Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066 - 6594 Vol. 4, No. 1 / 2012

OSCILLATION OF NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS OF FOURTH ORDER WITH SEVERAL DELAYS*

Saroj Panigrahi† Rakhee Basu‡

Abstract

In this paper, oscillatory and asymptotic behaviour of solutions of a class of nonlinear fourth order neutral differential equations with several delay of the form

$$(r(t)(y(t)+p(t)y(t-\tau))'')''+\sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i))=0$$

and

$$(E) \qquad (r(t)(y(t)+p(t)y(t-\tau))'')''+\sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i))=f(t)$$

are studied under the assumption

$$\int_0^\infty \frac{t}{r(t)}dt=\infty$$

*Accepted for publication in revised form on March 30, 2012.

†Corresponding author : panigrahi2008@gmail.com,spsm@uohyd.ernet.in, Department of Mathematics, University of Tennessee, Chattanooga TN 37403-2598 USA; Indo-US fellow 2011 in Mathematical and Computational Sciences. Research supported by Indo-US Science and Technology Forum (IUSSTF), Fullbright House, New Delhi, India

‡rakheebasu1983@gmail.com, Department of Mathematics and Statistics, University of Hyderabad, Hyderabad-500 046 INDIA; Research supported by Department of Science and Technology, New Delhi, through the letter no. SR/S4/MS : 541/08, dated September 30, 2008.

for various ranges of $p(t)$. Using Schauder's fixed point theorem, sufficient conditions are obtained for the existence of bounded positive solutions of (E). The results obtained in this paper generalize the results existing in the literature.

Mathematics Subject Classification 2000: 34 C 10, 34 C 15

Keywords: Neutral differential equations, nonlinear, oscillation, several delays, existence of positive solutions, asymptotic behaviour.

1 Introduction

Consider the fourth order nonlinear neutral delay differential equations with several delays of the form

$$(r(t)(y(t)+p(t)y(t-\tau))'')'' + \sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i)) = 0, \tag{1.1}$$

and its associated forced equations

$$(r(t)(y(t)+p(t)y(t-\tau))'')'' + \sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i)) = f(t), \tag{1.2}$$

where $r \in C([0,\infty),[0,\infty))$, $p \in C([0,\infty),\mathbb{R})$, $q_i \in C([0,\infty),[0,\infty))$ for $i = 1,....,m$, $f \in C([0,\infty),\mathbb{R})$, $G \in C(\mathbb{R},\mathbb{R})$ is nondecreasing with $uG(u) > 0$, for $u \neq 0$, $\tau > 0$, $\alpha_i > 0$ for $i = 1,...,m$.

The object of this work is to study oscillatory and asymptotic behaviour of solution of (1.1) and (1.2) under the assumption

$$(H_1) \qquad \int_0^\infty \frac{t}{r(t)} dt = \infty.$$

In [11], Parhi and Tripathy have studied the oscillatory and asymptotic behaviour of the fourth order nonlinear neutral delay differential equations of the form

$$(r(t)(y(t)+p(t)y(t-\tau))'')'' + q(t)G(y(t-\alpha)) = 0,$$

and

$$(r(t)(y(t)+p(t)y(t-\tau))'')''+q(t)G(y(t-\alpha))=f(t)$$

respectively under the same assumption (H_1). If $r(t)=1, m=1$ and $q_1(t)=q(t)$, then (H_1) is satisfied and equation (1.1) and (1.2) reduce to, respectively,

$$(y(t)+p(t)y(t-\tau))^{(iv)}+q(t)G(y(t-\alpha))=0, \tag{1.3}$$

and its associated forced equation

$$(y(t)+p(t)y(t-\tau))^{(iv)}+q(t)G(y(t-\alpha))=f(t). \tag{1.4}$$

In recent papers [9, 10] Parhi and Rath studied oscillatory and asymptotic behavior of solution of higher order neutral differential equations

$$(y(t)+p(t)y(t-\tau))^{(n)}+q(t)G(y(t-\alpha))=0, \tag{1.5}$$

and its associated forced equations

$$(y(t)+p(t)y(t-\tau))^{(n)}+q(t)G(y(t-\alpha)=f(t). \tag{1.6}$$

Clearly, equations (1.3) and (1.4) are particular cases of equations (1.5) and (1.6) respectively. However, equations (1.1) and (1.2) cannot be termed, in general, as particular cases of equations (1.5) and (1.6). Most of the results in [10] hold when n is even. Therefore, it is interesting to study the more general equations (1.1) and (1.2) under (H_1). It is interesting to observe that the nature of the function $r(t)$ influences the behaviour of solutions of (1.1) and (1.2). This behaviour can be easily observed in case of the homogeneous equation (1.1). By the use of new Lemma 1.4 which has been proved in Section 1, we have shown that all the solutions of (1.1) are oscillatory in Theorem 2.3. The results obtained in this papers are new and generalize the existing results in the literature (see [8–11]).

Moreover, the delay differential equations play an important role in modelling virtually every physical, technical, or biological process, from celestial motion, to bridge design, to interactions between neurons. Differential equations such as those used to solve real-life problem may not necessarily be directly solvable, that is do not have closed form solutions. Instead, solutions can be approximated by using numerical methods.

By a solution of (1.1)/(1.2) we understand a function $y \in C([-\rho, \infty), \mathbb{R})$ such that $y(t) + p(t)y(t-\tau)$ is twice continuously differentiable, $r(t)(y(t) + p(t)y(t-\tau))''$ is twice continuously differentiable and (1.1)/(1.2) is satisfied for $t \geq 0$, where $\rho = max\{\tau, \alpha_i\}$ for $i = 1, ..., m$, and $\sup\{|y(t)| : t \geq t_0\} > 0$ for every $t_0 \geq 0$. A solution of (1.1)/(1.2) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory.

We need the following lemmas for our use in the sequel.

Lemma 1.1. *[11] Let (H_1) hold. Let u be a twice continuously differentiable function on $[0, \infty)$ such that $r(t)u''(t)$ is twice continuously differentiable and $(r(t)u''(t))'' \leq 0$ for large t. If $u(t) > 0$ ultimately, then one of the cases (a) and (b) holds for large t, and if $u(t) < 0$ ultimately, then one of the cases (b), (c), (d) and (e) holds for large t, where*
(a) $u'(t) > 0$, $u''(t) > 0$, and $(r(t)u''(t))' > 0$
(b) $u'(t) > 0$, $u''(t) < 0$, and $(r(t)u''(t))' > 0$
(c) $u'(t) < 0$, $u''(t) < 0$, and $(r(t)u''(t))' > 0$
(d) $u'(t) < 0$, $u''(t) < 0$, and $(r(t)u''(t))' < 0$
(e) $u'(t) < 0$, $u''(t) > 0$, and $(r(t)u''(t))' > 0$.

Lemma 1.2. *[11] Let the conditions of Lemma 1.1 hold. If $u(t) > 0$ ultimately, then $u(t) > R_T(t)(r(t)u''(t))'$ for $t \geq T \geq 0$, where $R_T(t) = \int_T^t \frac{(t-s)(s-T)}{r(s)} ds$.*

Lemma 1.3. *[3] Let F, G, $P : [t_0, \infty) \to \mathbb{R}$ and $c \in \mathbb{R}$ be such that $F(t) = G(t) + P(t)G(t-c)$, for $t \geq t_0 + \max\{0, c\}$. Assume that there exists numbers $P_1, P_2, P_3, P_4 \in \mathbb{R}$ such that $P(t)$ is one of the following ranges:*

$$(1)\quad P_1 \leq P(t) \leq 0, \quad (2)\quad 0 \leq P(t) \leq P_2 < 1, \quad (3)\quad 1 < P_3 \leq P(t) \leq P_4.$$

Suppose that $G(t) > 0$ *for* $t \geq t_0$, $\liminf_{t\to\infty} G(t) = 0$ *and that* $\lim_{t\to\infty} F(t) = L \in \mathbb{R}$ *exists. Then* $L = 0$.

Lemma 1.4. *If* $q_i \in C([0,\infty),[0,\infty))$ *for* $i = 1, ..., m$ *and*

$$\liminf_{t\to\infty} \int_{t-\rho}^{t} \sum_{i=1}^{m} q_i(s)ds > \frac{1}{e}, \tag{1.7}$$

then

$$x'(t) + \sum_{i=1}^{m} q_i(t)x(t-\alpha_i) \leq 0, \tag{1.8}$$

cannot have an eventually positive solution for $t \geq 0$.

Proof. Assume for the sake of contradiction, the inequation (1.8) has an eventually positive solution $x(t)$ for $t \geq t_0$. Then there exists $t_i^* \geq t_0 + \alpha_i$ for every i such that for $t \geq t^* = \max_{i=1,2,...,m}\{t_i^*\}$, and

$$x(t) > 0, x(t-\alpha_i) > 0 \ for \ i = 1, ..., m.$$

From (1.8) we get

$$\begin{aligned} x'(t) &\leq -\sum_{i=1}^{m} q_i(t)x(t-\alpha_i) \\ &\leq 0. \end{aligned}$$

Therefore,

$$x(t-\alpha_i) \geq x(t), \qquad for \ \ i = 1, ..., m. \tag{1.9}$$

From (1.7) it follows that there exists $c > 0$ and $t_1 > t^*$ such that

$$\int_{t-\alpha_i}^{t} \sum_{i=1}^{m} q_i(s)ds \geq c > \frac{1}{e}, \tag{1.10}$$

for $t \geq t_1$ and $i = 1, 2, \ldots m$. From (1.8) and (1.9) it follows that

$$\begin{aligned} x'(t) &\leq -\sum_{i=1}^{m} q_i(t)x(t-\alpha_i) \\ &\leq -x(t)\sum_{i=1}^{m} q_i(t). \end{aligned}$$

Therefore

$$\frac{x'(t)}{x(t)} + \sum_{i=1}^{m} q_i(t) \quad \leq 0.$$

Integrating the preceeding inequality from $t - \alpha_i$ to t, we obtain

$$\begin{aligned} \ln\frac{x(t)}{x(t-\alpha_i)} &\leq -\int_{t-\alpha_i}^{t} \sum_{i=1}^{m} q_i(s)ds \leq -c, \\ \ln\frac{x(t)}{x(t-\alpha_i)} + c &\leq 0, \end{aligned} \tag{1.11}$$

for $t \geq t_1 + \alpha_i$. It is easy to verify that

$$e^c \geq ec \tag{1.12}$$

for $c \in \mathbb{R}$. From (1.11) and (1.12) it follows that

$$ecx(t) \leq x(t - \alpha_i). \tag{1.13}$$

Repeating the above procedure, it follows from induction that for any positive integer k

$$(ec)^k x(t) \leq x(t - \alpha_i), \tag{1.14}$$

for $t \geq \max_{i=1,2,\ldots m}\{t_1 + 2\alpha_i\}$. Choose k such that

$$\left(\frac{2}{c}\right)^2 < (ec)^k \tag{1.15}$$

which is possible as $ec > 1$. Fix $\tilde{t} \geq \max_{i=1,2,\ldots,m}\{t_1 + k\alpha_i\}$. From (1.10) it follows that there exists a $\xi_i \in (\tilde{t}, \tilde{t} + \alpha_i)$ for every i such that

$$\int_{\tilde{t}}^{\xi_i} \sum_{i=1}^{m} q_i(s)ds \geq \frac{c}{2}, \qquad \int_{\xi_i}^{\tilde{t}+\rho} \sum_{i=1}^{m} q_i(s)ds \geq \frac{c}{2}.$$

Integrating (1.8) from $[\tilde{t}, \xi_i]$ and $[\xi_i, \tilde{t} + \alpha_i]$, we have

$$x(\xi_i) - x(\tilde{t}) + \int_{\tilde{t}}^{\xi_i} \sum_{i=1}^{m} q_i(s)x(s - \alpha_i)ds \leq 0, \tag{1.16}$$

$$x(\tilde{t} + \alpha_i) - x(\xi_i) + \int_{\xi_i}^{\tilde{t}+\alpha_i} \sum_{i=1}^{m} q_i(s)x(s - \alpha_i)ds \leq 0. \tag{1.17}$$

As $x(t) > 0$ and is non-increasing, ignoring the first term from (1.16) and (1.17) we have

$$-x(\tilde{t}) + \int_{\tilde{t}}^{\xi_i} \sum_{i=1}^{m} q_i(s)x(s - \alpha_i)ds \leq 0, \tag{1.18}$$

and

$$-x(\xi_i) + \int_{\xi_i}^{\tilde{t}+\alpha_i} \sum_{i=1}^{m} q_i(s)x(s - \alpha_i)ds \leq 0. \tag{1.19}$$

Again using the fact that $x(t)$ decreasing in (1.18) and (1.19) we get

$$-x(\tilde{t}) + x(\xi_i - \alpha_i) \int_{\tilde{t}}^{\xi_i} \sum_{i=1}^{m} q_i(s)ds \leq 0,$$

and

$$-x(\xi) + x(\tilde{t}) \int_{\xi_i}^{\tilde{t}+\alpha_i} \sum_{i=1}^{m} q_i(s)ds \leq 0.$$

Therefore,

$$-x(\tilde{t}) + x(\xi_i - \alpha_i)\frac{c}{2} < 0. \tag{1.20}$$

Similarly from (1.19), we obtain

$$-x(\xi_i) + x(\tilde{t})\frac{c}{2} < 0. \tag{1.21}$$

From (1.20) and (1.21), it follows that

$$\frac{x(\xi_i)}{x(\xi_i - \alpha_i)} > \left(\frac{c}{2}\right)^2, \ for\ i = 1, 2, \ldots, m$$

which in turns implies

$$(ec)^k \leq \left(\frac{2}{c}\right)^2,$$

which is a contradiction to (1.15). Hence the Lemma is proved.

Theorem 1.5. *([3], Schauder's fixed point theorem) Let M be a closed, convex and non-empty subset of Banach Space X. Let $T : M \to M$ be a continuous function such that TM is relatively compact subset of X. Then T has at least one fixed point in M. That is, there exists an $x \in M$ such that $Tx = x$.*

2 Homogeneous Oscillations

In this section, sufficient conditions are obtained for oscillatory and asymptotic behaviour of all solutions or bounded solutions of (1.1) under the assumption (H_1).

Theorem 2.1. *Let $0 \leq p(t) \leq p < \infty$, $\tau \leq \alpha_i, i = 1, 2, ..., m$, and (H_1) hold. If*
(H_2) there exists $\lambda > 0$ such that $G(u) + G(v) \geq \lambda G(u + v), u > 0, v > 0$;
(H_3) $G(u)G(v) = G(uv)$ for $u, v \in \mathbb{R}$;
(H_4) G is sublinear and $\int_0^c \frac{du}{G(u)} < \infty$ for all $c > 0$;

(H_5) $\int_{T+\rho}^{\infty}\sum_{i=1}^{m}Q_i(t)G(R_T(t-\alpha_i))dt=\infty$, $Q_i(t)=\min\{q_i(t),q_i(t-\tau)\}; i=1,...,m$ *for* $t\geq\tau$

hold, then every solution of (1.1) *oscillates.*

Proof. Assume that (1.1) has a nonoscillatory solution on $[t_0,\infty)$, $t_0\geq 0$ and let it be $y(t)$. Hence $y(t)>0$ or <0 for $t\geq t_0$. Suppose that $y(t)>0$ for $t\geq t_0$. Setting

$$z(t)=y(t)+p(t)y(t-\tau), \tag{2.1}$$

we obtain

$$0<z(t)\leq y(t)+py(t-\tau), \tag{2.2}$$

and

$$(r(t)z''(t))''=-\sum_{i=1}^{m}q_i(t)G(y(t-\alpha_i))\leq 0,\not\equiv 0 \tag{2.3}$$

for $t\geq t_0+\rho$. By the Lemma 1.1, any one of the cases (a) and (b) holds. Upon using (H_2) and (H_3), Eq.(1.1) can viewed as

$$\begin{aligned}
0 &= (r(t)z''(t))''+\sum_{i=1}^{m}q_i(t)G(y(t-\alpha_i))+G(p)(r(t-\tau)z''(t-\tau))''\\
&+ G(p)\sum_{i=1}^{m}q_i(t-\tau)G(y(t-\tau-\alpha_i))\\
&\geq (r(t)z''(t))''+G(p)(r(t-\tau)z''(t-\tau))''\\
&+ \lambda\sum_{i=1}^{m}Q_i(t)G(y(t-\alpha_i)+ay(t-\alpha_i-\tau))\\
&= (r(t)z''(t))''+G(p)(r(t-\tau)z''(t-\tau))''+\lambda\sum_{i=1}^{m}Q_i(t)G(z(t-\alpha_i))
\end{aligned}$$

for $t\geq t_1>t_0+2\rho$. Therefore

$$\begin{aligned}
0 &\geq (r(t)z''(t))''+G(p)(r(t-\tau)z''(t-\tau))''\\
&+ \lambda\sum_{i=1}^{m}Q_i(t)G(R_T(t-\alpha_i)(r(t-\alpha_i)z''(t-\alpha_i))')
\end{aligned}$$

due to Lemma 1.2, for $t \geq T+\rho > t_1$. Hence

$$\begin{aligned} 0 &\geq (r(t)z''(t))'' + G(p)(r(t-\tau)z''(t-\tau))'' \\ &+ \lambda \sum_{i=1}^{m} Q_i(t) G(R_T(t-\alpha_i)) G((r(t-\alpha_i)z''(t-\alpha_i))'). \end{aligned}$$

Using the fact that $(r(t)z''(t)'$ is decreasing, we obtain

$$\begin{aligned} \lambda \sum_{i=1}^{m} Q_i(t) G(R_T(t-\alpha_i)) \leq & -[G((r(t)z''(t))')]^{-1}(r(t)z''(t))'' \\ & -G(p)[G((r(t-\tau)z''(t-\tau))')]^{-1}(r(t-\tau)z''(t-\tau))'' \end{aligned}$$

Because $\lim_{t\to\infty}(r(t)z''(t))' < \infty$, then using (H_4) the above inequality becomes

$$\int_{T+\rho}^{\infty} \sum_{i=1}^{m} Q_i(t) G(R_T(t-\alpha_i)) dt < \infty,$$

which contradicts (H_5).

Finally, we suppose that $y(t) < 0$ for $t \geq t_0$. Hence putting $x(t) = -y(t)$ for $t \geq t_0$, we obtain $x(t) > 0$ and

$$(r(t)(x(t)+p(t)x(t-\tau))'')'' + \sum_{i=1}^{m} q_i(t) G(x(t-\alpha_i)) = 0.$$

Proceeding as above, we get a contradiction. This completes the proof of the theorem.

Theorem 2.2. *Let $0 \leq p(t) \leq p < \infty$. Suppose (H_1), (H_2) hold. If*
$(H_3^{'})$ *$G(u)G(v) \geq G(uv)$ for $u, v > 0$;*
(H_6) *$G(-u) = -G(u), u \in \mathbb{R}$;*
(H_7) $\int_{\tau}^{\infty} \sum_{i=1}^{m} Q_i(t) dt = \infty$
hold, then every solution of (1.1) oscillates.

Proof. Let $y(t)$ be a non-oscillatory solution of (1.1). Let $y(t) > 0$ for $t \geq t_0$. The proof for the case $y(t) < 0, t \geq t_0$, is similar. Setting $z(t)$ as in (2.1), we obtain (2.2) and (2.3) for $t \geq t_0 + \rho$. From Lemma 1.1 it follows that one of the cases (a) and (b) holds. In both the cases (a) and (b), $z(t) > 0$ and $z'(t) > 0$, implies that $z(t) > k > 0$ for $t \geq t_1 > t_0 + \rho$. Proceeding as in the proof of Theorem 2.1 we obtain

$$\begin{aligned} 0 &\geq (r(t)z''(t))'' + G(p)(r(t-\tau)z''(t-\tau))'' + \lambda \sum_{i=1}^{m} Q_i(t)G(z(t-\alpha_i)) \\ &\geq (r(t)z''(t))'' + G(p)(r(t-\tau)z''(t-\tau))'' + \lambda \sum_{i=1}^{m} Q_i(t)G(k) \end{aligned}$$

for $t \geq t_2 > t_1 + \rho$. Because $\lim_{t\to\infty}(r(t)z''(t))' < \infty$, integrating the above inequality from t_2 to ∞, we obtain

$$\int_{t_2}^{\infty} \sum_{i=1}^{m} Q_i(t)dt < \infty,$$

which contradicts (H_7). Hence the theorem is proved.

Theorem 2.3. *Let* $0 \leq p(t) \leq p < 1$. *Suppose that* (H_1), (H_3) *hold and* $\tau \leq \alpha_i, i = 1, 2, ...m$. *If*

(H_8) $\liminf_{|x|\to 0} \dfrac{G(x)}{x} \geq \gamma > 0,$

and

(H_9) $\liminf_{t\to\infty} \int_{t-\alpha_i}^{t} \sum_{i=1}^{m} G(R_T(s-\alpha_i))q_i(s)ds > (e\gamma G(1-p))^{-1}$

hold, then all the solutions of (1.1) *oscillate.*

Remark 2.4. (H_9) implies that

(H_{10}) $\displaystyle\int_{T+\alpha_i}^{\infty} \sum_{i=1}^{m} G(R_T(s-\alpha_i))q_i(s)ds = \infty.$

Indeed, if

$$\int_{T+\alpha_i}^{\infty} \sum_{i=1}^{m} G(R_T(s-\alpha_i))q_i(s)ds = b < \infty,$$

then for $t > T + 2\alpha_i$,

$$\int_{t-\alpha_i}^{t} \sum_{i=1}^{m} G(R_T(s-\alpha_i))q_i(s)ds = \left(\int_{T+\alpha_i}^{t} - \int_{T+\alpha_i}^{t-\alpha_i}\right) \sum_{i=1}^{m} G(R_T(s-\alpha_i))q_i(s)ds,$$

implies that

$$\liminf_{t\to\infty} \int_{t-\alpha_i}^{t} \sum_{i=1}^{m} G(R_T(s-\alpha_i))q_i(s)ds \leq b - b = 0,$$

which contradicts (H_9).

Proof of Theorem 2.3. Suppose that $y(t)$ is a nonoscillatory solution of (1.1). Let $y(t) > 0$ for $t \geq t_0 > 0$. The case $y(t) < 0$ for $t \geq t_0$ is similar. Using (2.1) we obtain (2.2) and (2.3) for $t \geq \max_{i=1,2,\ldots,m}\{t_0 + \alpha_i\}$. Then any one of the cases (a) and (b) of Lemma 1.1 holds. In each case, $z(t)$ is nondecreasing. Hence

$$\begin{aligned} (1-p(t))z(t) &< z(t) - p(t)z(t-\tau) \\ &= y(t) - p(t)p(t-\tau)y(t-2\tau) < y(t), \end{aligned}$$

$t \geq \max_{i=1,2,\ldots,m}\{t_0 + 2\alpha_i\}$, that is,

$$y(t) > (1-p)z(t).$$

From (2.3), we obtain

$$\begin{aligned} 0 &= (r(t)z''(t))'' + \sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i)) \\ &\geq (r(t)z''(t))'' + \sum_{i=1}^{m} q_i(t)G(1-p)G(z(t-\alpha_i)) \qquad (2.4) \\ &\geq (r(t)z''(t))'' + G(1-p)\sum_{i=1}^{m} q_i(t)G(R_T(t-\alpha_i))G((r(t-\alpha_i)z''(t-\alpha_i))') \end{aligned}$$

due to Lemma 1.2 for

$t \geq \max_{i=1,2,\dots,m}\{T+\alpha_i\} \geq \max_{i=1,2,\dots,m}\{t_0+3\alpha_i\}$. Let $\lim_{t\to\infty}(r(t)z''(t))' = c$, $c \in [0,\infty)$. If $0 < c < \infty$, then there exists $c_1 > 0$ such that $(r(t)z''(t))' > c_1$ for $t \geq t_1 > \max_{i=1,2,\dots,m}\{T+\alpha_i\}$. For $t \geq t_2 > \max_{i=1,2,\dots,m}\{t_1+\alpha_i\}$

$$G(1-p)\sum_{i=1}^{m} q_i(t)G(R_T(t-\alpha_i))G(c_1) \leq -(r(t)z''(t))''.$$

Integrating the above inequality from t_2 to ∞, we get

$$\int_{t_2}^{\infty}\sum_{i=1}^{m} q_i(t)G(R_T(t-\alpha_i))dt < \infty,$$

a contradiction to (H_{10}). Hence $c = 0$. Consequently, (H_8) implies that $G((r(t)z''(t))') \geq \gamma(r(t)z''(t))'$ for $t \geq t_3 > t_2$. Hence (2.4) yields

$$(r(t)z''(t))'' + \gamma G(1-p)\sum_{i=1}^{m} q_i(t)G(R_T(t-\alpha_i))(r(t-\alpha_i)z''(t-\alpha_i))' \leq 0,$$

for $t \geq \max_{i=1,2,\dots,m}\{t_3+\alpha_i\}$. As $\tau \leq \alpha_i$ for $i = 1,\dots,m$, from Lemma 1.4 it follows that

$$u'(t) + \gamma G(1-p)\sum_{i=1}^{m} q_i(t)G(R_T(t-\alpha_i))u(t-\alpha_i) \leq 0$$

admits a positive solution $(r(t)z''(t))'$, which is a contradiction due to (H_9). Hence proof of theorem is complete.

Theorem 2.5. *Let $0 \leq p(t) \leq p < \infty$, $\tau \leq \alpha_i, i = 1,2,\dots,m$, and $(H_1)-(H_3)$ hold. Assume that*

(H_{11}) $\quad \dfrac{G(x_1)}{x_1^{\sigma}} \geq \dfrac{G(x_2)}{x_2^{\sigma}}$ *for* $x_1 \geq x_2 > 0$ *and* $\sigma \geq 1$;

and

(H_{12}) $\quad \displaystyle\int_{T+\rho}^{\infty}\sum_{i=1}^{m} Q_i(t)R_T^{\alpha}(t-\alpha_i)ds = \infty$

hold. Then every solution of (1.1) *oscillates.*

Proof. Proceeding as in the proof of Theorem 2.1, we obtain

$$(r(t)z''(t))'' + G(p)(r(t-\tau)z''(t-\tau))'' + \lambda \sum_{i=1}^{m} Q_i(t)G(z(t-\alpha_i)) \leq 0 \quad (2.5)$$

for $t \geq t_1 > t_0 + 2\rho$. Using the fact that $z(t)$ is nondecreasing, there exists $k > 0$ and $t_2 > 0$ such that $z(t) > k$ for $t \geq t_2 > t_1$. Using (H_{11}) and Lemma 1.2 we obtain, for $t > T + \rho \geq t_2 + \rho$,

$$\begin{aligned} G(z(t-\alpha_i)) &= (G(z(t-\alpha_i))/z^\sigma(t-\alpha_i))z^\sigma(t-\alpha_i) \\ &\geq (G(k)/k^\sigma)(z^\sigma(t-\alpha_i)) \\ &> (G(k)/k^\sigma)R_T^\sigma(t-\alpha_i)((r(t-\alpha_i)z''(t-\alpha_i))')^\sigma. \end{aligned}$$

Thus (2.5) yields

$$\begin{aligned} \lambda(G(k)/k^\sigma)\sum_{i=1}^{m} Q_i(t)R_T^\sigma(t-\alpha_i)((r(t-\alpha_i)z''(t-\alpha_i))')^\sigma \quad &\leq \quad -(r(t)z''(t))'' \\ &\quad -G(p)(r(t-\tau)z''(t-\tau))'', \end{aligned}$$

As $\tau \leq \alpha_i$ and $(r(t)z''(t))'$ is nonincreasing, therefore,

$$\begin{aligned} \lambda(G(k)/k^\sigma)\sum_{i=1}^{m} Q_i(t)R_T^\sigma(t-\alpha_i) &< -((r(t)z''(t))')^{-\sigma}(r(t)z''(t))'' \\ &\quad -G(p)((r(t-\tau)z''(t-\tau))')^{-\sigma}(r(t-\tau)z''(t-\tau))''. \end{aligned}$$

Since $\lim_{t\to\infty}(r(t)z''(t))'$ exists, then integrating the preceding inequality from $T+\rho$ to ∞, we obtain

$$\int_{T+\rho}^{\infty}\sum_{i=1}^{m} Q_i(t)R_T^\sigma(t-\alpha_i)dt < \infty,$$

a contradiction due to (H_{12}). Hence $y(t) < 0$ for $t \geq t_0$. Proceeding as in Theorem 2.1 we will arrive at contradiction. Thus the theorem is proved.

Theorem 2.6. *Let $-1 < p \leq p(t) \leq 0$. If (H_1), (H_3), (H_4) hold and if*

(H_{13}) $\displaystyle\int_0^{\infty}\sum_{i=1}^{m} q_i(t)dt = \infty,$

then every solution of (1.1) *either oscillates or tends to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a nonoscillatory solution of (1.1). In view of (H_3), without loss of generality we may consider that $y(t) > 0$ for $t \geq t_0 > 0$. Setting $z(t)$ as in (2.1), we obtain (2.3) for $t \geq t_0 + \rho$. Hence $z(t) > 0$ or < 0 for $t \geq t_0 > 0$. If $z(t) > 0$ for $t \geq t_1$, then any one of the cases (a) and (b) of Lemma 1.1 holds. Consequently, $z(t) > R_T(t)(r(t)z''(t))'$ for $t \geq T > t_1$ due to Lemma 1.2. Moreover, $z(t) \leq y(t)$ implies that $y(t) > R_T(t)(r(t)z''(t))'$ for $t \geq t_2 > T + \rho$ and $(r(t)z''(t))'$ is monotonic decreasing, then (2.3) yields, for $t \geq t_2 > T + \rho$,

$$(r(t)z''(t))'' \leq -\sum_{i=1}^{m} q_i(t)G(R_T(t-\alpha_i))G((r(t)z''(t))'). \tag{2.6}$$

Since R_T is nondecreasing, then

$$\int_{t_2}^{\infty} \sum_{i=1}^{m} q_i(t)dt < \infty,$$

a contradiction to (H_{13}). Hence $z(t) < 0$ for $t \geq t_1$. Therefore $y(t) < -p(t)y(t-\tau) < y(t-\tau)$ implies $y(t)$ is bounded, implies that, $z(t)$ is bounded and this implies any one of the cases (b) - (e) of Lemma 1.1 holds. Suppose case (b) holds. If $\lim_{t\to\infty} z(t) = \alpha(say)$, then $-\infty < \alpha \leq 0$.
If $-\infty < \alpha < 0$, then there exists $\beta < 0$ such that $z(t) < \beta$ for $t \geq t_3 > t_2$. Further, $z(t) > py(t-\tau)$. So, $\beta > py(t-\tau)$ implies $y(t-\alpha_i) > p^{-1}\beta > 0$ for $t \geq t_3 + \rho$.
Therefore, (2.3) yields

$$\sum_{i=1}^{m} q_i(t)G(p^{-1}\beta) \leq -(r(t)z''(t))''.$$

Since $\lim_{t\to\infty}(r(t)z''(t))'$ exists, then integrating the inequality above from $t_3+\rho$ to ∞, we obtain

$$\int_{t_3+\rho}^{\infty} \sum_{i=1}^{m} q_i(t)dt < \infty,$$

which is a contradiction. Therefore $\alpha = 0$. Consequently,

$$\begin{aligned} 0 = \lim_{t\to\infty} z(t) &\geq \limsup_{t\to\infty}(y(t) + py(t-\tau)) \\ &\geq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p(y(t-\tau)) \\ &= \limsup_{t\to\infty} y(t) + p \limsup_{t\to\infty} y(t-\tau) \\ &= (1+p)\limsup_{t\to\infty} y(t). \end{aligned}$$

Since $1 + p > 0$, then $\limsup_{t\to\infty} y(t) = 0$. Hence $\lim_{t\to\infty} y(t) = 0$.
In each of the cases (c) and (d), we have $\lim_{t\to\infty} z(t) = -\infty$, which contradicts the fact that $z(t)$ is bounded. Let case (e) hold, we have $(r(t)z''(t))' > 0$ for $t \geq t_1$. Integrating from t_1 to t, we get $z''(t) > (r(t_1)z''(t_1))/r(t)$. Multiplying the inequality through by t and then integrating it we obtain $z'(t) > 0$ for large t due to (H_1). This contradicts the fact that $z'(t) < 0$ in case (e). This completes the proof of the theorem.

Theorem 2.7. *Let $-\infty < p_1 \leq p(t) \leq p_2 \leq -1$. Assume that (H_1), (H_{13}) hold. Then every bounded solution of* (1.1) *either oscillates or tends to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a bounded non-oscillatory solution of (1.1). Then $y(t) > 0$ or < 0 for $t \geq t_0$. Let $y(t) > 0$ for $t \geq t_0$. Setting $z(t)$ as in (2.1) we obtain (2.3) for $t \geq t_0 + \rho$. Hence $z(t) > 0$ or $z(t) < 0$ for $t \geq t_1 > t_0 + \rho$. Let $z(t) > 0$ for $t \geq t_1$.Then by Lemma 1.1 one of the cases (a) and (b) hold and $y(t) > -p(t)y(t-\tau) > y(t-\tau)$, implies that $\lim\inf_{t\to\infty} y(t) > 0$. From (2.3) it follows that

$$\int_{t_2}^{\infty} \sum_{i=1}^{m} q_i(t)dt < \infty,$$

for $t \geq t_2 > t_1$, a contradiction. Hence $z(t) < 0$ for $t \geq t_1$. Since $y(t)$ is bounded, $z(t)$ is bounded. Hence as before we can show none of the cases (c), (d) and (e) of Lemma 1.1 occur.
Suppose that the case (b) of Lemma 1.1 holds. Let $z(t) < 0$ and $z'(t) > 0$

implies $-\infty < \lim_{t\to\infty} z(t) \leq 0$. If $-\infty < \lim_{t\to\infty} z(t) < 0$, then proceeding as in the proof of Theorem 2.6 before we arrive at a contradiction. Hence $\lim_{t\to\infty} z(t) = 0$. Consequently,

$$\begin{aligned} 0 = \lim_{t\to\infty} z(t) &\leq \liminf_{t\to\infty}(y(t) + p_2 y(t-\tau)) \\ &\leq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_2(y(t-\tau)) \\ &= \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} y(t-\tau) \\ &= (1+p_2)\limsup_{t\to\infty} y(t). \end{aligned}$$

Since $(1+p_2) < 0$, then $\limsup_{t\to\infty} y(t) = 0$, implies $\lim_{t\to\infty} y(t) = 0$. Thus the proof of the theorem is complete.

3 Non-homogeneous Oscillation

This section is devoted to study the oscillatory and asymptotic behavior of solutions of forced equations (1.2) with suitable forcing function. We have the following hypotheses regarding f(t):

(H_{14}) There exists $F \in C^2([0,\infty),\mathbb{R})$ such that $F(t)$ changes sign, with $rF'' \in C^2([0,\infty),\mathbb{R})$ and $(rF'')'' = f$;

(H_{15}) There exists $F \in C^2([0,\infty),\mathbb{R})$ such that $F(t)$ changes sign, with $-\infty < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \infty, rF'' \in C^2([0,\infty),\mathbb{R})$ and $(rF'')'' = f$;

(H_{16}) There exists $F \in C^2([0,\infty),\mathbb{R})$ such that $F(t)$ does not change sign, with $\lim_{t\to\infty} F(t) = 0$, $rF'' \in C^2([0,\infty),\mathbb{R})$ and $(rF'')'' = f$;

(H'_{16}) There exists $F \in C^2([0,\infty),\mathbb{R})$ such that $\lim_{t\to\infty} F(t) = 0$, $rF'' \in C^2([0,\infty),\mathbb{R})$ and $(rF'')'' = f$.

Theorem 3.1. *Let $0 \leq p(t) \leq p < \infty$. Assume that (H_1), (H_2), $(H_3^{'})$, (H_6) and (H_{14}) hold. If*

$$(H_{17}) \quad \int_\rho^\infty \sum_{i=1}^m Q_i(t)G(F^+(t-\alpha_i))dt = \infty = \int_\rho^\infty \sum_{i=1}^m Q_i(t)G(F^-(t-\alpha_i))dt,$$

where $F^{+}(t) = max\{0, F(t)\}$ *and* $F^{-}(t) = max\{-F(t), 0\}$, *then all solutions of* (1.2) *are oscillatory.*

Proof Let $y(t)$ be a non oscillatory solution of (1.2). Hence $y(t) > 0$ or $y(t) < 0$ for $t \geq t_0 > 0$. Suppose that $y(t) > 0$ for $t \geq t_0 > 0$. Setting $z(t)$ as in (2.1), we obtain (2.2) for $t \geq t_0 + \rho$. Let

$$w(t) = z(t) - F(t). \tag{3.1}$$

Hence for $t \geq t_0 + \rho$, (1.2) becomes

$$(r(t)w''(t))'' = -\sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i)) \leq 0, \not\equiv 0. \tag{3.2}$$

Thus $w(t)$ is monotonic and of constant sign on $[t_1, \infty]$, $t_1 > t_0 + \rho$. Since $F(t)$ changes sign, then $w(t) > 0$ for $t \geq t_1$. Hence one of the cases (a) and (b) of Lemma 1.1 holds for large t, as $w(t) > 0$ implies $z(t) > F^{+}(t)$. For $t \geq t_2 > t_1$, we have

$$\begin{aligned}
0 &= (r(t)w''(t))'' + \sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i)) + G(p)(r(t-\tau)w''(t-\tau))'' \\
&\quad + G(p)\sum_{i=1}^{m} q_i(t-\tau)G(y(t-\alpha_i-\tau)) \\
&\geq (r(t)w''(t))'' + G(p)(r(t-\tau)w''(t-\tau))'' + \lambda\sum_{i=1}^{m} Q_i(t)G(z(t-\alpha_i)) \\
&\geq (r(t)w''(t))'' + G(p)(r(t-\tau)w''(t-\tau))'' + \lambda\sum_{i=1}^{m} Q_i(t)G(F^{+}(t-\alpha_i)).
\end{aligned}$$

Integrating from $t_2 + \rho$ to ∞, we get

$$\int_{t_2+\rho}^{\infty} \sum_{i=1}^{m} Q_i(t)G(F^{+}(t-\alpha_i))dt < \infty,$$

which is a contradiction to (H_{17}).

If $y(t) < 0$ for $t \geq t_0$, we set $x(t) = -y(t)$ to obtain $x(t) > 0$ for $t \geq t_0$

and

$$(r(t)(x(t)+p(t)x(t-\tau))'')''+\sum_{i=1}^{m} q_i(t)G(x(t-\alpha_i)) = \tilde{f}(t),$$

where $\tilde{f}(t) = -f(t)$. If $\tilde{F}(t) = -F(t)$, then $\tilde{F}(t)$ changes sign, $\tilde{F}^+(t) = F^-(t)$ and $(r(t)\tilde{F}''(t))'' = f(t)$. Proceeding as above we obtain a contradiction. This completes the proof of the theorem.

Theorem 3.2. *Let $-1 < p \leq p(t) \leq 0$. Suppose that (H_1), (H_{15}) hold. If*

$$(H_{18}) \int_{\rho}^{\infty} \sum_{i=1}^{m} q_i(t)G(F^+(t-\alpha_i))dt = \infty = \int_{\rho}^{\infty} \sum_{i=1}^{m} q_i(t)G(F^-(t-\alpha_i+\tau))dt,$$

and

$$(H_{19}) \int_{\rho}^{\infty} \sum_{i=1}^{m} q_i(t)G(F^-(t-\alpha_i))dt = \infty = \int_{\rho}^{\infty} \sum_{i=1}^{m} q_i(t)G(F^+(t-\alpha_i+\tau))dt,$$

then every solution of (1.2) *oscillates.*

Proof. Proceeding as in the proof of the Theorem 3.1, we obtain $w(t) > 0$ or < 0 for $t \geq t_1 > t_0 + \rho$ when $y(t) > 0$ for $t \geq t_0$. If $w(t) > 0$ for $t \geq t_1$, then any one of the cases (a) and (b) of Lemma 1.1 holds for $t \geq t_1$. Further, $w(t) > 0$ implies that

$$y(t) > z(t) > F(t),$$

hence $y(t) > F^+(t)$. Consequently, we have from (3.2)

$$\sum_{i=1}^{m} q_i(t)G(F^+(t-\alpha_i)) \leq -(r(t)w''(t))'', \qquad t \geq t_1 + \rho.$$

Since $\lim_{t\to\infty} (r(t)w''(t))'$ exists, therefore we obtain

$$\int_{t_1+\rho}^{\infty} \sum_{i=1}^{m} q_i(t)G(F^+(t-\alpha_i))dt < \infty,$$

a contradiction to (H_{18}). Hence $w(t) < 0$ for $t \geq t_1$. Then one of the cases (b)-(e) of Lemma 1.1 holds. Let (b) holds. Since $w(t) < 0$ it follows that $p(t)y(t-\tau) < F(t)$, hence $y(t) > F^-(t+\tau)$ for $t \geq t_1$. From (3.2), we obtain

$$\begin{aligned}(r(t)w''(t))'' &= -\sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i)) \\ &\leq -\sum_{i=1}^{m} q_i(t)G(F^-(t-\alpha_i+\tau)).\end{aligned}$$

Integrating from $t_1+\rho$ to ∞, we obtain

$$\int_{t_1+\rho}^{\infty} \sum_{i=1}^{m} q_i(t)G(F^-(t-\alpha_i+\tau))dt < \infty,$$

which is a contradiction to (H_{18}).

Suppose $y(t)$ is unbounded. Then there exists an increasing sequence $\{\sigma_n\}_{n=1}^{\infty}$ such that $\sigma_n \to \infty$, $y(\sigma_n) \to \infty$ as $n \to \infty$ and

$$y(\sigma_n) = max\{y(t) : t_1 \leq t \leq \sigma_n\}.$$

We may choose n large enough such that $\sigma_n - \tau > t_1$. Therefore,

$$w(\sigma_n) > y(\sigma_n) + py(\sigma_n - \tau) - F(\sigma_n).$$

Since, $F(t)$ is bounded and $(1+p) > 0$, then $w(\sigma_n) > 0$ for large n, which is a contradiction.

Hence, $y(t)$ is bounded and so also $w(t)$ is bounded. Hence, none of the cases (c), (d) and (e) of Lemma 1.1 are possible.

Using the same type of reasoning as in Theorem 3.1, for the case $y(t) < 0$ for $t \geq t_0$, we obtain the desired contradiction. Hence the theorem is proved.

Theorem 3.3. *Let* $-\infty < p \leq p(t) \leq 0$. *If* (H_1), (H_3), (H_{15}), (H_{18}) *and* (H_{19}) *hold, then every solution of* (1.2) *either oscillates or tends to* $\pm\infty$ *as* $t \to \infty$.

Proof Proceeding same as the proof of Theorem 3.2 we obtain a contradiction for $w(t) > 0$ for $t \geq t_1 > t_0 + \rho$. Hence $w(t) < 0$ for $t \geq t_1$. Therefore one of the cases (b)-(e) of Lemma 1.1 holds. Suppose case (b) holds. Since $w(t) < 0$, then $py(t-\tau) < F(t)$ implies $y(t) > (-p^{-1})F^-(t+\tau)$ for $t \geq t_1$. From (3.2) we have

$$\begin{aligned}(r(t)w''(t))'' &= -\sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i)) \\ &\leq -\sum_{i=1}^{m} q_i(t)G(-p^{-1})G(F^-(t-\alpha_i+\tau)).\end{aligned}$$

Integrating from $t_1 + \rho$ to ∞

$$\int_{t_1+\rho}^{\infty} \sum_{i=1}^{m} q_i(t)G(F^-(t-\alpha_i+\tau))dt < \infty,$$

a contradiction. In cases (c) and (d), $\lim_{t\to\infty} w(t) = -\infty$. In case (e), if we take $-\infty < \lim_{t\to\infty} w(t) < 0$, then we get a contradiction due to (H_1). Thus $\lim_{t\to\infty} w(t) = -\infty$ in each of the cases (c)-(e), and $py(t-\tau) < w(t) + F(t)$, implies that,

$$\limsup_{t\to\infty}(py(t-\tau)) \leq \lim_{t\to\infty} w(t) + \limsup_{t\to\infty} F(t),$$

that is, $p \liminf_{t\to\infty} y(t) = -\infty$ due to (H_{15}). Hence $\lim_{t\to\infty} y(t) = \infty$. The proof for the case $y(t) < 0$ for $t \geq t_0$ is similar. Hence the proof of the theorem is complete.

Corollary 3.4. *Let $-\infty < p \leq p(t) \leq 0$. If (H_1), (H_3), (H_{15}), (H_{18}) and (H_{19}) hold, then every bounded solution of (1.2) oscillates.*

Theorem 3.5. *Let $0 < p(t) \leq p < \infty$. If (H_1), (H_2), $(H_3^{'})$, (H_6) and (H_{16}) hold, If*

$$(H_{20}) \qquad \int_{\rho}^{\infty} \sum_{i=1}^{m} Q_i(t)G(|F(t-\alpha_i)|)dt = \infty,$$

then every bounded solution of (1.2) oscillates or tends to zero as $t \to \infty$.

Proof. Proceeding as in the proof of Theorem 3.1 we obtain $w(t) > 0$ or < 0 for $t \geq t_1 > t_0 + \rho$. Let $w(t) > 0$ for $t \geq t_1$ implies $z(t) > F(t)$. Suppose $F(t) > 0$ for $t \geq t_2 > t_1$. Therefore

$$\begin{aligned} 0 &= (r(t)w''(t))'' + \sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i)) + G(p)(r(t-\tau)w''(t-\tau))'' \\ &\qquad + G(p)\sum_{i=1}^{m} q_i(t-\tau)G(y(t-\alpha_i-\tau)) \\ &\geq (r(t)w''(t))'' + G(p)(r(t-\tau)w''(t-\tau))'' + \lambda\sum_{i=1}^{m} Q_i(t)G(z(t-\alpha_i)) \\ &\geq (r(t)w''(t))'' + G(p)(r(t-\tau)w''(t-\tau))'' + \lambda\sum_{i=1}^{m} Q_i(t)G(F(t-\alpha_i)) \end{aligned}$$

for $t \geq t_2 + \rho$. Integrating the last inequality from $t_2 + \rho$ to ∞ we obtain

$$\int_{t_2+\rho}^{\infty} \sum_{i=1}^{m} Q_i(t)G(F(t-\alpha_i))dt < \infty,$$

a contradiction. Hence $F(t) < 0$ for $t \geq t_2$. Now (3.2) implies that

$$\int_{\rho}^{\infty} \sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i))dt < \infty,$$

due to Lemma 1.1. Hence $\liminf_{t\to\infty} y(t) = 0$ because of (H_{20}) implies that $\int_{\rho}^{\infty} \sum_{i=1}^{m} q_i(t)dt = \infty.$

Further, $w(t)$ is bounded and monotonic, then $\lim_{t\to\infty} w(t)$ exists and hence $\lim_{t\to\infty} z(t)$ exists implies $\lim_{t\to\infty} z(t) = 0$ (see [3, Lemma 1.5.2]). As $z(t) \geq y(t)$, then $\lim_{t\to\infty} y(t) = 0$. Suppose $w(t) < 0$ for $t \geq t_1$. Hence $y(t) < F(t)$. Hence $\lim_{t\to\infty} y(t) = 0$. Hence the theorem is proved.

Theorem 3.6. *Let* $-1 < p \leq p(t) \leq 0$. *Suppose that* (H_1), (H_{13}), (H_{16}) *hold. Then every solution of* (1.2) *either oscillates or tends to zero as* $t \to \infty$.

Proof. Proceeding as in the proof of the Theorem 3.1, we obtain $w(t) > 0$ or < 0 for $t \geq t_1 > t_0 + \rho$. When $w(t) > 0$ for $t \geq t_1$, then any one of the cases (a) and (b) of Lemma 1.1 holds for $t \geq t_1$. From (3.2) it follows that

$$\int_{t_2+\rho}^{\infty} \sum_{i=1}^{m} q_i(t) G(y(t-\alpha_i))dt < \infty, \tag{3.3}$$

for $t_2 > t_1$. Hence $\liminf_{t\to\infty} y(t) = 0$ and $\lim_{t\to\infty} z(t) = 0$. On the other hand $\lim_{t\to\infty} w(t) = \infty$ in case (a) of Lemma 1.1. Hence $\lim_{t\to\infty} z(t) = \infty$. Therefore, $y(t) \geq z(t)$ implies that $\lim_{t\to\infty} y(t) = \infty$, a contradiction. In case (b), $\lim_{t\to\infty} w(t) = \alpha$, where $0 < \alpha \leq \infty$. If $\alpha = \infty$ then we get a contradiction as above. If $0 < \alpha < \infty$, then $\lim_{t\to\infty} z(t) = \alpha$. From [3; Lemma 1.5.2] it follows that $\alpha = 0$, which is a contradiction. Hence $w(t) < 0$ for $t \geq t_1$.

We claim that $y(t)$ is bounded. Suppose $y(t)$ is unbounded, then there exists an increasing sequence $\{\sigma_n\}_{n=1}^{\infty}$ such that $\sigma_n \to \infty$, $y(\sigma_n) \to \infty$ as $n \to \infty$ and

$$y(\sigma_n) = \max\{y(t) : t_1 \leq t \leq \sigma_n\}.$$

We may choose n large enough such that $\sigma_n - \tau > t_1$. Therefore,

$$w(\sigma_n) > y(\sigma_n) + py(\sigma_n - \tau) - F(\sigma_n) \quad \geq \quad (1+p)y(\sigma_n) - F(\sigma_n).$$

Since, $F(t)$ is bounded and $(1+p) > 0$, then $w(\sigma_n) > 0$ for large n, which is a contradiction. Thus $w(t)$ is bounded.

In each of the cases (c) and (d) of Lemma 1.1, $\lim_{t\to\infty} w(t) = -\infty$, a contradiction.

In each of the cases (b) and (e) of Lemma 1.1, (3.3) holds. Hence $\liminf_{t\to\infty} y(t) = 0$ and $\lim_{t\to\infty} w(t)$ exists. Consequently, $\lim_{t\to\infty} z(t) = \infty$ exists. From [3 ; Lemma

1.5.2] it follows that $\lim_{t\to\infty} z(t) = 0$.

$$\begin{aligned} 0 = \lim_{t\to\infty} \quad z(t) &= \limsup_{t\to\infty}(y(t) + p(t)y(t-\tau)) \\ &\geq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p(y(t-\tau)) \\ &= \limsup_{t\to\infty} y(t) + p \limsup_{t\to\infty} y(t-\tau) \\ &= (1+p) \limsup_{t\to\infty} y(t). \end{aligned}$$

Since $(1+p) > 0$, then $\lim_{t\to\infty} y(t) = 0$. Hence the theorem is proved.

In the following sufficient conditions are obtained for the existence of bounded positive solutions of (1.2).

Theorem 3.7. *Let $0 \leq p(t) \leq p < 1$ and (H_{15}) holds with*

$$-\frac{3}{8}(1-p) < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \frac{1}{4}(1-p).$$

and G is Lipschitzian on the intervals of the form $[a, b]$, $0 < a < b < \infty$. If

$$\int_0^\infty \frac{s}{r(s)} \int_s^\infty t \sum_{i=1}^m q_i(t) dt ds < \infty,$$

then (1.2) *admits a positive bounded solution on* $[a, b]$.

Proof It is possible to choose $t_0 > 0$ large enough such that for $t \geq t_0 > 0$,

$$\int_{t_0}^\infty \frac{t}{r(t)} \int_t^\infty s \sum_{i=1}^m q_i(s) ds dt < \frac{1-p}{4L},$$

where $L = max\{L_1, G(1)\}$ and L_1 is Lipschitz constant of G on $[\frac{1}{8}(1-p), 1]$. Let $X = BC([t_0, \infty), \mathbb{R})$. Then X is a Banach Space with respect to supremum norm defined by

$$||x|| = \sup_{t\geq t_0}\{|x(t)|\}.$$

Let

$$S = \{x \in X : \frac{1}{8}(1-p) \leq x(t) \leq 1, t \geq t_0\}.$$

Hence S is a complete metric space. For $y \in S$, we define

$$Ty(t) = \begin{cases} Ty(t_0+\rho), \quad t \in [t_0, t_0+\rho], \\ \\ -p(t)y(t-\tau) + \dfrac{3+p}{4} + F(t) \\ -\displaystyle\int_t^\infty (\frac{s-t}{r(s)} \int_s^\infty (u-s) \sum_{i=1}^m q_i(u) G(y(u-\alpha_i)) du) ds, \quad t \geq t_0+\rho. \end{cases}$$

Hence

$$Ty(t) < \frac{3+p}{4} + \frac{1-p}{4} = 1,$$

and

$$Ty(t) > -p + \frac{3+p}{4} - \frac{3}{8}(1-p) - \frac{1}{4}(1-p) = \frac{1}{8}(1-p) \quad for \quad t \geq t_0 + \rho.$$

Hence $Ty \in S$, that is, $T : S \to S$.

Next, we show that T is continuous. Let $y_k(t) \in S$ such that $\lim_{k\to\infty} ||y_k(t) - y(t)|| = 0$ for all $t \geq t_0$. Because S is closed, $y(t) \in S$. Indeed,

$$\begin{aligned} |(Ty_k) - (Ty)| &\leq p(t)|y_k(t-\tau) - y(t-\tau)| \\ &+ \left| \int_t^\infty \frac{s-t}{r(s)} \int_s^\infty (u-s) \sum_{i=1}^m q_i(u)[G(y_k(u-\alpha_i)) \right. \\ &\left. -G(y(u-\alpha_i))]duds \right| \\ &\leq p|y_k(t-\tau) - y(t-\tau)| \\ &+ \int_t^\infty \frac{s}{r(s)} \int_s^\infty u \sum_{i=1}^m q_i(u)|G(y_k(u-\alpha_i)) \\ &-G(y(u-\alpha_i))|duds \\ &\leq p||y_k - y|| + ||y_k - y|| \frac{1-p}{4} \end{aligned}$$

implies that

$$||(Ty_k) - (Ty)|| \leq ||y_k - y||[p + \frac{1-p}{4}] \to 0$$

as $k \to \infty$. Hence T is continuous.

In order to apply Schauder's fixed point Theorem (see [2]) we need to show that Ty is precompact. Let $y \in S$. For $t_2 \geq t_1$,

$$\begin{aligned}
(Ty)(t_2) - (Ty)(t_1) &= p(t_2)y(t_2 - \tau) - p(t_1)y(t_1 - \tau) \\
&+ \int_{t_1}^{\infty} \frac{s - t_1}{r(s)} \int_s^{\infty} (u - s) \sum_{i=1}^{m} q_i(u)G(y(u - \alpha_i))duds \\
&- \int_{t_2}^{\infty} \frac{s - t_2}{r(s)} \int_s^{\infty} (u - s) \sum_{i=1}^{m} q_i(u)G(y(u - \alpha_i))duds,
\end{aligned}$$

that is,

$$\begin{aligned}
|(Ty)(t_2) - (Ty)(t_1)| &\leq |p(t_2)y(t_2 - \tau) - p(t_1)y(t_1 - \tau)| \\
&+ |\int_{t_2}^{\infty} \frac{s - t_2}{r(s)} \int_s^{\infty} (u - s) \sum_{i=1}^{m} q_i(u)G(y(u - \alpha_i))duds \\
&- \int_{t_1}^{\infty} \frac{s - t_1}{r(s)} \int_s^{\infty} (u - s) \sum_{i=1}^{m} q_i(u)G(y(u - \alpha_i))duds| \\
&\leq |p(t_2)y(t_2 - \tau) - p(t_1)y(t_1 - \tau)| \\
&+ |\int_{t_2}^{\infty} \frac{s - t_1}{r(s)} \int_s^{\infty} (u - s) \sum_{i=1}^{m} q_i(u)G(y(u - \alpha_i))duds \\
&- \int_{t_1}^{\infty} \frac{s - t_1}{r(s)} \int_s^{\infty} (u - s) \sum_{i=1}^{m} q_i(u)G(y(u - \alpha_i))duds| \\
&= |p(t_2)y(t_2 - \tau) - p(t_1)y(t_1 - \tau)| \\
&+ |\int_{t_1}^{t_2} \frac{s - t_1}{r(s)} \int_s^{\infty} (u - s) \sum_{i=1}^{m} q_i(u)G(y(u - \alpha_i))duds| \\
&\to 0 \qquad\qquad as \quad t_2 \to t_1.
\end{aligned}$$

Thus Ty is precompact. By Schauder's fixed point theorem T has a fixed point, that is, $Ty = y$. Consequently, $y(t)$ is a solution of (1.2) on $[\frac{1}{8}(1 - p), 1]$. This completes the proof of the theorem.

Remark 3.8. Theorems similar to Theorem 3.6 can be proved in other ranges of $p(t)$.

4 Examples and Discussion

Example 4.1. Consider

$$(y(t)+y(t-\pi))^{(iv)}+y(t-3\pi)+y(t-2\pi)=0, \tag{4.1}$$

where $r(t)=1$, $p(t)=1$, $q_1(t)=q_2(t)=1$, $\tau=\pi$, $m=2$, $\alpha_1=3\pi$, $\alpha_2=2\pi$, $G(u)=u$. Clearly, $(H_1),(H_2),(H_3^{'}),(H_6)$ and

$$(H_7) \qquad \int_{\pi}^{\infty}[Q_1(t)+Q_2(t)]dt=\infty,$$

hold, where $Q_1(t)=Q_2(t)=1$. Hence Theorem 2.2 can be applied to (4.1), that is, every solution of (4.1) oscillates. Indeed, $y(t)=\sin t$ is such a solution of (4.1).

Example 4.2. Consider

$$(y(t)+y(t-\pi))^{(iv)}+y^{\frac{1}{3}}(t-3\pi)+y^{\frac{1}{3}}(t-2\pi)=0, \tag{4.2}$$

where $r(t)=1$, $p(t)=1$, $q_1(t)=q_2(t)=1$, $\tau=\pi$, $m=2$, $\alpha_1=3\pi$, $\alpha_2=2\pi$, $G(u)=u^{1/3}$. Clearly, $(H_1),(H_2),(H_3^{'}),(H_6)$ and

$$(H_7) \qquad \int_{\pi}^{\infty}[Q_1(t)+Q_2(t)]dt=\infty,$$

hold, where $Q_1(t)=Q_2(t)=1$. Hence Theorem 2.2 can be applied to (4.2), that is, every solution of (4.2) oscillates. Indeed, $y(t)=\sin t$ is such a solution of (4.2).

Example 4.3. Consider

$$(y(t)-y(t-\pi))^{(iv)}+4y(t)+4e^{-\pi}y(t-2\pi)=0, \tag{4.3}$$

where $r(t)=1$, $p(t)=-1$, $q_1(t)=4$, $q_2(t)=4e^{-\pi}$, $\tau=\pi$, $m=2$, $\alpha_1=0$, $\alpha_2=2\pi$, $G(u)=u$. Clearly, (H_1) and

$$(H_{13}) \qquad \int_{0}^{\infty}[q_1(t)+q_2(t)]dt=\infty$$

hold. Hence by Theorem 2.7 every bounded solution of (4.3) either oscillates or converges to zero as $t \to \infty$. In particular, $y(t) = e^{-t}\sin t$ is such a solution of (4.3).

Example 4.4. Consider

$$(e^{-t}(y(t) + 2y(t-\pi))'')'' + e^t y(t-3\pi) + e^t y(t-2\pi) = 2e^{-t}\cos t, \quad (4.4)$$

where $r(t) = e^{-t}$, $p(t) = 2$, $q_1(t) = q_2(t) = e^t$, $\tau = \pi$, $\alpha_1 = 3\pi$, $\alpha_2 = 2\pi$, $G(u) = u$ and $f(t) = 2e^{-t}\cos t$. Indeed, if we choose $F(t) = \sin t$, then $(r(t)F''(t))'' = f(t)$. Since

$$F(t-\alpha_1) = -\sin t \text{ and } F(t-\alpha_2) = \sin t.$$

$$F^+(t-\alpha_1) = \begin{cases} 0, & t \in [2n\pi, (2n+1)\pi] \\ -\sin t, & t \in [(2n+1)\pi, (2n+2)\pi], \end{cases}$$

$$F^+(t-\alpha_2) = \begin{cases} \sin t, & t \in [2n\pi, (2n+1)\pi] \\ 0, & t \in [(2n+1)\pi, (2n+2)\pi], \end{cases}$$

$$F^-(t-\alpha_1) = \begin{cases} \sin t, & t \in [2n\pi, (2n+1)\pi] \\ 0, & t \in [(2n+1)\pi, (2n+2)\pi], \end{cases}$$

and

$$F^-(t-\alpha_2) = \begin{cases} 0, & t \in [2n\pi, (2n+1)\pi] \\ -\sin t, & t \in [(2n+1)\pi, (2n+2)\pi], \end{cases}$$

for $n = 0, 1, 2, ...$, then $(H_1), (H_2), (H_3^{'})$ and (H_6) are satisfied. Now

$$\int_{3\pi}^{\infty} [Q_1(t)F^+(t-3\pi) + Q_2(t)F^+(t-2\pi)]dt = I_1 + I_2,$$

where

$$I_1 = \int_{3\pi}^{\infty} e^{t-\pi}F^+(t-3\pi)dt = -e^{-\pi}\sum_{n=1}^{\infty}\int_{(2n+1)\pi}^{(2n+2)\pi} e^t \sin t dt$$
$$= \frac{e^{-\pi}}{2}(e^{\pi}+1)\sum_{n=1}^{\infty} e^{(2n+1)\pi} = \infty,$$

and

$$I_2 = \int_{3\pi}^{\infty} e^{t-\pi} F^+(t-2\pi)dt = e^{-\pi} \sum_{n=2}^{\infty} \int_{2n\pi}^{(2n+1)\pi} e^t \sin t dt$$
$$= \frac{e^{-\pi}}{2}(e^{\pi}+1) \sum_{n=2}^{\infty} e^{2n\pi} = \infty$$

Hence

$$\int_{3\pi}^{\infty} [Q_1(t)F^-(t-3\pi) + Q_2(t)F^-(t-2\pi)]dt = \frac{e^{-\pi}}{2}(e^{\pi}+1) \sum_{n=2}^{\infty} e^{2n\pi}$$
$$+\frac{e^{-\pi}}{2}(e^{\pi}+1) \sum_{n=1}^{\infty} e^{(2n+1)\pi} = \infty.$$

Hence Theorem 3.1 can be applied to (4.4), that is, every solution of (4.4) oscillates. Indeed, $y(t) = -\sin t$ is such a solution of (4.4).

Example 4.5. Consider

$$(y(t) - \frac{1}{2}y(t-\pi))'''' + 4y(t) + 2e^{-\pi}y(t-2\pi) + 4y(t-\pi) = -4e^{-(t-\pi)}\sin t, \quad (4.5)$$

where $r(t) = 1$, $p(t) = -\frac{1}{2}$, $q_1(t) = 4$, $q_2(t) = 2e^{-\pi}$, $q_3(t) = 4$, $\tau = \pi$, $\alpha_1 = 0$, $\alpha_2 = 2\pi$, $\alpha_3 = \pi$, $G(u) = u$ and $f(t) = -4e^{-(t-\pi)}\sin t$. Indeed, if we choose $F(t) = e^{-(t-\pi)}\sin t$, then $(r(t)F''(t))'' = f(t)$ and $\lim_{t\to\infty} F(t) = 0$.
Clearly, (H_1) is satisfied. Now

$$\int_0^{\infty} [q_1(t) + q_2(t) + q_3(t)]dt = \infty.$$

Hence (H_{13}) is also satisfied. Hence Theorem 3.6 can be applied to (4.5), that is, every solution of (4.5) oscillates or tends to zero as $t \to \infty$. Indeed, $y(t) = e^{-t}\sin t$ is such a solution of (4.5).

Acknowledgements The first author wishes to thank the Department of Mathematics of the University of Tennessee Chattanooga for its generous support of this work was completed.

The authors express their appreciations to the referees for several valuable suggestions which have improved the manuscript.

References

[1] Q. Chuanxi, G. Ladas, A. Peterson, *Oscillation in differential equations with positive and negative coefficients,* Canad. Math. Bull. 33 (1990), 442 - 450.

[2] R. E. Edwards, *Functional Analysis,* Holt, Rinehart and Winston Inc. New York, 1965.

[3] I. Gyori, G. Ladas, *Oscillation Theory of Delay Differential Equation with Application,* Claredon Press, Oxford, 1991.

[4] W. T. Li, H. S. Quan, *Oscillation of higher order neutral differential equations with positive and negative coefficients,* Ann. Differential Equations, II (1995), 70 - 76.

[5] 5. W. T. Li, J. Yan, *Oscillation of first order neutral differential equations with positive and negative coefficients,* Collect. Math. 50 (1999), 199 - 209.

[6] O. O'calan, *Oscillation of forced neutral differential equations with positive and negative coefficients,* Compu. Math. Appl. 54 (2007), 1411 - 1421.

[7] O. O'calan, *Oscillation of neutral differential equation with positive and negative coefficients,* J. Math. Anal. Appl. 331 (2007), 644 - 654.

[8] N. Parhi, S. Chand, *On forced first order neutral differential equations with positive and negative coefficients,* Math. Slovaca, 50 (2000), 183 - 202.

[9] N. Parhi and R. N. Rath, *On oscillations of solutions of forced nonlinear neutral differential equations of higher order,* Czechoslovak Math. J. 53 (2003), 805-825.

[10] N. Parhi, R. N. Rath, *On oscillation criteria for forced nonlinear higher order neutral differential equations,* Math. Slovaca 54 (2004), 369-388.

[11] N. Parhi, A. K. Tripathy, *On oscillatory fourth order nonlinear neutral differential equations - II,* Math. Slovaca, 55 (2005), 183 - 202.