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CONVERGENCE OF THE FRACTIONAL PARTS OF THE RANDOM VARIABLES TO THE TRUNCATED EXPONENTIAL DISTRIBUTION*

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Abstract

Using the stochastic approximations, in this paper it was studied the convergence in distribution of the fractional parts of the sum of random variables to the truncated exponential distribution with parameter λ. This fact is feasible by means of the Fourier-Stieltjes sequence (FSS) of the random variable.

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1 Introduction

The aim of this paper is to extend the results of Wilms [9] about convergence of the fractional parts of the random variables.

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This theory was analysed by Wilms in [9], where the study of convergence of the fractional parts of the sum of random variables it was directed towards the uniform distribution on the interval $[0,1]$. Moreover, he identifed the necessary and sufficient conditions for the convergence of the product of the random variables, not necessary independent and identically distributed, towards the same uniform distribution on the interval $[0,1]$. The Fourier-Stieltjes sequence, (see Definition 1), play an important role in the study of fractional parts of random variables. Also, Wilms obtain conditions under which fractional parts of products of independent and identically distributed random variables are uniform distribution on the interval $[0,1]$. After a survey of some results by Schatte in [6], Wilms extend the results of Schatte on sums of independent and identically distributed lattice random variables. Furthermore, Schatte in [6], gives rates for the convergence of distribution function of the fractional parts of the sum of random variables to distribution function of random variables with continouous uniform distribution on the interval $[0,1]$.

The novelty of this paper consist in the identification of the conditions (Theorems 4, 5, 6) when the distribution of the fractional parts of the sum of random variables converge to the truncated exponential distribution.

2 Notations, definitions and auxiliary results

Let $(\Omega\,,\mathcal{F}\,,\mathbb{P})$ be the probability space and a random variable X, $X:\Omega\to\mathbb{R}$ measurable function. The distribution of random variable X is the measure of probability $\mathbb{P}_X$ defined on $\mathcal{B}(\mathbb{R})$ Borel and $\mathbb{P}_X(B)=\mathbb{P}\,(X\in B)$. The distribution function of the random variable X is $F_X(x)=\mathbb{P}(X<x)$, $x\in\mathbb{R}$, or $F_X(x)=\int\limits_{-\infty}^{x} f_X(y)\mathrm{d}y$ where f_X represents the density of probability of the random variable X.

Throughout the paper, for $A\subset\mathbb{R}$, $\mathcal{F}(A)=\{F_X \mid \mathbb{P}\,(X\in A)=1\}$.

For the random variable X, the fractional part of X is defined as follows: $\{X\}=X-[X]$, where $[X]$ represents the integer part of X.

The distribution function of the random variable $\{X\}$ for any $x\in[0,1]$ is

$$F_{\{X\}}(x)=\sum_{m=-\infty}^{\infty}\mathbb{P}(m\le X<m+x)=\sum_{m=-\infty}^{\infty}\left(F_X(m+x)-F_X(m)\right)\ .$$

where F_X is the distribution function of random variable X.

The random variable X has truncated exponential distribution of parameter λ (denoted by $X \sim \text{Exp}^*(\lambda)$), if its distribution function $F_X \in \mathcal{F}([0,1))$ and

$$F_X(x) = \begin{cases} 0 \ , \ x < 0 \\ \frac{1-e^{-\lambda x}}{1-e^{-\lambda}} \ , \ x \in [0,1] \\ 1 \ , \ x > 1 \end{cases} .$$

Moreover, if random variable X has the density f_X, then:

$$F_{\{X\}}(x) = \sum_{j=-\infty}^{\infty} \int_j^{j+x} f_X(y)\mathrm{d}y = \sum_{j=-\infty}^{\infty} \int_0^x f_X(j+t)\mathrm{d}t = \int_0^x \sum_{j=-\infty}^{\infty} f_X(j+t)\mathrm{d}t \ ,$$

that is $h_{\{X\}}(x) = \sum\limits_{j=-\infty}^{\infty} f_X(j+x)$, $x \in [0,1]$ is the density of probability of the random variable $\{X\}$. For example, if $X \sim \text{Exp}^*(\lambda)$, then

$$F_{\{X\}}(x) = \left(1 - e^{-\lambda x}\right) / \left(1 - e^{-\lambda}\right), \ \ x \in [0,1].$$

So, the characteristic function of the random variable X, $\varphi_X : \mathbb{R} \to \mathbb{C}$ is defined by:

$$\varphi_X(t) := \mathbb{E}e^{itX} = \int_{-\infty}^{+\infty} e^{itx}\mathrm{d}F_X(x) \ , \ \ (t \in \mathbb{R}) \ .$$

Definition 1. The Fourier-Stieltjes sequence (FSS) of the random variable X is the function $c_X : \mathbb{Z} \to \mathbb{C}$ defined by

$$c_X(k) := \varphi_X(2\pi k) \ , \ k \in \mathbb{Z} \ .$$

Proposition 1. ([9]) *For any random variable X the following relation occurs:*

$$c_X(k) = c_{\{X\}}(k) \ , \ \forall \ k \in \mathbb{Z} \ .$$

The properties that characterizes the Fourier-Stieltjes sequence, it was presented a books [1], [3] and [5].

Theorem 1. (of continuity, [5]) *Let $(F_n) \in \mathcal{F}([0,1))$ be a sequence of the random variables, and let (c_n) be FSS respectively.*

(i). *Let* $F \in \mathcal{F}([0,1))$ *be with* c *FSS respectively. If* $F_n \overset{n\to\infty}{\to} F$*, then* $\lim_{n\to\infty} c_n(k) = c(k)$*, pentru* $k \in \mathbb{Z}$*.*

(ii). *If* $\lim_{n\to\infty} c_n(k) = c(k)$ *is for* $k \in \mathbb{Z}$*, then is* $F \in \mathcal{F}([0,1))$ *so that* $F_n \overset{n\to\infty}{\to} F$*. Then the sequence* c *is FSS of* F*.*

We define the convolution F of distribution functions $F_1, F_2 \in \mathcal{F}([0,1))$ such that $F \in \mathcal{F}([0,1))$.

Denote by $F_1 \equiv F_{\{X_1\}}$, $F_2 \equiv F_{\{X_2\}}$ and $F \equiv F_{\{X_1+X_2\}}$.

To this end, let $F_1 * F_2$ denote the convolution in the customary sense, [2], i.e.

$$(F_1 * F_2)(x) = \int_{-\infty}^{\infty} F_1(x-y)\mathrm{d}F_2(y) = \int_{-\infty}^{\infty} F_2(x-y)\mathrm{d}F_1(y),$$

with $F_1 * F_2 \in \mathcal{F}([0,2))$ if $F_1, F_2 \in \mathcal{F}([0,1))$ and $x \in [0,1]$.

Definition 2. Let $F, F_1, F_2 \in \mathcal{F}([0,1))$. The function

$$(F_1 \otimes F_2)(x) = \frac{(F_1 * F_2)(x)}{(F_1 * F_2)(1) - (F_1 * F_2)(0)},\ x \in [0,1]$$

is said to be the *truncated convolution.*

In particular, if $F_1, F_2 \in \mathcal{F}([0,1))$ is the distribution functions of random variables X_1 and X_2 independent and identically exponential distributed, then the distribution function of the sum of random variables X_1 and X_2 in fractional part is $F(x) = \frac{(F_1*F_2)(x)}{(F_1*F_2)(1)-(F_1*F_2)(0)} = \frac{1-(1+\lambda x)e^{-\lambda x}}{1-(1+\lambda)e^{-\lambda}}$, with $F \in \mathcal{F}([0,1))$, $\forall\, x \in [0,1]$.

Theorem 2. (of convolution, [9]) *Let* $F, F_1, F_2 \in \mathcal{F}([0,1))$ *be with FSS* c, c_1, c_2*. Then*

$$c = c_1 c_2 \iff F = F_1 \otimes F_2\ .$$

Corollary 1. ([9]) *Let* X *and* Y *be two independent random variables with* $F_X, F_Y \in \mathcal{F}([0,1))$*. Then*

$$c_{\{X+Y\}} = c_X c_Y .$$

The next result characterizes FSS with the help of the repartition function F_X.

Proposition 2. ([9]) *Let* $F_X \in \mathcal{F}([0,1))$ *be. Then*

$$c_F(k) = 1 - 2\pi i k \int_0^1 F_X(x) e^{2\pi i k x} \mathrm{d}x \ , \quad k \in \mathbb{Z} \ .$$

The next theorem characterizes the convergence in distribution (denoted by " $\xrightarrow{d}$ ") by means of FSS.

Theorem 3. ([9]) *Let* (X_m) *be a sequence of independent random variables and* $S_n := \sum\limits_{m=1}^{n} X_m$, $n \in \mathbb{N}$. *Let* S *be the random variable with* $F_S \in \mathcal{F}([0,1))$. *Then* $\{S_n\} \xrightarrow{d} S$ *if and only if* $\prod\limits_{m=1}^{n} c_{\{X_m\}}(k) \to c_S(k)$ *if* $n \to \infty$ *for any* $k \in \mathbb{Z}$.

There are a couple of intermediate results.

Proposition 3. ([7]) *Let* (a_n) *be a sequence of real numbers,* $a_n > 0$, *for all* $n \in \mathbb{N}$. *Then* $\prod\limits_{n=0}^{\infty} a_n$ *is convergent if and only if* $\sum\limits_{n=0}^{\infty} (1 - a_n)$ *is convergent.*

Proposition 4. ([8]) *Let* X_n *be a sequence of independent random variables. We assume that* $\sum\limits_{m=1}^{\infty} Var X_m$ *is finite.*

(i). *Then* $\sum\limits_{m=1}^{n} (X_m - \mathbb{E}X_m)$ *converges almost certainly for* $n \to \infty$.

(ii). *If* $\sum\limits_{m=1}^{\infty} \mathbb{E}X_m$ *is convergent, then* $\sum\limits_{m=1}^{n} X_m$ *converges almost certainly if* $n \to \infty$.

3 The convergence in the distribution of the fractional part

In this section we shall give sufficient conditions for the fractional parts of the independent and identically distributed random variables. In Theorem

4, supposing that $\sum_{m=1}^{n} VarX_m$ is convergent, we show that the existence of limit $\lim_{n\to\infty}\left\{\sum_{m=1}^{n}\mathbb{E}X_m\right\}$ is necessary and sufficient for the convergence of the distribution of $\left\{\sum_{m=1}^{n}\mathbb{E}X_m\right\}$ if $n\to\infty$. Theorem 5 states necessary and sufficient conditions, using FSS for the convergence $\left\{\sum_{m=1}^{n}X_m\right\}$ to the distribution $\mathrm{Exp}^*(\lambda)$ if $n\to\infty$. We also neet conditions of convergence in Teorema 6.

The Fourier-Stieltjes sequence of the random variable X, $X\sim\mathrm{Exp}^*(\lambda)$, is presented in the following theoretical result:

Proposition 5. *If* $X\sim\mathrm{Exp}^*(\lambda)$ *and the distribution function* $F_X\in\mathcal{F}([0,1))$, *then*

$$c_{\mathrm{Exp}^*(\lambda)}(k)=\frac{\lambda}{2\pi ik-\lambda}\left(e^{2\pi ik\lambda}-1\right),\quad\forall\, k\in\mathbb{Z}_0.$$

Proof. According to the definition FSS,

$$\begin{aligned}c_{\mathrm{Exp}^*(\lambda)}(k) &= \int_0^1 e^{2\pi ikx}\mathrm{d}\left(1-e^{-\lambda x}\right)=\lambda\int_0^1 e^{(2\pi ik-\lambda)x}\mathrm{d}x\\ &= \frac{\lambda}{2\pi ik-\lambda}\left(e^{2\pi ik-\lambda}-1\right).\end{aligned}$$

□

Next, we shall present the original results that inform us under what circumstances the sum $\left\{\sum_{m=1}^{n}X_m\right\}$ converges in distribution towards truncated exponential distribution.

Theorem 4. *Let* (X_m) *be a sequence of independent and identically distributed random variables, so that* $\sum_{m=1}^{\infty}VarX_m$ *is finite and* $X_1\sim\mathrm{Exp}^*(\lambda)$. *Then* $\left\{\sum_{m=1}^{n}X_m\right\}$ *converges in distribution if and only if* $\lim_{n\to\infty}\left\{\sum_{m=1}^{n}\mathbb{E}X_m\right\}$ *exists.*

Proof. First, we shall proof sufficiency. Since $X_1 \sim \text{Exp}^*(\lambda)$, it results that $\mathbb{E}X_m = \frac{1}{\lambda}$.

On the other hand, $\left\{\sum\limits_{m=1}^{n} X_m\right\} = \left\{\sum\limits_{m=1}^{n} \left(X_m - \frac{1}{\lambda}\right) + \left\{\sum\limits_{m=1}^{n} \frac{1}{\lambda}\right\}\right\}$. According to Proposition 4, $\sum\limits_{m=1}^{n} \left(X_m - \frac{1}{\lambda}\right) \overset{a.s}{\to} X$, for $n \to \infty$. Then $\left\{\sum\limits_{m=1}^{n} X_m\right\} \overset{d}{\to} \left\{X + \frac{1}{\lambda}\right\}$.

As a necessity, let $\left\{\sum\limits_{m=1}^{n} X_m\right\}$ be shall converge for $n \to \infty$. Similarly, we have $\left\{\sum\limits_{m=1}^{n} \frac{1}{\lambda}\right\} = \left\{\sum\limits_{m=1}^{n} \left(\frac{1}{\lambda} - X_m\right) + \left\{\sum\limits_{m=1}^{n} X_m\right\}\right\}$. From Proposition 4, $\sum\limits_{m=1}^{n} \left(\frac{1}{\lambda} - X_m\right)$ converges almost certainly when $n \to \infty$. Therefore, $\left\{\sum\limits_{m=1}^{n} \frac{1}{\lambda}\right\}$ exists. $\square$

The following theorems provide the necessary and sufficient conditions for the fractional parts of the sums of the independent random variables identically towards the truncated exponential distribution of parameter λ.

Theorem 5. *Let (X_m) be a sequence of independent and identically random variables with (c_m), the corresponding FSS , and also $S_n = \sum\limits_{m=1}^{n} X_m$, $n \in \mathbb{N}$.*

(i). $\{S_n\} \overset{d}{\to} \text{Exp}^*(\lambda) \Longleftrightarrow \prod\limits_{m=1}^{n} c_m(k) \to \frac{\lambda}{2\pi ik-\lambda}\left(e^{2\pi ik-\lambda} - 1\right)$, $n \to \infty$,

(ii). *We suppose $c_m(k) \neq 0$, $\forall\, k \in \mathbb{Z}_0$, $m \in \mathbb{N}$, $\{S_n\}$ does not converge to Exp$^*(\lambda)$ is equivalent to*

$$\sum_{m=1}^{\infty} (1 - |c_m(k)|) \quad \textit{is divergent}, \ \forall\, k \in \mathbb{Z}_0\ .$$

Proof. [i]. Based on the Theorem 3,

$$\{S_n\} \overset{d}{\to} \text{Exp}^*(\lambda) \Longleftrightarrow c_{\{S_n\}}(k) \overset{n\to\infty}{\to} c_{\text{Exp}^*(\lambda)}(k) \overset{\text{Proposition 5}}{\Longleftrightarrow}$$

$$\Longleftrightarrow \prod_{m=1}^{n} c_m(k) \to \frac{\lambda}{2\pi ik - \lambda}\left(e^{2\pi ik-\lambda} - 1\right)\ ,\ n \to \infty\ .$$

[ii]. According to Proposition 3, $\sum_{m=1}^{\infty}(1-|c_m(k)|)$ divergent $\Longleftrightarrow$ $\prod_{m=1}^{\infty} c_m(k)$ is divergent, that is $\{S_n\} \xrightarrow{d}$ F, with $F \neq \mathrm{Exp}^*(\lambda)$. $\square$

Corollary 2. *If the sequence (c_m), of the random variables sequence (X_m) meets the condition of* $\sum_{m=1}^{\infty}(1-|c_m(k)|)$ to be convergent, $\forall\ k \in \mathbb{Z}_0$, *then* $\{S_n\} \xrightarrow{d} \mathrm{Exp}^*(\lambda)$, *where* $S_n = \sum_{m=1}^{n} X_m$, $n \in \mathbb{N}$.

Theorem 6. *Let (X_m) be a sequence of independent and identically distributed random variables with the characteristic function φ, $X_1 \sim Exp^*(\lambda)$, and $0 < VarX_1 < \infty$. Let (a_m) be a sequence of real numbers so that* $\lim_{m\to\infty} a_m = 0$. *We define $V_n := \sum_{m=1}^{n} a_m X_m$, $n \in \mathbb{N}$.*

(i). *If $\sum_{m=1}^{\infty} a_m^2$ is convergent, then* $\{V_n\} \xrightarrow{d} \mathrm{Exp}^*(\lambda)$.

(ii). *We suppose that there is $k \in \mathbb{Z}_0$, $\varphi(2\pi k a_m) \neq 0$. If $\sum_{m=1}^{\infty} a_m^2$ is divergent, then $\{V_n\}$ don't converge to* $\mathrm{Exp}^*(\lambda)$.

Proof. [i]. Let $k \in \mathbb{Z}_0$ be fixed. By Corollary 2, it is sufficient to show that $\sum_{m=1}^{\infty}(1-|\varphi(2\pi k a_m)|)$ is convergent.

It is known that if $X \sim \mathrm{Exp}^*(\lambda)$, then $\varphi_X(t) = \lambda/(\lambda - it)$, from where

$$|\varphi_X(t)| = \frac{\lambda}{\sqrt{\lambda^2+t^2}} = \left(1+\left(\frac{t}{\lambda}\right)^2\right)^{-\frac{1}{2}}.$$

If we consider the development in binomial series $(1+x)^{-\frac{1}{2}} = 1 - \frac{x}{2} + \frac{3}{8}x^2 + ...$, then we obtain $\left(1+\left(\frac{t}{\lambda}\right)^2\right)^{-\frac{1}{2}} = 1 - \frac{1}{2\lambda}\, t^2 + o(t^2)$, from where the following $\frac{1}{4\lambda}\, t^2 < 1 - |\varphi_X(t)| < \frac{1}{\lambda}\, t^2$.

Then

$$\sum_{m=1}^{\infty}(1-|\varphi(2\pi k a_m)|) < \sum_{m=1}^{\infty}\frac{1}{\lambda}\, 4\pi^2 k^2 a_m^2 = \frac{4\pi^2 k^2}{\lambda}\sum_{m=1}^{\infty} a_m^2.$$

As $\sum\limits_{m=1}^{\infty} a_m^2$ is convergent, it means that $\sum\limits_{m=1}^{\infty} (1 - |\varphi(2\pi k a_m)|)$ is convergent.

[ii]. Since $1 - |\varphi_X(t)| > \frac{t^2}{4\lambda}$, we have $\sum\limits_{m=1}^{\infty} (1 - |\varphi(2\pi k a_m)|) > \frac{\pi^2 k^2}{\lambda} \sum\limits_{m=1}^{\infty} a_m^2$.

Results that $\sum\limits_{m=1}^{\infty} (1 - |\varphi(2\pi k a_m)|)$ is divergent because $\sum\limits_{m=1}^{\infty} a_m^2$ is divergent. Next Theorem 5(ii) is taken into account . $\square$

Example

Let (X_m) be a sequence of independent and identically distributed random variables, $X_1 \sim \mathrm{Exp}^*(\lambda)$ and $a_m = m^{-b}$, $b > 0$; $\sum\limits_{m=1}^{\infty} a_m^2 = \sum\limits_{m=1}^{\infty} \frac{1}{m^{2b}} = \begin{cases} = \infty \ , \ b \leq \frac{1}{2} \\ < \infty \ , \ b > \frac{1}{2} \end{cases}$. We have the following situations for the sequence $V_n = \sum\limits_{m=1}^{n} \frac{1}{m^b} X_m$:

(1) $b \leq \frac{1}{2}$, $\sum\limits_{m=1}^{\infty} a_m^2 = \infty \overset{Theorem\ 6}{\Longrightarrow} \{V_n\}$ does not converge to $\mathrm{Exp}^*(\lambda)$.

(2) $\frac{1}{2} < b \leq 1$, $\sum\limits_{m=1}^{\infty} a_m^2 < \infty \overset{Theorem\ 6}{\Longrightarrow} \{V_n\} \overset{d}{\to} \mathrm{Exp}^*(\lambda)$ or

$$\left\{\frac{1}{1^b} X_1 + \frac{1}{2^b} X_2 + ... + \frac{1}{n^b} X_n\right\} \overset{n\to\infty}{\to} \mathrm{Exp}^*(\lambda)$$

(3) $b > 1$, according to Theorem 4, $\lim_{n\to\infty} \sum\limits_{m=1}^{n} \mathbb{E}X_m = \lim_{n\to\infty} \frac{n}{\lambda} = \infty$, that is $\{V_n\} \overset{d}{\to} F$, with $F \neq \mathrm{Exp}^*(\lambda)$.

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