathbf{m})\,\big|A^{1/2}u_{\varepsilon h}(t)\big|\,\big|A^{1/2}u'_\varepsilon(t+h)\big|$$

$$+\Big(\mathbf{m}L_1(\mathbf{m})\left|A^{1/2}\alpha\right| + ||B'(0)||\,|\alpha|\Big)\left(e^{-t/\varepsilon}\right)_h.$$

As

$$|z_{\varepsilon h}(t) + 2\varepsilon z'_{\varepsilon h}(t)| \leq 2v(t),$$

where

$$v^2(t) = \varepsilon^2 \left|z'_{\varepsilon h}(t)\right|^2 + \frac{1}{2}\left|z_{\varepsilon h}(t)\right|^2 + \varepsilon\left(Az_{\varepsilon h}(t), z_{\varepsilon h}(t)\right) + \varepsilon\left(z_{\varepsilon h}(t), z'_{\varepsilon h}(t)\right),$$

integrating the equality (3.26) on (t_0, t), we obtain

$$v^2(t) + \int_{t_0}^t \left(Az_{\varepsilon h}(s), z_{\varepsilon h}(s)\right) ds$$

$$\leq v^2(t_0) + 2\int_{t_0}^t \left(k_1(s,h,\varepsilon) + \gamma_0\left|A^{1/2}z_{\varepsilon h}(s)\right|\right) v(s)\, ds, \quad t > t_0 \geq 0, \quad (3.27)$$

where

$$k_1(t,h,\varepsilon) = k(t,h,\varepsilon) + |f'_h(t)| + \left(e^{-t/\varepsilon}\right)_h |A\alpha|.$$

Applying Lemma of Brézis to the inequality (3.27), we get

$$v(t) + \Big(\int_{t_0}^t \left(Az_{\varepsilon h}(s), z_{\varepsilon h}(s)\right) ds\Big)^{1/2}$$

$$\leq v(t_0) + \int_{t_0}^t k_1(s,h,\varepsilon)\, ds + \gamma_0 \int_{t_0}^t \left|A^{1/2}z_{\varepsilon h}(s)\right| ds, \quad t > t_0 \geq 0. \quad (3.28)$$

Applying Lemma 1.1 to the inequality (3.28), we deduce that

$$v(t) + \Big(\int_0^t \left|A^{1/2}z_{\varepsilon h}(s)\right|^2 ds\Big)^{1/2}$$

$$\leq 2\, e^{4\gamma_0^2 t}\Big(v(0) + \int_0^t k_1(s,h,\varepsilon)\, ds\Big), \quad \forall t \geq 0. \quad (3.29)$$

Due to (3.4), we get

$$\int_0^t h^{-1}\, k_1(s,h,\varepsilon)\, ds \leq C\, e^{4\, L^2(\mathbf{m})\, t}\, \mathbf{m_3}, \quad \forall \varepsilon \in (0,1], \quad \forall t \geq 0.$$

Then, from (3.29), it follows that

$$h^{-1}\,|z_{\varepsilon h}| + h^{-1}\Big(\int_0^t \big|A^{1/2} z_{\varepsilon h}(\tau)\big|^2\, d\tau\Big)^{1/2}$$

$$\leq C\, e^{\gamma\, t}\Big(h^{-1}\, E^{1/2}(z_{\varepsilon h},0) + \mathbf{m_3}\Big), \quad \forall \varepsilon \in (0,1], \quad \forall t \geq 0. \tag{3.30}$$

Next we calculate the limits

$$\lim_{h\downarrow 0} h^{-2} E(z_{\varepsilon h},0) = |f'(0) - B'(u_0)\, u_1 - A(\alpha + u_1)|^2,$$

$$\lim_{h\downarrow 0} h^{-2}\int_0^t \big|A^{1/2} z_{\varepsilon h}(\tau)\big|^2\, d\tau = \int_0^t \big|A^{1/2} z'_\varepsilon(\tau)\big|^2\, d\tau.$$

Passing to the limit in (3.28) as $h \downarrow 0$ and using the last two relationships, we get

$$||z'_\varepsilon||_{C([0,t];H)} + ||A^{1/2} z'_\varepsilon||_{L^2(0,t;H)} \leq C\, e^{\gamma\, t}\, \mathbf{m_3}, \quad \forall \varepsilon \in (0,1], \quad \forall t \geq 0. \tag{3.31}$$

In what follows, we denote by

$$\mathcal{E}(u,t) = \varepsilon |u'(t)|^2 + |u(t)|^2 + \Big(Au(t), u(t)\Big) + 2(1-\varepsilon)\int_0^t |u'(\tau)|^2 d\tau$$

$$+2\varepsilon\Big(u(t), u'(t)\Big) + 2\int_0^t \Big(Au(\tau), u(\tau)\Big) d\tau. \tag{3.32}$$

Then we have

$$\frac{d}{dt}\mathcal{E}(z_\varepsilon, t) = 2\Big(\mathcal{F}(t,\varepsilon), z_\varepsilon(t) + z'_\varepsilon(t)\Big), \quad \text{a. e.} \quad t \geq 0.$$

Integrating the last equality, we obtain

$$\mathcal{E}(z_\varepsilon, t) = \mathcal{E}(z_\varepsilon, 0) + 2\int_0^t \Big(\mathcal{F}(s,\varepsilon), z_\varepsilon(s) + z'_\varepsilon(s)\Big)\, ds, \quad \forall t \geq 0. \tag{3.33}$$

Taking into account (3.20), **(HB3)** and (3.3), (3.4), (3.31), we get

$$\int_0^t \Big|\Big(\mathcal{F}(s,\varepsilon), z_\varepsilon(s) + z'_\varepsilon(s)\Big)\Big|\, ds$$

$$\leq \int_0^t \Big(\mathbf{m}\, L_1(\mathbf{m})\, \big|A^{1/2} u_\varepsilon'(s)\big| + |f'(s)| + |A\alpha|\, e^{-s/\varepsilon}\Big)\times$$

$$\times\Big(|u_\varepsilon'(s)| + |\alpha|\, e^{-s/\varepsilon} + |z_\varepsilon'(s)|\Big)\, ds \leq C\, e^{2\,\gamma\, t}\, \mathbf{m_3}^2, \forall \varepsilon \in (0,1], \forall t \geq 0. \quad (3.34)$$

For $\mathcal{E}(z_\varepsilon, 0)$ we have the estimate

$$\mathcal{E}(z_\varepsilon, 0) \leq |\alpha + u_1|^2 + \big|A^{1/2}(\alpha + u_1)\big|^2 \leq C\, \big|A^{1/2}(\alpha + u_1)\big|^2. \quad (3.35)$$

From (3.30), using the estimates (3.34) and (3.35), we deduce that

$$\big|A^{1/2} z_\varepsilon\big|_{C([0,t];H)} \leq C\, e^{\gamma\, t}\, \mathbf{m_3}, \quad \forall \varepsilon \in (0,1], \quad \forall t \geq 0. \quad (3.36)$$

From estimates (3.31), (3.36), the estimate (3.21) follows. □

4 The relationship between the solutions to the problems (P_ε) and (P_0) in the linear case

Now we are going to present the relationship between the solutions to the problem (P_ε) and the corresponding solutions to the problem (P_0) in the linear case, i. e. $B = 0$. This relationship was established in the work [21]. To this end we define the kernel of transformation which realizes this relationship.

For $\varepsilon > 0$, let us denote by

$$K(t,\tau,\varepsilon) = \frac{1}{2\sqrt{\pi}\varepsilon}\Big(K_1(t,\tau,\varepsilon) + 3K_2(t,\tau,\varepsilon) - 2K_3(t,\tau,\varepsilon)\Big),$$

where

$$K_1(t,\tau,\varepsilon) = \exp\Big\{\frac{3t - 2\tau}{4\varepsilon}\Big\}\lambda\Big(\frac{2t - \tau}{2\sqrt{\varepsilon t}}\Big),$$

$$K_2(t,\tau,\varepsilon) = \exp\Big\{\frac{3t + 6\tau}{4\varepsilon}\Big\}\lambda\Big(\frac{2t + \tau}{2\sqrt{\varepsilon t}}\Big),$$

$$K_3(t,\tau,\varepsilon) = \exp\Big\{\frac{\tau}{\varepsilon}\Big\}\lambda\Big(\frac{t + \tau}{2\sqrt{\varepsilon t}}\Big), \quad \lambda(s) = \int_s^\infty e^{-\eta^2}\, d\eta.$$

The properties of the kernel $K(t,\tau,\varepsilon)$ are collected in the following lemma.

Lemma 4.1. *The function $K(t,\tau,\varepsilon)$ possesses the following properties:*

(i) $K \in C([0,\infty)\times[0,\infty))\cap C^2((0,\infty)\times(0,\infty))$;

(ii) $K_t(t,\tau,\varepsilon) = \varepsilon K_{\tau\tau}(t,\tau,\varepsilon) - K_\tau(t,\tau,\varepsilon), \quad \forall t>0, \quad \forall \tau>0$;

(iii) $\varepsilon K_\tau(t,0,\varepsilon) - K(t,0,\varepsilon) = 0, \quad \forall t \geq 0$;

(iv) $K(0,\tau,\varepsilon) = \dfrac{1}{2\varepsilon}\exp\Big\{-\dfrac{\tau}{2\varepsilon}\Big\}, \quad \forall \tau \geq 0$;

(v) *For every fixed $t>0$ and every $q,s\in\mathbb{N}$, there exist constants $C_1(q,s,t,\varepsilon) > 0$ and $C_2(q,s,t)>0$ such that*

$$\big|\partial_t^s\partial_\tau^q K(t,\tau,\varepsilon)\big| \leq C_1(q,s,t,\varepsilon)\exp\{-C_2(q,s,t)\tau/\varepsilon\}, \quad \forall \tau>0;$$

Moreover, for $\gamma\in\mathbb{R}$ there exist C_1, C_2 and ε_0, all of them positive and depending on γ, such that the following estimates are fulfilled:

$$\int_0^\infty e^{\gamma\tau}\,\big|K_t(t,\tau,\varepsilon)\big|\,d\tau \leq C_1\,\varepsilon^{-1}\,e^{C_2 t}, \quad \forall\varepsilon\in(0,\varepsilon_0], \quad \forall t\geq 0,$$

$$\int_0^\infty e^{\gamma\tau}\,\big|K_\tau(t,\tau,\varepsilon)\big|\,d\tau \leq C_1\,\varepsilon^{-1}\,e^{C_2 t}, \quad \forall\varepsilon\in(0,\varepsilon_0], \quad \forall t\geq 0,$$

$$\int_0^\infty e^{\gamma\tau}\,\big|K_{\tau\tau}(t,\tau,\varepsilon)\big|\,d\tau \leq C_1\,\varepsilon^{-2}\,e^{C_2 t}, \quad \forall\varepsilon\in(0,\varepsilon_0], \quad \forall t\geq 0,$$

(vi) $K(t,\tau,\varepsilon)>0, \quad \forall t\geq 0, \quad \forall\tau\geq 0$;

(vii) *For every continuous function $\varphi:[0,\infty)\to H$ with $|\varphi(t)|\leq M\exp\{\gamma\,t\}$ the following equality is true:*

$$\lim_{t\to 0}\Big|\int_0^\infty K(t,\tau,\varepsilon)\varphi(\tau)d\tau - \int_0^\infty e^{-\tau}\varphi(2\varepsilon\tau)d\tau\Big| = 0,$$

for every $\varepsilon\in\big(0,(2\,\gamma)^{-1}\big)$;

(viii)

$$\int_0^\infty K(t,\tau,\varepsilon)d\tau = 1, \quad \forall t\geq 0,$$

(ix) *Let $\gamma > 0$ and $q \in [0,1]$. There exist C_1, C_2 and ε_0 all of them positive and depending on γ and q, such that the following estimates are fulfilled:*

$$\int_0^\infty K(t,\tau,\varepsilon)\, e^{\gamma\tau} |t-\tau|^q \, d\tau \le C_1 \, e^{C_2 t} \, \varepsilon^{q/2}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t > 0.$$

If $\gamma \le 0$ and $q \in [0,1]$, then

$$\int_0^\infty K(t,\tau,\varepsilon)\, e^{\gamma\tau} \, |t-\tau|^q \, d\tau \le C \, \varepsilon^{q/2} \left(1+\sqrt{t}\right)^q, \quad \forall \varepsilon \in (0,1], \quad \forall t \ge 0;$$

(x) *Let $p \in (1,\infty]$ and $f : [0,\infty) \to H$, $f(t) \in W^{1,p}_\gamma(0,\infty;H)$. If $\gamma > 0$, then there exist C_1, C_2 and ε_0 all of them positive and depending on γ and p, such that*

$$\Big| f(t) - \int_0^\infty K(t,\tau,\varepsilon) f(\tau) d\tau \Big|$$

$$\le C_1 \, e^{C_2 t} \, ||f'||_{L^p_\gamma(0,\infty;H)} \; \varepsilon^{(p-1)/2p}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \ge 0.$$

If $\gamma \le 0$, then

$$\Big| f(t) - \int_0^\infty K(t,\tau,\varepsilon) f(\tau) d\tau \Big|$$

$$\le C(\gamma,p) \, \|f'\|_{L^p_\gamma(0,\infty;H)} \left(1+\sqrt{t}\right)^{\frac{p-1}{p}} \; \varepsilon^{(p-1)/2p}, \quad \forall \varepsilon \in (0,1], \quad \forall t \ge 0.$$

(xi) *For every $q > 0$ and $\alpha \ge 0$ there exists a constant $C(q,\alpha) > 0$ such that*

$$\int_0^t \int_0^\infty K(\tau,\theta,\varepsilon) \, e^{-q\,\theta/\varepsilon} \, |\tau-\theta|^\alpha \, d\theta \, d\tau \le C(q,\alpha) \, \varepsilon^{1+\alpha}, \ \forall \varepsilon > 0, \ \forall t \ge 0;$$

(xii) *Let $f \in W^{1,\infty}_\gamma(0,\infty;H)$ with $\gamma \ge 0$. There exist positive constants C_1, C_2 and ε_0, depending on γ, such that*

$$\Big| \int_0^\infty K_t(t,\tau,\varepsilon) f(\tau) d\tau \Big| \le C_1 \, e^{C_2 t} \|f'\|_{L^\infty_\gamma(0,\infty;H)}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \ge 0.$$

Theorem 4.1. *Let $B = 0$. Let us assume that $A : D(A) \subset H \to H$ is a positive definite operator and $f \in L^\infty_\gamma(0,\infty;H)$ for some $\gamma \geq 0$. If u_ε is the strong solution to the problem* (3.1), (3.2), *with $u_\varepsilon \in W^{2,\infty}_\gamma(0,\infty;H) \cap L^\infty_\gamma(0,\infty;H)$, $Au_\varepsilon \in L^\infty_\gamma(0,\infty;H)$, then for every $0 < \varepsilon < (4\gamma)^{-1}$ the function w_ε, defined by*

$$w_\varepsilon(t) = \int_0^\infty K(t,\tau,\varepsilon)\, u_\varepsilon(\tau)\, d\tau,$$

is the strong solution in H to the problem

$$\begin{cases} w'_\varepsilon(t) + Aw_\varepsilon(t) = F_0(t,\varepsilon), & \text{a. e.} \quad t > 0, \\ w_\varepsilon(0) = \varphi_\varepsilon, \end{cases}$$

where

$$F_0(t,\varepsilon) = \frac{1}{\sqrt{\pi}}\Big[2\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big) - \lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big]u_1 + \int_0^\infty K(t,\tau,\varepsilon)\, f(\tau)\, d\tau,$$

$$\varphi_\varepsilon = \int_0^\infty e^{-\tau}\, u_\varepsilon(2\varepsilon\tau)\, d\tau.$$

5 Limits of solutions to the problem (P_ε) as $\varepsilon \to 0$

In this section we will prove the convergence estimates for the difference of solutions to the problems (P_ε) and (P_0). These estimates will be uniform relative to small values of the parameter ε.

Theorem 5.1. *Let $T > 0$ and $p \in (1,\infty]$. Let us assume that the operators A_0, A_1 satisfy* **(H1)**, **(H2)** *and the operator B verifies* **(HB1)** *and* **(HB2)**. *If $u_0,\ u_{0\varepsilon} \in D(A_0)$, $u_{1\varepsilon} \in D(A_0^{1/2})$ and $f,\ f_\varepsilon \in W^{1,p}(0,T;H)$, then there exist $C = C(T,p,\omega_0,\omega_1,L(\mu)) > 0$, $\varepsilon_0 = \varepsilon_0(\omega_0,\omega_1,L(\mu))$, $\varepsilon_0 \in (0,1)$, such that*

$$||u_\varepsilon - v||_{C([0,T];H)}$$

$$\le C\left(\mathbf{M}_2(T,u_{0\varepsilon},u_{1\varepsilon},f_\varepsilon)\,\varepsilon^{\beta}+|u_{0\varepsilon}-u_0|+||f_\varepsilon-f||_{L^p(0,T;H)}\right),\forall\varepsilon\in(0,\varepsilon_0], \tag{5.1}$$

$$||A_0^{1/2}u_\varepsilon-A_0^{1/2}v||_{L^2(0,T;H)}$$
$$\le C\left(\mathbf{M}_2(T,u_{0\varepsilon},u_{1\varepsilon},f_\varepsilon)\,\varepsilon^{\beta}+|u_{0\varepsilon}-u_0|+||f_\varepsilon-f||_{L^p(0,T;H)}\right),\forall\varepsilon\in(0,\varepsilon_0], \tag{5.2}$$

where u_ε and v are strong solutions to problems (P_ε) and (P_0) respectively, $\beta=\min\{1/4,\,(p-1)/2p\}$,

$$\mu(T,u_{0\varepsilon},u_{1\varepsilon},f_\varepsilon)=C\big(\big|A_0^{1/2}u_{0\varepsilon}\big|+\big|\mathcal{B}(u_{0\varepsilon})\big|^{1/2}+|u_{1\varepsilon}|+||f_\varepsilon||_{W^{1,p}(0,T;H)}\big),$$

$$\mathbf{M}_2(T,u_{0\varepsilon},u_{1\varepsilon},f_\varepsilon)$$
$$=\big|A_0u_{0\varepsilon}\big|+\big|A_1u_{0\varepsilon}\big|+\big|A_0^{1/2}u_{1\varepsilon}\big|+\big|B(u_{0\varepsilon})\big|+\big|\mathcal{B}(u_{0\varepsilon})\big|^{1/2}+||f_\varepsilon||_{W^{1,p}(0,T;H)}.$$

If $B=0$, then in (5.1) *and* (5.2), $C=C(T,p,\omega_0,\omega_1)$, $\varepsilon_0=\varepsilon_0(\omega_0,\omega_1)$ *and*

$$\mathbf{M_2}(T,u_{0\varepsilon},u_{1\varepsilon},f_\varepsilon)=\big|A_0u_{0\varepsilon}\big|+\big|A_0^{1/2}u_{1\varepsilon}\big|+\big|A_1u_{0\varepsilon}\big|+||f_\varepsilon||_{W^{1,p}(0,T;H)}.$$

In this case $\beta=(p-1)/2p$ in (5.1) *and $\beta=\min\{1/4,\,(p-1)/2p\}$ in* (5.2).

Proof. During the proof, we will agree to denote all constants $C(T,p,\omega_0,\omega_1,L(\mu))$, $\mathcal{M}_1(T,u_{0\varepsilon},u_{1\varepsilon},f_\varepsilon)$, $\varepsilon_0(\omega_0,\omega_1,L)$, $\gamma(\omega_0,\omega_1,L(\mu))$ by C, $\mathcal{M}_1$, ε_0 and γ, respectively.

First of all, let us observe that, from (**H1**) and (**H2**), we obtain

$$\big((A_1+\omega_1A_0)u,u\big)=\big(A_1u,u\big)+\omega_1\big(A_0u,u\big)\ge-\omega_1\big(A_0u,u\big)+\omega_1\big(A_0u,u\big)=0.$$

Thus $A_1+\omega_1A_0$ is positive, which implies

$$\Big|\Big(A_1u,v\Big)\Big|\le\Big|\Big((A_1+\omega_1\,A_0)u,v\Big)\Big|+\omega_1\,|A_0^{1/2}u|\,|A_0^{1/2}v|$$

$$=\Big((A_1+\omega_1\,A_0)^{1/2}u,(A_1+\omega_1\,A_0)^{1/2}\,v\Big)+\omega_1\,|A_0^{1/2}u|\,|A_0^{1/2}v|$$

$$\le\Big((A_1+\omega_1\,A_0)u,u\Big)^{1/2}\Big((A_1+\omega_1\,A_0)v,v\Big)^{1/2}+\omega_1\,|A_0^{1/2}u|\,|A_0^{1/2}v|$$

$$\leq \Big(2\omega_1\,(A_0 u,u)\Big)^{1/2}\,\Big(2\omega_1\,(A_0 v,v)\Big)^{1/2}$$

$$+\omega_1\left|A_0^{1/2}u\right|\,\left|A_0^{1/2}v\right| \leq 3\,\omega_1\,\left|A_0^{1/2}u\right|\,\left|A_0^{1/2}v\right|, \quad \forall u,v\in D(A_0). \tag{5.3}$$

If $f,\ f_\varepsilon \in W^{k,p}(0,T;H)$ with $k\in\mathbb{N}$ and $p\in(1,\infty]$, then $f,\ f_\varepsilon\in C([0,T];H)$. Moreover, there exist extensions $\tilde f,\ \tilde f_\varepsilon \in W^{k,p}(0,\infty;H)$ such that

$$\begin{cases} ||\tilde f||_{C([0,\infty);H)} + ||\tilde f||_{W^{k,p}(0,\infty;H)} \leq C(T,p)\,||f||_{W^{k,p}(0,T;H)}, \\ ||\tilde f_\varepsilon||_{C([0,\infty);H)} + ||\tilde f_\varepsilon||_{W^{k,p}(0,\infty;H)} \leq C(T,p)\,||f_\varepsilon||_{W^{k,p}(0,T;H)}. \end{cases} \tag{5.4}$$

Let us denote by $\tilde u_\varepsilon$ the unique strong solution to the problem (P_ε), defined on $(0,\infty)$ instead of $(0,T)$ and $\tilde f_\varepsilon$ instead of f_ε.

From Lemma 3.1, it follows that $\tilde u_\varepsilon \in W^{2,\infty}_\gamma(0,\infty;H)\cap W^{1,2}_\gamma(0,\infty;D(A_0))$, $A_0^{1/2}\tilde u_\varepsilon\in L^\infty_\gamma(0,\infty;H)$, $A_0\tilde u_\varepsilon\in L^\infty_\gamma(0,\infty;H)$ with $\gamma=\gamma(\omega_0,\omega_1,L(\mu))$. Moreover, due to this lemma and (5.4), the following estimates

$$||A_0^{1/2}\tilde u_\varepsilon||_{C([0,t];H)} + ||\tilde u_\varepsilon'||_{L^2(0,t;H)} \leq C\,\mu, \quad \forall\varepsilon\in(0,1], \quad \forall t\geq 0, \tag{5.5}$$

$$||\tilde u_\varepsilon'||_{C([0,\,t];H)} + ||A_0^{1/2}\tilde u_\varepsilon'||_{L^2(0,\,t;\,H)} \leq C\,e^{12\,L^2(\mu)\,t}\,\mathbf{M}_2, \quad \forall\varepsilon\in(0,1], \quad \forall t\geq 0, \tag{5.6}$$

$$\big|\big|A_0\tilde u_\varepsilon\big|\big|_{C([0,\,t];\,H)} \leq C\,\mathbf{M}_2\,e^{(6\,L^2(\mu)+1)\,t}, \quad \forall\varepsilon\in(0,1/2], \quad \forall t\geq 0,$$

are valid. By Theorem 4.1, the function w_ε, defined by

$$w_\varepsilon(t) = \int_0^\infty K(t,\tau,\varepsilon)\,\tilde u_\varepsilon(\tau)\,d\tau,$$

is the strong solution in H to the problem

$$\begin{cases} w_\varepsilon'(t) + \big(A_0+\varepsilon A_1\big)w_\varepsilon(t) = F(t,\varepsilon), \quad t>0, \\ w_\varepsilon(0)=w_0, \end{cases} \tag{5.7}$$

for every $\varepsilon\in(0,\varepsilon_0]$, where

$$F(t,\varepsilon) = f_0(t,\varepsilon)\,u_{1\varepsilon} + \int_0^\infty K(t,\tau,\varepsilon)\,\tilde f_\varepsilon(\tau)\,d\tau - \int_0^\infty K(t,\tau,\varepsilon)\,B\big(\tilde u_\varepsilon(\tau)\big)\,d\tau,$$

$$f_0(t,\varepsilon)=\frac{1}{\sqrt{\pi}}\Big[2\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big)-\lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big],\quad w_0=\int_0^\infty e^{-\tau}\tilde{u}_\varepsilon(2\varepsilon\tau)d\tau.$$

Since A_0 is closed, then from the estimates (5.5), we deduce that

$$||A_0^{1/2}w_\varepsilon||_{C([0,\,t;H)}\leq C\,\mu,\quad \forall\varepsilon\in(0,1],\quad \forall t\geq 0. \tag{5.8}$$

Proof of the estimate (5.1). Using properties (**vi**), (**viii**), (**x**), from Lemma 4.1, and (5.5), we obtain that

$$||\tilde{u}_\varepsilon-w_\varepsilon||_{C([0,t];\,H)}\leq C\,\mu\,\varepsilon^{1/4},\quad \forall\varepsilon\in(0,\varepsilon_0],\quad \forall t\geq 0. \tag{5.9}$$

In what follows, let us observe that

$$\begin{aligned}\Big|A_0^{1/2}\big(\tilde{u}_\varepsilon(t)-w_\varepsilon(t)\big)\Big| &\leq \int_0^\infty K(t,\tau,\varepsilon)\,\Big|A_0^{1/2}\big(\tilde{u}_\varepsilon(t)-\tilde{u}_\varepsilon(\tau)\big)\Big|\,d\tau\\ &\leq \int_0^\infty K(t,\tau,\varepsilon)\,\Big|\int_\tau^t\Big|A_0^{1/2}\tilde{u}_\varepsilon'(s)\Big|\,ds\Big|\,d\tau\\ &\leq \int_0^\infty K(t,\tau,\varepsilon)\,|t-\tau|^{1/2}\,\Big|\int_\tau^t\Big|A_0^{1/2}\tilde{u}_\varepsilon'(s)\Big|^2\,ds\Big|^{1/2}\,d\tau\\ &\leq C\,e^{\gamma\,t}\,\mathbf{M}_2\,\varepsilon^{1/4},\quad \forall\varepsilon\in(0,\varepsilon_0],\quad \forall t\geq 0.\end{aligned} \tag{5.10}$$

Denote by $R(t,\varepsilon)=\tilde{v}(t)-w_\varepsilon(t)$, where $\tilde{v}$ is the strong solution to the problem (P_0) with $\tilde{f}$ instead of f, $T=\infty$ and w_ε is the strong solution of (5.7). Then, due to Theorem 2.2, $R(\cdot,\varepsilon)\in W_\gamma^{1,\infty}(0,\infty;\,H)$ and R is the strong solution in H to the problem

$$\begin{cases}R'(t,\varepsilon)+A_0R(t,\varepsilon)=\varepsilon A_1\omega_\varepsilon(t)+B(w_\varepsilon(t))-B(\tilde{v}(t))+\mathcal{F}(t,\varepsilon), \ \textit{a.e. } t>0,\\ R(0,\varepsilon)=R_0,\end{cases}$$

where $R_0=u_0-w_0$ and

$$\begin{aligned}\mathcal{F}(t,\varepsilon)=\tilde{f}(t)-\int_0^\infty K(t,\tau,\varepsilon)\tilde{f}_\varepsilon(\tau)\,d\tau\\ -f_0(t,\varepsilon)\,u_{1\varepsilon}-B\big(w_\varepsilon(t)\big)+\int_0^\infty K(t,\tau,\varepsilon)B\big(\tilde{u}_\varepsilon(\tau)\big)\,d\tau.\end{aligned} \tag{5.11}$$

Taking the inner product in H by R and then integrating, from **(H1)** and **(HB1)**, we obtain

$$|R(t,\varepsilon)|^2+2\int_{t_0}^t \left|A_0^{1/2}R(s,\varepsilon)\right|^2 ds \le |R(t_0,\varepsilon)|^2+2\,\varepsilon\int_{t_0}^t \Big(A_1 w_\varepsilon(s), R(s,\varepsilon)\Big)\, ds$$

$$+2\int_{t_0}^t \big|\mathcal{F}(s,\varepsilon)+B\big(w_\varepsilon(s)\big)-B\big(\tilde v(s)\big)\big|\, |R(s,\varepsilon)|\, ds, \quad \forall t\ge t_0\ge 0.$$

Using (5.3), from the last equality, we deduce

$$|R(t,\varepsilon)|^2+\int_{t_0}^t \left|A_0^{1/2}R(s,\varepsilon)\right|^2 ds$$

$$\le |R(t_0,\varepsilon)|^2+2\int_{t_0}^t \big|\mathcal{F}(s,\varepsilon)+B\big(w_\varepsilon(s)\big)-B\big(\tilde v(s)\big)\big|\, |R(s,\varepsilon)|\, ds$$

$$+9\,\omega_1^2\,\varepsilon^2\int_{t_0}^t \left|A_0^{1/2}w_\varepsilon(s)\right|^2 ds, \quad \forall t\ge t_0\ge 0. \tag{5.12}$$

Applying Lemma of Brézis to (5.12), we get

$$|R(t,\varepsilon)|+\left(\int_{t_0}^t \left|A_0^{1/2}R(s,\varepsilon)\right|^2 ds\right)^{1/2}$$

$$\le \sqrt{2}\,|R(t_0,\varepsilon)|+\sqrt{2}\int_{t_0}^t |\mathcal{F}(s,\varepsilon)+B(w_\varepsilon(s))-B(\tilde v(s))|\, ds$$

$$+3\,\sqrt{2}\,\omega_1\varepsilon\left(\int_{t_0}^t \left|A_0^{1/2}w_\varepsilon(s)\right|^2 ds\right)^{1/2}\Big), \quad \forall t\ge t_0\ge 0. \tag{5.13}$$

Using **(HB1)**, we get the estimate

$$\big|B\big(w_\varepsilon(t)\big)-B\big(\tilde v(t)\big)\big|\le L(\mu)\big|A_0^{1/2}(w_\varepsilon(t)-\tilde v(t))\big| = L(\mu)\big|A_0^{1/2}R(t,\varepsilon)\big|,$$

which, together with (5.8) and (5.13), gives

$$|R(t,\varepsilon)|+\left(\int_{t_0}^t \left|A_0^{1/2}R(s,\varepsilon)\right|^2 ds\right)^{1/2}\le \sqrt{2}\Big(\,|R(t_0,\varepsilon)|$$

$$+\int_{t_0}^{t}\big(|\mathcal{F}(s,\varepsilon)|+C\,\varepsilon\big)ds+L(\mu)\int_{t_0}^{t}\Big|A_0^{1/2}R(s,\varepsilon)\Big|\,ds\Big),\quad \forall t\geq t_0\geq 0. \tag{5.14}$$

Applying Lemma 1.1 to the inequality (5.14), we get

$$|R(t,\varepsilon)|+\Big(\int_0^t\Big|A_0^{1/2}R(s,\varepsilon)\Big|^2\,ds\Big)^{1/2}\leq 2e^{12L^2(\mu)t}\Big(|R_0|$$

$$+\int_0^t\big(|\mathcal{F}(s,\varepsilon)|+C\,\varepsilon\big)ds\Big),\quad \forall t\geq 0. \tag{5.15}$$

From (5.6), it follows that

$$\big|R_0\big|\leq|u_{0\varepsilon}-u_0|+\int_0^\infty e^{-\tau}\big|\tilde{u}_\varepsilon(2\varepsilon\tau)-u_{0\varepsilon}\big|\,d\tau\leq|u_{0\varepsilon}-u_0|+$$

$$\int_0^\infty e^{-\tau}\int_0^{2\varepsilon\tau}\big|\tilde{u}_\varepsilon'(s)\big|\,ds\,d\tau\leq|u_{0\varepsilon}-u_0|+$$

$$C\,\varepsilon\,\mathbf{M}_2\int_0^\infty\tau\,e^{-\tau+\gamma\,\varepsilon\,\tau}\,d\tau\leq|u_{0\varepsilon}-u_0|+C\,\mathbf{M}_2\,\varepsilon,\quad\forall\varepsilon\in(0,\varepsilon_0]. \tag{5.16}$$

In what follows, we will estimate $\big|\mathcal{F}(t,\varepsilon)\big|$. Using the property ($\mathbf{x}$) from Lemma 4.1 and (5.4), we have

$$\Big|\tilde{f}(t)-\int_0^\infty K(t,\tau,\varepsilon)\,\tilde{f}_\varepsilon(\tau)\,d\tau\Big|\leq|\tilde{f}(t)-\tilde{f}_\varepsilon(t)|+\Big|\tilde{f}_\varepsilon(t)-\int_0^\infty K(t,\tau,\varepsilon)\,\tilde{f}_\varepsilon(\tau)\,d\tau\Big|$$

$$\leq|\tilde{f}(t)-\tilde{f}_\varepsilon(t)|+C(T,p)\|f_\varepsilon'\|_{L^p(0,T\,;\,H)}\,\varepsilon^{(p-1)/2\,p},\ \forall\varepsilon\in(0,\varepsilon_0],\ \forall t\in[0,T]. \tag{5.17}$$

Since

$$e^\tau\lambda(\sqrt{\tau})\leq C,\quad\forall\tau\geq 0,$$

the estimates

$$\int_0^t\exp\Big\{\frac{3\tau}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{\tau}{\varepsilon}}\Big)d\tau\leq C\,\varepsilon\int_0^\infty e^{-\tau/4}\,d\tau\leq C\varepsilon,\quad\forall t\geq 0,$$

$$\int_0^t\lambda\Big(\frac{1}{2}\sqrt{\frac{\tau}{\varepsilon}}\Big)\,d\tau\leq\varepsilon\int_0^\infty\lambda\Big(\frac{1}{2}\sqrt{\tau}\Big)\,d\tau\leq C\,\varepsilon,\quad\forall t\geq 0,$$

hold. Then

$$\Big| \int_0^t f_0(\tau,\varepsilon)\, d\tau \Big| \le C\,\varepsilon, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \ge 0. \tag{5.18}$$

In what follows we will estimate the difference

$$I(t,\varepsilon) = \int_0^\infty K(t,\tau,\varepsilon)\, B\big(\tilde u_\varepsilon(\tau)\big)\, d\tau - B\big(w_\varepsilon(t)\big) = I_1(t,\varepsilon) + I_2(t,\varepsilon), \tag{5.19}$$

where, due to the property (**viii**) from Lemma 4.1, we have

$$I_1(t,\varepsilon) = \int_0^\infty K(t,\tau,\varepsilon)\, \Big(B\big(\tilde u_\varepsilon(\tau)\big) - B\big(w_\varepsilon(\tau)\big)\Big)\, d\tau,$$

$$I_2(t,\varepsilon) = \int_0^\infty K(t,\tau,\varepsilon)\, \Big(B\big(w_\varepsilon(\tau)\big) - B\big(w_\varepsilon(t)\big)\Big)\, d\tau.$$

Using **(HB1)** and (5.5), (5.8), (5.10), we deduce the estimates

$$|I_1(t,\varepsilon)| \le L(\mu) \int_0^\infty K(t,\tau,\varepsilon)\, \big| A_0^{1/2}\tilde u_\varepsilon(\tau) - A_0^{1/2} w_\varepsilon(\tau)\big|\, d\tau$$

$$\le C\,\mathbf{M}_2\, e^{\gamma t}\, \varepsilon^{1/4}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \ge 0, \tag{5.20}$$

$$\big| B\big(w_\varepsilon(t)\big) - B\big(w_\varepsilon(\tau)\big)\big| \le L(\mu)\, \big| A_0^{1/2} w_\varepsilon(t) - A_0^{1/2}\tilde u_\varepsilon(t)\big|$$

$$+ L(\mu)\, \big| A_0^{1/2} w_\varepsilon(\tau) - A_0^{1/2}\tilde u_\varepsilon(\tau)\big| + L(\mu)\, \big| A_0^{1/2}\tilde u_\varepsilon(t) - A_0^{1/2}\tilde u_\varepsilon(\tau)\big|$$

$$\le C\,\mathbf{M}_2\, \varepsilon^{1/4} \left(e^{\gamma t} + e^{\gamma \tau}\right) + L(\mu) \Big| \int_\tau^t |A_0^{1/2}\tilde u_\varepsilon'(s)|\, ds \Big|, \ \forall \varepsilon \in (0,\varepsilon_0],\ \forall t \ge 0,\ \forall \tau \ge 0.$$

Using the last estimate, (5.6) and properties (**viii**), (**ix**) from Lemma 4.1, for $I_2(t,\varepsilon)$ we get the estimate

$$|I_2(t,\varepsilon)| \le C\,\mathbf{M}_2\, e^{\gamma t} \varepsilon^{1/4}$$

$$+ L(\mu) \int_0^\infty K(t,\tau,\varepsilon)\, |t-\tau|^{1/2} \Big| \int_\tau^t |A_0^{1/2}\tilde u_\varepsilon'(s)|^2\, ds \Big|^{1/2} d\tau$$

$$\le C\,\mathbf{M}_2\, e^{\gamma t}\, \varepsilon^{1/4}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \ge 0. \tag{5.21}$$

From (5.19), using (5.20) and (5.21), for $I(t,\varepsilon)$, we get the estimate

$$|I(t,\varepsilon)| \leq C\,\mathbf{M}_2\, e^{\gamma\, t}\,\varepsilon^{1/4}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \geq 0. \tag{5.22}$$

Using (5.4), (5.17), (5.18) and (5.22), from (5.11), we obtain

$$\int_0^t \big|\mathcal{F}(\tau,\varepsilon)\big|\, d\tau \leq C\,\Big(\mathbf{M}_2\,\varepsilon^{\beta} + ||f_\varepsilon - f||_{L^p(0,T;H)}\Big), \forall \varepsilon \in (0,\varepsilon_0], \forall t \in [0,T]. \tag{5.23}$$

From (5.15), using (5.16) and (5.23), we get the estimate

$$||R||_{C([0,\,t];\,H)} + ||A_0^{1/2}R||_{L^2(0,t;H)}$$

$$\leq C\,\Big(\mathbf{M}_2\,\varepsilon^{\beta} + |u_{0\varepsilon} - u_0| + ||f_\varepsilon - f||_{L^p(0,T;H)}\Big), \ \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \in [0,T]. \tag{5.24}$$

Consequently, from (5.9) and (5.24), we deduce

$$||\tilde{u}_\varepsilon - \tilde{v}||_{C([0,t];H)} \leq ||\tilde{u}_\varepsilon - w_\varepsilon||_{C([0,t];H)} + ||R||_{C([0,t];H)}$$

$$\leq C\,\Big(\mathbf{M}_2\,\varepsilon^{\beta} + |u_{0\varepsilon} - u_0| + ||f_\varepsilon - f||_{L^p(0,T;H)}\Big), \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \in [0,T]. \tag{5.25}$$

Since $u_\varepsilon(t) = \tilde{u}_\varepsilon(t)$ and $v(t) = \tilde{v}(t)$, for all $t \in [0,T]$, then the estimate (5.1) follows from (5.25).

Proof of the estimate (5.2). From (5.10), it follows that

$$\big|\big|A_0^{1/2}u_\varepsilon - A_0^{1/2}w_\varepsilon\big|\big|_{C([0,T];H)} \leq C\,\mathbf{M}_2\,\varepsilon^{1/4}, \quad \forall \varepsilon \in (0,\varepsilon_0]. \tag{5.26}$$

Since $u_\varepsilon(t) = \tilde{u}_\varepsilon(t)$ and $v(t) = \tilde{v}(t)$, for all $t \in [0,T]$, the estimate (5.2) is a simple consequence of (5.26) and (5.24). □

Remark 5.1. *If in conditions of Theorem* 5.1 $f, f_\varepsilon \in W^{1,\infty}(0,T;H)$, *then in* (5.1), (5.2), $\beta = 1/4$.

Theorem 5.2. *Let $T > 0$ and $p \in (1, \infty]$. Let us assume that A_0, A_1 satisfy* **(H1)**, **(H2)**, *and B verifies* **(HB1)**, **(HB2)** *and* **(HB3)**. *If $u_0, u_{0\varepsilon}, A_0 u_0, A_0 u_{0\varepsilon}, A_1 u_{0\varepsilon}, B u_{0\varepsilon}, u_{1\varepsilon}, f(0), f_\varepsilon(0) \in D(A_0)$ and $f, f_\varepsilon \in W^{2,p}(0,T;H)$, then there exist $C = C(T, p, \omega_0, \omega_1, L(\mu), L_1(\mu_1), ||B'(0)||) > 0$, $\varepsilon_0 \in (0,1)$, $\varepsilon_0 = \varepsilon_0(\omega_0, \omega_1, L(\mu))$, such that*

$$||u_\varepsilon' - v' + h_\varepsilon e^{-t/\varepsilon}||_{C([0,T];H)} + \big|\big|A_0^{1/2}\big(u_\varepsilon' - v' + h_\varepsilon e^{-t/\varepsilon}\big)\big|\big|_{L^2(0,T;H)}$$

$$\leq C\left(\mathbf{M}_3^3(T, u_{0\varepsilon}, u_{1\varepsilon}, f_\varepsilon)\,\varepsilon^\beta + \mathbf{D}_\varepsilon \mathbf{M}_4\right), \quad \forall \varepsilon \in (0, \varepsilon_0], \tag{5.27}$$

where u_ε and v are strong solutions to (P_ε) and (P_0) respectively,

$$h_\varepsilon = f_\varepsilon(0) - (A_0 + \varepsilon A_1)u_{0\varepsilon} - B(u_{0\varepsilon}) - u_{1\varepsilon}, \quad \beta = \min\{1/4, (p-1)/2p\},$$

$$\mu_1 = C\big(\mu + |(A_0 + \varepsilon A_1)u_{0\varepsilon}|\big),$$

$$\mathbf{M}_3(T, u_{0\varepsilon}, u_{1\varepsilon}, f_\varepsilon) = |A_0 u_{0\varepsilon}| + |A_1 u_{0\varepsilon}| + |(A_0 + \varepsilon A_1)u_{1\varepsilon}| +$$

$$+ |\mathcal{B}(u_{0\varepsilon})|^{1/2} + |(A_0 + \varepsilon A_1)h_\varepsilon| + ||f_\varepsilon||_{W^{2,p}(0,T;H)} + 1.$$

$$\mathbf{M}_4(T, u_0, f) = |A_0 u_0| + |B(u_0)| + ||f||_{W^{1,p}(0,T;H)}.$$

$$\mathbf{D}_\varepsilon = ||f_\varepsilon - f||_{W^{1,p}(0,T;H)} + |A_0(u_{0\varepsilon} - u_0)| + |B(u_{0\varepsilon}) - B(u_0)|.$$

If $B = 0$, then

$$||u_\varepsilon' - v' + h_\varepsilon e^{-t/\varepsilon}||_{C([0,T];H)} \leq C\Big(\mathbf{M}_3(T, u_{0\varepsilon}, u_{1\varepsilon}, f_\varepsilon)\,\varepsilon^{(p-1)/2p} + \mathbf{D}_\varepsilon\Big), \forall \varepsilon \in (0, \varepsilon_0],$$

$$\big|\big|A^{1/2}\big(u_\varepsilon' - v' + h_\varepsilon e^{-t/\varepsilon}\big)\big|\big|_{L^2(0,T;H)} \leq C\Big(\mathbf{M}_3(T, u_{0\varepsilon}, u_{1\varepsilon}, f_\varepsilon)\,\varepsilon^\beta + \mathbf{D}_\varepsilon\Big), \forall \varepsilon \in (0, \varepsilon_0]$$

with $C = C(T, \omega_0, \omega_1, p)$, $\varepsilon_0 = \varepsilon_0(\omega_0, \omega_1)$, $h_\varepsilon = f_\varepsilon(0) - (A_0 + \varepsilon A_1)u_{0\varepsilon} - u_{1\varepsilon}$,

$$\mathbf{M}_3(T, u_{0\varepsilon}, u_{1\varepsilon}, f_\varepsilon) = |A_0 u_{0\varepsilon}| + |A_1 u_{0\varepsilon}| + |(A_0 + \varepsilon A_1)u_{1\varepsilon}|$$

$$+ |(A_0 + \varepsilon A_1)h_\varepsilon| + ||f_\varepsilon||_{W^{2,p}(0,T;H)} + 1.$$

$$\mathbf{D}_\varepsilon = ||f_\varepsilon - f||_{W^{1,p}(0,T;H)} + |A_0(u_{0\varepsilon} - u_0)|.$$

Proof. In the proof of this theorem, we will agree to denote all constants $C(T,p,\omega_0,\omega_1,L(\mu),L_1(\mu_1),||B'(0)||)$, $\gamma(\omega_0,\omega_1,L(\mu),L_1(\mu))$, $\varepsilon_0(\omega_0,\omega_1,L(\mu))$,

$\mathbf{M}_3(T,u_{0\varepsilon},u_{1\varepsilon},f_\varepsilon)$ by C, γ, ε_0 and $\mathbf{M}_3$ respectively. Also we preserve for $\tilde{v}(t)$, $\tilde{u}_\varepsilon(t)$, $\tilde{f}(t)$ and $\tilde{f}_\varepsilon(t)$ the same notations as in Theorem 5.1.

By Lemma 3.2, we have that the function

$$\tilde{z}_\varepsilon(t) = \tilde{u}'_\varepsilon(t) + h_\varepsilon e^{-t/\varepsilon}, \ \ with \ \ h_\varepsilon = f_\varepsilon(0) - u_{1\varepsilon} - (A_0+\varepsilon A_1)u_{0\varepsilon} - B(u_{0\varepsilon}),$$

is the strong solution in H to the problem

$$\begin{cases} \varepsilon\tilde{z}''_\varepsilon(t) + \tilde{z}'_\varepsilon(t) + (A_0+\varepsilon A_1)\tilde{z}_\varepsilon(t) = \tilde{\mathcal{F}}(t,\varepsilon), \quad t>0, \\ \tilde{z}_\varepsilon(0) = f_\varepsilon(0) - (A_0+\varepsilon A_1)u_{0\,\varepsilon} - B(u_{0\varepsilon}), \quad \tilde{z}'_\varepsilon(0)=0, \end{cases}$$

where

$$\tilde{\mathcal{F}}(t,\varepsilon) = \tilde{f}'_\varepsilon(t) - \big(B(\tilde{u}_\varepsilon(t))\big)' + e^{-t/\varepsilon}\,(A_0+\varepsilon A_1)h_\varepsilon$$

and $\tilde{z}_\varepsilon$ possesses the properties

$$\tilde{z}_\varepsilon \in W^{1,\infty}_\gamma(0,\infty;H)\cap W^{1,2}_\gamma(0,\infty;H), \quad A^{1/2}\tilde{z}_\varepsilon \in W^{1,\,2}_\gamma(0,\infty;H).$$

Moreover, by this lemma and the second inequality from (5.4), the following estimate

$$||A_0^{1/2}\tilde{z}_\varepsilon||_{C([0,\,t];\,H)} + ||\tilde{z}'_\varepsilon||_{C([0,\,t];\,H)} + \big|\big|A_0^{1/2}\tilde{z}'_\varepsilon\big|\big|_{L^2(0,\,t;\,H)}$$
$$\le C\,\mathbf{M}_3^2\,e^{\gamma(\mu)\,t}, \quad \forall\varepsilon\in(0,1], \quad \forall t\ge 0, \tag{5.28}$$

holds.

Since $\tilde{z}'_\varepsilon(0)=0$, from Theorem 4.1, the function $w_{1\varepsilon}(t)$, defined by

$$w_{1\varepsilon}(t) = \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{z}_\varepsilon(\tau)\,d\tau, \tag{5.29}$$

verifies in H the following conditions

$$\begin{cases} w'_{1\varepsilon}(t) + (A_0+\varepsilon A_1)w_{1\varepsilon}(t) = F_1(t,\varepsilon), \quad a.\ e. \quad t>0, \\ w_{1\varepsilon}(0) = \varphi_{1\varepsilon}, \end{cases}$$

for every $0 < \varepsilon \leq \varepsilon_0$, where

$$F_1(t,\varepsilon) = \int_0^\infty K(t,\tau,\varepsilon)\tilde{f}\,'(\tau)\,d\tau - \int_0^\infty K(t,\tau,\varepsilon)\,\big(B(\tilde{u}_\varepsilon)\big)'(\tau)\,d\tau$$

$$-\int_0^\infty K(t,\tau,\varepsilon)\,e^{-\tau/\varepsilon}\,d\tau\,(A_0+\varepsilon A_1)h, \quad \varphi_{1\varepsilon} = \int_0^\infty e^{-\tau}\,\tilde{z}_\varepsilon(2\varepsilon\tau)\,d\tau.$$

Moreover, since A_0 is closed, we have

$$\left|A_0^{1/2} w_{1\,\varepsilon}(t)\right| \leq \int_0^\infty K(t,\tau,\varepsilon)\,\left|A_0^{1/2}\tilde{z}_\varepsilon(\tau)\right|\,d\tau$$

$$\leq C\,\mathbf{M}_3^2\,e^{\gamma(\mu)\,t},\ \forall \varepsilon \in (0,\varepsilon_0],\ \forall t \in [0,T]. \tag{5.30}$$

Using (5.29), the property (**viii**) and (**ix**) from Lemma 4.1 and (5.28), we get the estimate

$$\left|\tilde{z}_\varepsilon(t) - w_{1\varepsilon}(t)\right| \leq \int_0^\infty K(t,\tau,\varepsilon)\,\left|\tilde{z}_\varepsilon(t) - \tilde{z}_\varepsilon(\tau)\right|d\tau$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon)\left|\int_\tau^t \left|\tilde{z}_\varepsilon'(s)\right|ds\right|d\tau \leq C\,\mathbf{M}_3^2\int_0^\infty K(t,\tau,\varepsilon)\,\left|e^{\gamma\,t} - e^{\gamma\,\tau}\right|d\tau$$

$$\leq C\,\mathbf{M}_3^2\int_0^\infty K(t,\tau,\varepsilon)\,|t-\tau|\,\big(e^{\gamma\,\tau} + e^{\gamma\,t}\big)\,d\tau$$

$$\leq \mathbf{M}_3^2\,e^{\gamma(\mu)\,t}\,\varepsilon^{1/2}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \geq 0,$$

which implies

$$\big|\big|\tilde{z}_\varepsilon - w_{1\varepsilon}\big|\big|_{C([0,t];\,H)} \leq C\,\mathbf{M}_3^2\,e^{\gamma\,t}\,\varepsilon^{1/2}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \geq 0. \tag{5.31}$$

Similar to the proof of (5.10), using (5.28), we get

$$\big|\big|A_0^{1/2}\big(\tilde{z}_\varepsilon - w_{1\varepsilon}\big)\big|\big|_{C([0,t];H)} \leq C\,\mathbf{M}_3^2\,e^{\gamma\,t}\,\varepsilon^{1/4}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \geq 0. \tag{5.32}$$

Let $v_1(t) = \tilde{v}'(t)$, where $\tilde{v}$ is the strong solution to the problem (P_0) with $\tilde{f}$ instead of f and $T = \infty$. Let us denote by $R_1(t,\varepsilon) = v_1(t) - w_{1\varepsilon}(t)$. The function $R_1(t,\varepsilon)$ verifies in H the following equalities

$$\begin{cases} R_1'(t,\varepsilon) + A_0 R_1(t,\varepsilon) = \mathcal{F}_1(t,\varepsilon) - I(t,\varepsilon) + \varepsilon A_1 \omega_{1\varepsilon}(t), \quad t > 0, \\ R_1(0,\varepsilon) = R_{10}, \end{cases}$$

where

$$R_{10} = f(0) - A_0 u_0 - B(u_0) - \varphi_{1\varepsilon},$$

$$\mathcal{F}_1(t,\varepsilon) = \tilde{f}\,'(t) - \int_0^\infty K(t,\tau,\varepsilon)\, \tilde{f}\,'(\tau)\, d\tau + \int_0^\infty K(t,\tau,\varepsilon)\, e^{-\tau/\varepsilon}\, d\tau\, (A_0 + \varepsilon A_1) h_\varepsilon,$$

$$I(t,\varepsilon) = \big(B(v)\big)'(t) - \int_0^\infty K(t,\tau,\varepsilon)\, \big(B(\tilde{u}_\varepsilon)\big)'(\tau) d\tau. \tag{5.33}$$

Due to estimate (5.28), it follows that $R_{10} \in H$. In addition, $\mathcal{F}_1 \in L^1(0,T;H)$ for each $T > 0$. According to Theorem 2.2, $A_0^{1/2} \tilde{v} \in W^{1,2}(0,T;H)$. Therefore, due to condition **(HB3)** and the estimates (2.21), (2.22), we have that $\big(B(\tilde{v})\big)' \in L^1(0,T;H)$ for each $T > 0$, because

$$\Big|\Big(B\big(\tilde{v}(t)\big)\Big)'\Big| \leq ||B\,'(0)||\, |\tilde{v}'(t)| + L_1(\mu)\, \big|A_0^{1/2} \tilde{v}(t)\big|\, \big|A_0^{1/2} \tilde{v}'(t)\big|, \quad a.\ e. \quad t > 0.$$

Similarly, due to **(HB3)** and the estimates (3.3), (3.4), we deduce that $\big(B(\tilde{u}_\varepsilon)\big)' \in L^2_\gamma(0,\infty;H)$. Using the property (**ix**) from Lemma 4.1, we conclude that $I \in L^1(0,T;H)$ for each $T > 0$.

Accordingly, using (5.30), similarly to (5.13) we obtain

$$|R_1(t,\varepsilon)| + ||A_0^{1/2} R_1||_{L^2(t_0,t;\,H)} \leq \sqrt{2} |R_1(t_0,\varepsilon)|$$

$$+ \sqrt{2} \int_{t_0}^t |\mathcal{F}_1(\tau,\varepsilon) - I(\tau,\varepsilon)|\, ds$$

$$+ 3\sqrt{2}\, \omega_1\, \varepsilon \int_{t_0}^t |A_0^{1/2} \omega_{1\varepsilon}(s)|^2 ds\Big)^{1/2}, \quad \forall t \geq t_0 \geq 0. \tag{5.34}$$

Using the properties (**viii**), (**ix**) from Lemma 4.1 and the inequalities (5.4), we get

$$\Big|\tilde{f}'(t) - \int_0^\infty K(t,\tau,\varepsilon)\, \tilde{f}'_\varepsilon(\tau)\, d\tau\Big|$$

$$\leq |\tilde{f}'(t) - \tilde{f}'_\varepsilon(t)| + \int_0^\infty K(t,\tau,\varepsilon)\, \big|\tilde{f}'_\varepsilon(\tau) - \tilde{f}'_\varepsilon(t)\big|\, d\tau$$

$$\leq |\tilde{f}'(t) - \tilde{f}'_\varepsilon(t)| + ||\tilde{f}''_\varepsilon||_{L^p(0,\infty;\,H)} \int_0^\infty K(t,\tau,\varepsilon)\, |t-\tau|^{(p-1)/p}\, d\tau \leq |\tilde{f}'(t) - \tilde{f}'_\varepsilon(t)|$$

$$+C(T,p)\,||f_\varepsilon''||_{L^p(0,T;\,H)}\,\varepsilon^{(p-1)/2p},\quad \forall\varepsilon\in(0,\varepsilon_0],\quad \forall t\in[0,T], \tag{5.35}$$

In what follows, we will evaluate the difference

$$I(t,\varepsilon)=\big(B\big(\tilde v(t)\big)\big)'-\int_0^\infty K(t,\tau,\varepsilon)\,\big(B(\tilde u_\varepsilon)\big)'(\tau)d\tau=I_1(t,\varepsilon)+I_2(t,\varepsilon), \tag{5.36}$$

where

$$I_1(t,\varepsilon)=\big(B(\tilde v(t))\big)'-\big(B(\tilde u_\varepsilon(t))\big)',$$

$$I_2(t,\varepsilon)=\big(B\big(\tilde u_\varepsilon(t)\big)\big)'-\int_0^\infty K(t,\tau,\varepsilon)\,\big(B\big(\tilde u_\varepsilon(\tau)\big)\big)'\,d\tau.$$

Using **(HB3)** and (5.5), (2.22), (5.2), we obtain the inequality

$$\big|I_1(t,\varepsilon)\big|=\big|B'(\tilde v(t))\,\tilde v'(t)-B'(\tilde u_\varepsilon(t))\,\tilde u_\varepsilon'(t)\big|$$

$$\le\big|B'\big(\tilde u_\varepsilon(t)\big)\big(\tilde v'(t)-\tilde u_\varepsilon'(t)\big)\big|+\Big|\Big(B'(\tilde u_\varepsilon(t))-B'(\tilde v(t))\Big)\,\tilde v'(t)\Big|$$

$$\le\ \mu_2(T)\,\big|A_0^{1/2}\big(\tilde v'(t)-\tilde u_\varepsilon'(t)\big)\big|$$

$$+L_1(\mu_1)\Big|A_0^{1/2}\Big(\tilde u_\varepsilon(t)-\tilde v(t\Big)\Big|\,\big|A_0^{1/2}\tilde v'(t)\big|,\quad \forall\varepsilon\in(0,\varepsilon_0],\quad a.\ e.\quad t\in(0,T),$$

where $\mu_2(T)=L_1(\mu)\mu+||B'(0)||$. Since

$$v'(t)-\tilde u_\varepsilon'(t)=R_1(t,\varepsilon)+w_{1\varepsilon}-\tilde z_\varepsilon(t)+h_\varepsilon\,e^{-t/\varepsilon},$$

due to (2.22), (5.2) and (5.32), we get

$$\int_{t_0}^t\big|I_1(s,\varepsilon)\big|\,ds\le C\left(\varepsilon^\beta\,\mathbf{M}_3^2+\mathbf{D}_\varepsilon\right)\mathbf{M}_4+\mu_2(T)\int_{t_0}^t\big|A^{1/2}R_1(s,\varepsilon)\big|\,ds, \tag{5.37}$$

for every $\varepsilon\in(0,\varepsilon_0]$, $0\le t_0\le t\le T$.

Now we are going to evaluate $I_2(t,\varepsilon)$. As

$$\Big|\big(B(\tilde u_\varepsilon)\big)'(t)-\big(B(\tilde u_\varepsilon)\big)'(\tau)\Big|\le I_{21}(t,\tau,\varepsilon)+I_{22}(t,\tau,\varepsilon), \tag{5.38}$$

where

$$I_{21}(t,\tau,\varepsilon)=\big|B'\big(\tilde u_\varepsilon(\tau)\big)\,\big(\tilde u_\varepsilon'(t)-\tilde u_\varepsilon'(\tau)\big)\big|,$$

$$I_{22}(t,\tau,\varepsilon) = \Big| \Big(B'\big(\tilde{u}_\varepsilon(t)\big) - B'\big(\tilde{u}_\varepsilon(\tau)\big) \Big)\, \tilde{u}'_\varepsilon(t) \Big|,$$

At the beginning, let us estimate $I_{21}(t,\tau,\varepsilon)$. Using **(HB3)** and (5.5), (5.28), we obtain

$$I_{21}(t,\tau,\varepsilon) \le L_1(\mu) \left| A_0^{1/2} \tilde{u}_\varepsilon(t) \right| \left| A_0^{1/2} \big(\tilde{u}'_\varepsilon(t) - \tilde{u}'_\varepsilon(\tau) \big) \right|$$

$$+ \big| |B'(0)| \big| \, |\tilde{u}'_\varepsilon(t) - \tilde{u}'_\varepsilon(\tau)| \le C\, \mu_2(T) \left| A_0^{1/2} \big(\tilde{u}'_\varepsilon(t) - \tilde{u}'_\varepsilon(\tau) \big) \right|$$

$$\le C\, \mu_2(T) \Big(\left| A_0^{1/2} \big(\tilde{z}_\varepsilon(t) - \tilde{z}_\varepsilon(\tau) \big) \right| + \left| A_0^{1/2} h_\varepsilon \right| \big(e^{-t/\varepsilon} + e^{-\tau/\varepsilon} \big) \Big)$$

$$\le C\, \mu_2(T) \Big(\Big| \int_\tau^t \left| A_0^{1/2} \tilde{z}'_\varepsilon(s) \right| ds \Big| + \left| A_0^{1/2}\, h_\varepsilon \right| \big(e^{-t/\varepsilon} + e^{-\tau/\varepsilon} \big) \Big)$$

$$\le C\, \mu_2(T) \Big(\Big(|t-\tau|^{1/2} \Big| \int_\tau^t \left| A_0^{1/2} \tilde{z}'_\varepsilon(s) \right|^2 ds \Big|^{1/2} + \left| A_0^{1/2} h_\varepsilon \right| \big(e^{-t/\varepsilon} + e^{-\tau/\varepsilon} \big) \Big)$$

$$\le C\, \mu_2(T)\, \mathbf{M}_3^2 \Big(\big(e^{\gamma(L(\mu))\, t} + e^{\gamma(L(\mu))\, \tau} \big) |t-\tau|^{1/2}$$

$$+ e^{-t/\varepsilon} + e^{-\tau/\varepsilon} \Big), \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall \tau \ge 0, \quad \forall t \ge 0.$$

From the last estimate, due to properties **(viii)** and **(ix)** from Lemma 4.1, we get

$$\int_0^\infty K(t,\tau,\varepsilon)\, I_{21}(t,\tau,\varepsilon)\, d\tau \le C\, \mu_2(T)\, \mathbf{M}_3^2 \Big(\varepsilon^{1/4} e^{-t/\varepsilon} +$$

$$+ \int_0^\infty K(t,\tau,\varepsilon)\, e^{-\tau/\varepsilon}\, d\tau \Big), \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \in [0,T]. \tag{5.39}$$

Now, let us estimate $I_{22}(t,\tau,\varepsilon)$. Due to **(HB3)** and (5.6), (5.28), we obtain

$$I_{22}(t,\tau,\varepsilon) \le L_1(\mu) \left| A_0^{1/2} \big(\tilde{u}_\varepsilon(t) - \tilde{u}_\varepsilon(\tau) \big) \right| \left| A_0^{1/2} \tilde{u}'_\varepsilon(t) \right|$$

$$\le L_1(\mu) \Big| \int_\tau^t \left| A_0^{1/2} \tilde{u}'_\varepsilon(s) \right| ds \Big| \Big(\left| A_0^{1/2} \tilde{z}_\varepsilon(t) \right| + \left| A_0^{1/2} h_\varepsilon \right| e^{-t/\varepsilon} \Big) \le C\, \mathbf{M}_3^3 \big(e^{\gamma(L(\mu))\, t}$$

$$+ e^{\gamma(L(\mu))\, \tau} \big) \times \big(e^{\gamma(L(\mu))\, \tau} + e^{-t/\varepsilon} \big) |t-\tau|^{1/2}, \quad \forall \varepsilon \in (0,1], \quad \forall \tau \ge 0, \quad \forall t \ge 0.$$

From this estimate, due to property **(ix)** from Lemma 4.1, we deduce that

$$\int_0^\infty K(t,\tau,\varepsilon)\, I_{22}(t,\tau,\varepsilon)\, d\tau \leq C\,\mathbf{M}_3^3\,\varepsilon^{1/4}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \in [0,T]. \tag{5.40}$$

From (5.38), using (5.39), (5.40) and property **(xi)**, from Lemma 4.1, we get

$$\int_{t_0}^t |I_2(\tau,\varepsilon)|\, d\tau \leq C\,\mathbf{M}_3^3\,\varepsilon^{1/4},\ \forall \varepsilon \in (0,\varepsilon_0],\ \forall t_0 \in [0,T],\ \forall t \in [0,T],\ \forall t > t_0. \tag{5.41}$$

From (5.36), (5.37) and (5.41), it follows that

$$\int_{t_0}^t \big|I(s,\varepsilon)\big|\, ds \leq C\left(\mathbf{M}_3^3\,\varepsilon^\beta + \mathbf{D}_\varepsilon \mathbf{M}_4\right)$$

$$+\mu_2(T) \int_{t_0}^t \big|A_0^{1/2} R_1(s,\varepsilon)\big|\, ds,\ \forall \varepsilon \in (0,\varepsilon_0],\ \forall t_0 \in [0,T],\ \forall t \in [0,T],\ t > t_0.$$

Applying Lemma 5.2 to (5.34) and using (5.30) and the last estimate, we get

$$|R_1(t,\varepsilon)| + ||A_0^{1/2} R_1||_{L^2(0,t;\,H)}$$

$$\leq C\left(|R_1(0,\varepsilon)| + \int_0^t |\mathcal{F}_1(\tau,\varepsilon)|\, ds + \mathbf{M}_3^3\,\varepsilon^\beta + \mathbf{D}_\varepsilon \mathbf{M}_4\right), \quad \forall t \geq t_0 \geq 0. \tag{5.42}$$

For R_{10}, due to (5.28), we have

$$|R_{10}| \leq |f(0) - f_\varepsilon(0)| + |A_0(u_0 - u_{0\varepsilon})| + \varepsilon |A_1 u_{0\varepsilon}| + |B(u_{0\varepsilon}) - B(u_0)|$$

$$+ \int_0^\infty e^{-\tau}\, |\tilde{z}_\varepsilon(2\varepsilon\tau) - \tilde{z}_\varepsilon(0)|\, d\tau \leq |f(0) - f_\varepsilon(0)| + |A_0(u_0 - u_{0\varepsilon})| + \varepsilon |A_1 u_{0\varepsilon}|$$

$$+|B(u_{0\varepsilon}) - B(u_0)| + \int_0^\infty e^{-\tau} \int_0^{2\varepsilon\tau} |\tilde{z}'_\varepsilon(s)|\, ds\, d\tau \leq |f(0) - f_\varepsilon(0)| + |A_0(u_0 - u_{0\varepsilon})|$$

$$+\varepsilon |A_1 u_{0\varepsilon}| + |B(u_{0\varepsilon}) - B(u_0)| + C\,\mathbf{M}_3^2\,\varepsilon \int_0^\infty \tau\, e^{-\tau + 2\gamma\varepsilon\tau}\, d\tau$$

$$\leq C\,\mathbf{D}_\varepsilon + C\,\mathbf{M}_3^2\,\varepsilon, \quad \forall \varepsilon \in (0,\varepsilon_0]. \tag{5.43}$$

From (5.33), using (5.35) and property **(xi)** from Lemma 4.1, we get

$$\int_{t_0}^t |\mathcal{F}_1(s,\varepsilon)|\, ds \leq C\,\mathbf{M}_3^3\,\varepsilon^\beta + \mathbf{D}_\varepsilon \mathbf{M}_4,$$

$\forall \varepsilon \in (0, \varepsilon_0], \quad \forall t_0 \in [0, T], \quad \forall t \in [0, T], \quad t > t_0.$

Using the last estimate and (5.43), from (5.42), we obtain

$$||R_1(t, \varepsilon)||_{C([0,t];\, H)} + \Big(\int_0^t \big|A_0^{1/2} R_1(s, \varepsilon)\big|^2 ds\Big)^{1/2}$$

$$\leq C \Big(\mathbf{M}_3^3\, \varepsilon^\beta + \mathbf{D}_\varepsilon \mathbf{M}_4 \Big), \quad \forall \varepsilon \in (0, \varepsilon_0], \quad \forall t \in [0, T],$$

which together with (5.31), (5.32) imply (5.27). □

6 Example

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with C^m boundary $\partial\Omega$. In the real Hilbert space $L^2(\Omega)$, with the usual inner product

$$(u, v) = \int_\Omega u(x)\, v(x)\, dx,$$

we consider the following Cauchy problem

$$\begin{cases} \varepsilon \partial_t^2 u_\varepsilon + \partial_t u_\varepsilon + \big(A_0 + \varepsilon A_1\big) u_\varepsilon + B\big(u_\varepsilon\big) = f(x, t), \; x \in \Omega, \; t > 0, \\ u_\varepsilon(x, 0) = u_{0\varepsilon}(x), \quad \partial_t\, u_\varepsilon(x, 0) = u_{1\varepsilon}(x), \quad x \in \overline{\Omega}, \\ \dfrac{\partial^j u_\varepsilon}{\partial \nu^j}\Big|_{\partial\Omega} = 0, \quad j = 0, 1, \ldots, m-1, \quad t \geq 0, \end{cases} \tag{6.1}$$

where $\partial_x = (\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n})$ and $A_0(x, \partial_x)$, $A_1(x, \partial_x)$ are differential operators of orders m and q, respectively, of the following type: $D(A_0) = H^{2m}(\Omega) \cap H_0^m(\Omega)$,

$$A_0(x, \partial_x) u(x) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|}\, \partial^\alpha \Big(a_\alpha(x) \partial^\alpha u(x) \Big), \; u \in D(A_0), \; a_\alpha \in C^m(\overline{\Omega}) \tag{6.2}$$

and $D(A_1) = H^{2r}(\Omega) \cap H_0^r(\Omega)$,

$$A_1(x, \partial_x) u(x) = \sum_{|\alpha| \leq r} (-1)^{|\alpha|}\, \partial^\alpha \Big(c_\alpha(x) \partial^\alpha u(x) \Big), \; u \in D(A_1), \; c_\alpha \in C^r(\overline{\Omega}), \tag{6.3}$$

where

$$\alpha=(\alpha_1,\dots,\alpha_n),\ \alpha_i\in\mathbb{N},\ |\alpha|=\alpha_1+\dots+\alpha_n,\ \partial^\alpha=\Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1}\dots\Big(\frac{\partial}{\partial x_n}\Big)^{\alpha_n}.$$

We will suppose that operators $A_i,\ i=0,1$ are self-adjoint, i. e.

$$\int_\Omega\Big(A_i(x,\partial_x)u(x)\Big)v(x)\,dx=\int_\Omega u(x)\Big(A_i(x,\partial_x)v(x)\Big)dx,\ \forall u,v\in D(A_i). \tag{6.4}$$

Moreover, we will suppose that

$$\sum_{|\alpha|\le m}\Big(a_\alpha(x)\,\xi^\alpha,\,\xi^\alpha\Big)_{\mathbb{R}^n}\ge a_0\,||\xi||^{2m},\ \forall x\in\bar\Omega,\ \forall\xi=(\xi_i)_1^n\in\mathbb{R}^n,\ a_0>0 \tag{6.5}$$

Conditions (6.4) and (6.5) assure the strong ellipticity of the operator A_0. For $r\le m$, conditions (6.2)-(6.5) imply (**H1**) and (**H2**).

Define the operator B by:

$$D(B)=L^2(\Omega)\cap L^{2(q+1)}(\Omega),\quad Bu=b\,|u|^q u.$$

If $b>0$, then B is a Fréchet derivative of convex and positive functional $\mathcal{B}$, which is defined as follows

$$D(\mathcal{B})=L^{q+2}(\Omega)\cap L^2(\Omega),\quad \mathcal{B}u=\frac{b}{q+2}\int_\Omega|u(x)|^{q+2}\,dx,$$

and the Fréchet's derivative of operator B is defined by the relationships

$$D\big(B'(u)\big)=\{v\in L^2(\Omega):u^q\,v\in L^2(\Omega)\},\quad B'(u)v=b\,(q+1)|u|^q\,v.$$

First of all, let us observe that

$$\Big||t|^q\,t-|\tau|^q\,\tau\Big|=\Big|\int_\tau^t\frac{d}{ds}\big(|s|^q s\big)ds\Big|=(q+1)\Big|\int_\tau^t|s|^q ds\Big|$$

$$\le(q+1)|t-\tau|^{1/2}\Big|\int_\tau^t|s|^{2q}ds\Big|^{1/2}=\frac{q+1}{\sqrt{2q+1}}|t-\tau|^{1/2}\Big||t|^{2q+1}-|\tau|^{2q+1}\Big|^{1/2}$$

$$\le(q+1)|t-\tau|\Big(|t|^{2q}+|\tau|^{2q}\Big)^{1/2}.$$

Then, if $n > 2m$ and $q \in [0, 2m/(n-2m)]$, using Hölder's inequality, Sobolev-Rellich-Kondrachov embedding theorem and condition (6.5), we obtain

$$||Bu_1 - Bu_2||^2_{L^2(\Omega)} = b^2 \int_\Omega \Big| |u_1(x)|^q u_1(x) - |u_2(x)|^q u_2(x)\Big|^2 dx$$

$$\leq C(q,b) \int_\Omega \big|u_1(x) - u_2(x)\big|^2 \left(\big|u_1(x)\big|^{2q} + \big|u_2(x)\big|^{2q}\right) dx$$

$$\leq C(q,n,b)||u_1 - u_2||^2_{L^{2n/(n-2m)}(\Omega)} \left(||u_1||^{2q}_{L^{qn/m}(\Omega)} + ||u_2||^{2q}_{L^{qn/m}(\Omega)}\right)$$

$$\leq C(q,b,n,\Omega)||u_1 - u_2||^2_{H^m_0(\Omega)} \left(||u_1||^{2q}_{H^m_0(\Omega)} + ||u_2||^{2q}_{H^m_0(\Omega)}\right)$$

$$\leq C(q,n,b,\Omega)\Big|A_0^{1/2}(u_1 - u_2)\Big|^2 \left(|A_0^{1/2}u_1|^{2q} + |A_0^{1/2}u_2|^{2q}\right), \ \forall u_1, u_2 \in D(A_0^{1/2}). \tag{6.6}$$

In the same way, if $n = 2m$, $m > 1$ and $q \in [(m-1)/2m, \infty)$, we obtain

$$||Bu_1 - Bu_2||^2_{L^2(\Omega)} \leq C(q,b) \int_\Omega \big|u_1(x) - u_2(x)\big|^2 \left(\big|u_1(x)\big|^{2q} + \big|u_2(x)\big|^{2q}\right) dx$$

$$\leq C(q,n,b)||u_1 - u_2||^2_{L^{2m}(\Omega)} \left(||u_1||^{2q}_{L^{2mq/(m-1)}(\Omega)} + ||u_2||^{2q}_{L^{2mq/(m-1)}(\Omega)}\right)$$

$$\leq C(q,b,n,\Omega)||u_1 - u_2||^2_{H^m_0(\Omega)} \left(||u_1||^{2q}_{H^m_0(\Omega)} + ||u_2||^{2q}_{H^m_0(\Omega)}\right)$$

$$\leq C(q,n,b,\Omega)\Big|A_0^{1/2}(u_1 - u_2)\Big|^2 \left(|A_0^{1/2}u_1|^{2q} + |A_0^{1/2}u_2|^{2q}\right), \ \forall u_1, u_2 \in D(A_0^{1/2}). \tag{6.7}$$

Similarly, we prove the inequality (6.6) in the case when $n < 2m$ and $q \geq 0$. Due to inequalities (6.6) and (6.7), if Ω is bounded with C^m boundary $\partial\Omega$, the condition (6.5) is fulfilled and q verifies

$$\begin{cases} q \in [0, 2m/(n-2m)], & if \quad n > 2m, \\ q \in [(m-1)/2m), \infty), & if \quad n = 2m, \quad m > 1, \\ q \in [0, \infty), & if \quad n = 2, \ m = 1, \\ q \in [0, \infty), & if \quad n < 2m, \end{cases} \tag{6.8}$$

then the operator B verifies (**HB1**).

If $n > 2m$ and $q \in (1, 2m/(n-2m)]$, then, in the same way as the inequality (6.6) was proved, we deduce that

$$||\big(B'(u_1) - B'(u_2)\big)v||^2_{L^2(\Omega)} = b^2(q+1)^2 \int_\Omega \Big|\,|u_1(x)|^q - |u_2(x)|^q\Big|^2 |v(x)|^2\,dx$$

$$\leq C(q,b) \int_\Omega |u_1(x) - u_2(x)|^2 \Big(|u_1(x)|^{2(q-1)} + |u_2(x)|^{2(q-1)}\Big) |v(x)|^2\,dx$$

$$\leq C(q,b)\,||v||^2_{L^{2n/(n-2m)}(\Omega)}\,||u_1 - u_2||^2_{L^{2n/(n-(n-2m)q)}(\Omega)}$$

$$\times\Big(||u_1||^{2(q-1)}_{L^{2n/(n-2m)}(\Omega)} + ||u_2||^{2(q-1)}_{L^{2n/(n-2m)}(\Omega)}\Big) \leq$$

$$\leq C(n,q,b,\Omega)\,||u_1 - u_2||^2_{H_0^m(\Omega)}\,||v||^2_{H_0^m(\Omega)}\Big(||u_1||^{2(q-1)}_{H_0^m(\Omega)} + ||u_2||^{2(q-1)}_{H_0^m(\Omega)}\Big)$$

$$\leq C(n,q,b,\Omega)\,\Big|A_0^{1/2}(u_1 - u_2)\Big|^2\,|A_0^{1/2}v|^2$$

$$\Big(|A_0^{1/2}u_1|^{2(q-1)} + |A_0^{1/2}u_2|^{2(q-1)}\Big), \quad \forall u_1, u_2, v \in D(A_0^{1/2}). \tag{6.9}$$

Similarly, if $n = 2m$ and $q \in (1, m)$, then, we deduce that

$$||\big(B'(u_1) - B'(u_2)\big)v||^2_{L^2(\Omega)}$$

$$\leq C(q,b) \int_\Omega |u_1(x) - u_2(x)|^2 \Big(|u_1(x)|^{2(q-1)} + |u_2(x)|^{2(q-1)}\Big) |v(x)|^2\,dx$$

$$\leq C(q,b)\,||v||^2_{L^{2m/(m-q)}(\Omega)}\,||u_1 - u_2||^2_{L^{2m}(\Omega)} \times \Big(||u_1||^{2(q-1)}_{L^{2m}(\Omega)} + ||u_2||^{2(q-1)}_{L^{2m}(\Omega)}\Big)$$

$$\leq C(n,q,b,\Omega)\,||u_1 - u_2||^2_{H_0^m(\Omega)}\,||v||^2_{H_0^m(\Omega)}\Big(||u_1||^{2(q-1)}_{H_0^m(\Omega)} + ||u_2||^{2(q-1)}_{H_0^m(\Omega)}\Big)$$

$$\leq C(n,q,b,\Omega)\,\Big|A_0^{1/2}(u_1 - u_2)\Big|^2\,|A_0^{1/2}v|^2\Big(|A_0^{1/2}u_1|^{2(q-1)} + |A_0^{1/2}u_2|^{2(q-1)}\Big), \tag{6.10}$$

for every $u_1, u_2, v \in D(A_0^{1/2})$.

Also, if $n = 2m$, $m > 2$ and $q \geq (3m-2)/2m$, then

$$||\big(B'(u_1) - B'(u_2)\big)v||^2_{L^2(\Omega)}$$

$$\leq C(q,b) \int_\Omega |u_1(x) - u_2(x)|^2 \Big(|u_1(x)|^{2(q-1)} + |u_2(x)|^{2(q-1)}\Big) |v(x)|^2\,dx$$

$$\leq C(q,b)\,||v||^2_{L^{2m}(\Omega)}\,||u_1-u_2||^2_{L^{2m}(\Omega)}$$
$$\times\Big(||u_1||^{2(q-1)}_{L^{2m(q-1)/(m-2)}(\Omega)}+||u_2||^{2(q-1)}_{L^{2m(q-1)/(m-2)}(\Omega)}\Big)$$
$$\leq C(n,q,b,\Omega)\,||u_1-u_2||^2_{H_0^m(\Omega)}\,||v||^2_{H_0^m(\Omega)}\Big(||u_1||^{2(q-1)}_{H_0^m(\Omega)}+||u_2||^{2(q-1)}_{H_0^m(\Omega)}\Big)$$
$$\leq C(n,q,b,\Omega)\left|A_0^{1/2}(u_1-u_2)\right|^2|A_0^{1/2}v|^2\Big(|A_0^{1/2}u_1|^{2(q-1)}+|A_0^{1/2}u_2|^{2(q-1)}\Big), \tag{6.11}$$

for every $u_1,u_2,v\in D(A_0^{1/2})$.

Similarly, we prove the inequality (6.9), in the case when $n<2m$ and $q\geq 1$. Therefore, if Ω is bounded with C^m boundary $\partial\Omega$, (6.5) is fulfilled and q verifies

$$\begin{cases} q\in[1,2m/(n-2m)], & if \quad n>2m,\\ q\in[1,\infty), & if \quad n\leq 2m, \end{cases} \tag{6.12}$$

then, due to (6.9), (6.10), (6.11), the operator B verifies (**HB3**).

The unperturbed Cauchy problem associated to (6.1) is

$$\begin{cases} \partial_t u_\varepsilon(x,t)+A_0(x,\partial_x)u_\varepsilon(x,t)+B(u_\varepsilon(x,t))=f(x,t),\ x\in\Omega,\ t>0,\\ u_\varepsilon(x,0)=u_{0\varepsilon}(x), \quad x\in\overline{\Omega},\\ \dfrac{\partial^j u_\varepsilon}{\partial\nu^j}\Big|_{\partial\Omega}=0, \quad j=0,\,1,\,\ldots,m-1, \quad t\geq 0, \end{cases} \tag{6.13}$$

According to Theorem 5.1, we have

Theorem 6.1. *Let $\Omega\subset\mathbb{R}^n$ be an open bounded set with C^m boundary $\partial\Omega$. Let us assume that $T>0$, $p\in(1,\infty]$, $r\leq m$, $b>0$, q verifies* (6.8) *and* (6.4)-(6.5) *are fulfilled. If $u_0,u_{0\varepsilon}\in H^{2m}(\Omega)\cap H_0^m(\Omega)$, $u_{1\varepsilon}\in H_0^m(\Omega)$ and $f,f_\varepsilon\in W^{1,p}(0,T;L^2(\Omega))$ then there exist $C=C(T,p,a_0,b,n,m,q,\Omega,\mu)>0$ and $\varepsilon_0=\varepsilon_0(a_0,n,m,\Omega,\mu)$, $\varepsilon_0\in(0,1)$, such that*

$$||u_\varepsilon-v||_{C([0,T];L^2(\Omega))}+||u_\varepsilon-v||_{L^2(0,T;\,H_0^m(\Omega))}$$
$$\leq C\Big(\mathbf{M}_2(T,u_{0\varepsilon},u_{1\varepsilon},f_\varepsilon)\,\varepsilon^\beta+||f_\varepsilon-f||_{L^p(0,T;L^2(\Omega))}+||u_{0\varepsilon}-u_0||_{L^2(\Omega)}\Big),$$

$$\forall \varepsilon \in [0, \varepsilon_0],$$

where u_ε *and* v *are the strong solutions to the problems* (6.1) *and* (6.13), *respectively,*

$$\mathbf{M}_2(T, u_{0\varepsilon}, u_{1\varepsilon}, f_\varepsilon)$$

$$= ||A_0^{1/2} u_{1\varepsilon}||_{L^2(\Omega)} + ||A_0 u_{0\varepsilon}||_{L^2(\Omega)} + ||A_1 u_{0\varepsilon}||_{L^2(\Omega)} + ||f_\varepsilon||_{W^{1,p}(0,T;L^2(\Omega))},$$

$$\mu(T, u_{0\varepsilon}, u_{1\varepsilon}, f_\varepsilon) = C\Big(||u_{1\varepsilon}||_{L^2(\Omega)} + ||A_0^{1/2} u_{0\varepsilon}||_{L^2(\Omega)} + ||f_\varepsilon||_{W^{1,p}(0,T;L^2(\Omega))}\Big),$$

$$\beta = \min\{1/4,\ (p-1)/2p\}.$$

Using Theorem 5.2, we can prove

Theorem 6.2. *Let* $\Omega \subset \mathbb{R}^n$ *be an open bounded set with* C^m *boundary* $\partial\Omega$. *Let us assume that* $T > 0$, $p \in (1, \infty]$, $r \leq m$, $b > 0$, q *verifies* (6.12) *and* (6.4)-(6.5) *are fulfilled. If* $u_\varepsilon, u_{0\varepsilon}, A_0 u_0, h_\varepsilon \in H^{2m}(\Omega) \cap H_0^m(\Omega)$, $u_{1\varepsilon} \in H_0^m(\Omega)$ *and* $f, f_\varepsilon \in W^{2,p}(0,T; L^2(\Omega))$ *then there exist* $C = C(T, p, a_0, b, n, m, q, \Omega, \mu, \mu_1) > 0$ *and* $\varepsilon_0 = \varepsilon_0(a_0, n, m, \Omega, \mu)$, $\varepsilon_0 \in (0,1)$, *such that*

$$||u_\varepsilon' - v' + h_\varepsilon e^{-t/\varepsilon}||_{C([0,T];L^2(\Omega))} + ||u_\varepsilon' - v' + h_\varepsilon e^{-t/\varepsilon}||_{L^2(0,T;\, H_0^m(\Omega))}$$

$$\leq C\Big(\mathbf{M}_2(T, u_{0\varepsilon}, u_{1\varepsilon}, f_\varepsilon)\, \varepsilon^\beta + \mathbf{D}_\varepsilon \mathbf{M}_4\Big), \quad \forall \varepsilon \in [0, \varepsilon_0],$$

where u_ε *and* v *are the strong solutions to the problems* (6.1) *and* (6.13), *respectively,*

$h_\varepsilon = f_\varepsilon(0) - (A_0 + \varepsilon A_1) u_{0\varepsilon} - B(u_{0\varepsilon}) - u_{1\varepsilon}$, $\mu_1 = C(\mu + ||(A_0 + \varepsilon A_1) u_{0\varepsilon}||_{L^2(\Omega)})$,

$$\mathbf{M}_2(T, u_{0\varepsilon}, u_{1\varepsilon}, f_\varepsilon)$$

$$= ||A_0^{1/2} u_{1\varepsilon}||_{L^2(\Omega)} + ||A_0 u_{0\varepsilon}||_{L^2(\Omega)} + ||A_1 u_{0\varepsilon}||_{L^2(\Omega)} + ||f_\varepsilon||_{W^{1,p}(0,T;L^2(\Omega))},$$

$$\mu(T, u_{0\varepsilon}, u_{1\varepsilon}, f_\varepsilon) = C\Big(||u_{1\varepsilon}||_{L^2(\Omega)} + ||A_0^{1/2} u_{0\varepsilon}||_{L^2(\Omega)} + ||f_\varepsilon||_{W^{1,p}(0,T;L^2(\Omega))}\Big),$$

$$\beta = \min\{1/4,\ (p-1)/2p\}.$$

$$\mathbf{D}_\varepsilon = ||f_\varepsilon - f||_{W^{1,p}(0,T;L^2(\Omega))} + ||u_{0\varepsilon} - u_0||_{L^2(\Omega)} + ||B(u_{0\varepsilon}) - B(u_0)||_{L^2(\Omega)}.$$

Remark 6.1. *If* $\Omega = \mathbb{R}^n$ *with* $n > 2m$, $q \in [1, 2m/(n-2m)]$ *and there exists* $c_0 > 0$ *such that*

$$\Big| \sum_{|\alpha| \leq r} \Big(c_\alpha(x)\, \xi^\alpha,\, \xi^\alpha \Big)_{\mathbb{R}^n} \Big| \leq c_0 \sum_{|\alpha| \leq m} \Big(a_\alpha(x)\, \xi^a,\, \xi^\alpha \Big)_{\mathbb{R}^n}, \; \forall x \in \bar{\Omega},\, \forall \xi = (\xi_i)_1^n \in \mathbb{R}^n$$

the statements of Theorems 6.1 *and* 6.2 *remain also valid.*

References

[1] Barbu, V. (2010). *Nonlinear Differential Equations of Monotone Types in Banach Spaces.* Springer.

[2] Brézis, H. (1973). *Operateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert.* North-Holland: Elsevier.

[3] D'Acunto, B. (1993). Hyperbolic-parabolic singular perturbations. *Rend. Mat. Appl,* no.1, p. 229-254.

[4] Engel, K.-J. (1992). On singular perturbations of second order Cauchy problems. *Pacific J. Math.* 152, no. 1, p. 79-91.

[5] Esham, B. F., Weinacht, R. J. (1988). Hyperbolic-parabolic singular perturbations for scalar nonlinearities. *Appl. Anal.* 29, no. 1-2, p. 19-44.

[6] Evans, L. C. (1998). *Partial Differential Equations.* American Mathematical Society.

[7] Fattorini, H.O. (1987). The hyperbolic singular perturbation problem: an operator approach. *J. Differential Equations* 70, no. 1, p. 1-41.

[8] Gallay, Th., Raugel, G. (1998). Scaling variables and asymptotic expansions in damped wave equations. *J. Differential Equations* 150, no. 1, p. 42-97.

[9] Ghisi, M., Gobbino, M. (2006). Global-in-time uniform convergence for linear hyperbolic-parabolic singular perturbations. *Acta Math. Sinica (English Series)* 22, no. 4, p. 1161-1170.

[10] Gobbino, M. (2001). Singular perturbation hyperbolic - parabolic for degenerate nonlinear equations of Kirchhoff type. *Nonlinear Anal.* 44, no. 3, p. 361-374.

[11] Hajouj, B. (2000). Perturbations singulières d'équations hyperboliques du second ordre non linéaires. *Ann. Math. Blaise Pascal*, 7, no. 1, p. 1-22.

[12] Hajouj, B., Madaune-Tort, M. (2001). Perturbations singulières pour une équation hyperbolique dégénérée. *Ann. Fac. Sci. Toulouse Math.* vol. (6)10, no. 2, p. 313-345.

[13] Horodnii, M. F. (2003). Stability of bounded solutions of differential equations with small parameter in a Banach space. *Ucrainian Math. J.* 55, no. 7, p. 1071-1085.

[14] Hsiao, G., Weinacht, R. (1983). Singular perturbations for a semilinear hyperbolic equation. *SIAM J. Math. Anal.* 14, no. 6, p. 1168-1179.

[15] Jager, E. M. (1975). Singular perturbations of hyperbolic type. *Nieuw Arch. Wisk.* (3)23, no. 2, p. 145-172.

[16] Kisynski, J. (1963). Sur les équations hyperboliques avec petit parametre. *Colloq. Math.* 10, p. 331-343.

[17] Kubesov, N. A. (1994). Asymptotic behavior of the solution of a mixed problem with large initial velocity for a singularly perturbed equation

that degenerates into a parabolic equation. *Vestnik Nats. Akad. Respub. Kazakhstan* 1, p. 71-74. (Russian).

[18] Milani, A. (2000). On singular perturbations for IBV problems. *Ann.Fac. Sci. Toulouse Math.* (6)9, no. 3, p. 467-468.

[19] Najman, B. (1993). Time singular limit of semilinear wave equations with damping. *J. Math. Anal. Appl.* 174, p. 95-117.

[20] Najman, B.(1998). Convergence estimate for second order Cauchy problems with a small parameter. *Czechoslovak Mathematical Journal* 48 (123), p. 737-745.

[21] Perjan, A. (2003). Linear singular perturbations of hyperbolic-parabolic type. *Bul. Acad. Stiinte Repub. Mold. Mat.* 2(42), p. 95-112.

[22] Perjan, A. (2006). Limits of solutions to the semilinear wave equations with small parameter. *Bul. Acad. Stiinte Repub. Mold. Mat.* 1(50), p. 65-84.

[23] Perjan, A., Rusu, G. (2009). Singularly perturbed Cauchy problem for abstract linear differential equations of second order in Hilbert spaces. *Ann. Acad. Rom. Sci. Ser. Math. Appl.* 1, no. 1, p. 31-61.

[24] Perjan, A., Rusu, G. (2011). Convergence estimates for abstract second-order singularly perturbed Cauchy problems with Lipschitz nonlinearities. *Asymptotic Analysis.* vol. 74, no. 3-4, p. 135-165.

[25] Rusu, G. (2009). Singular limits of solutions to the Cauchy problem for second order linear differential equations in Hilbert spaces. *Bul. Acad. Stiinte Repub. Mold. Mat.* 3(61), p. 81-95.

[26] Vainberg, M. M. (1972). *The variational method and the method of monotone operators.* Nauka. (Russian).

[27] Vrabie, I. I. (2003). *C_0-Semigroups and Applications.* North-Holland: Elsevier.

[28] Zlamal, M. (1960). The mixed problem for hyperbolic equations with a small parameter. *Czechoslovak Math. J.* 10(85), p. 83-120. (Rusian).