Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066 - 6594 Vol. 4, No. 1 / 2012

ON LIAPUNOV-TYPE INTEGRAL INEQUALITIES FOR EVEN ORDER DYNAMIC EQUATIONS ON TIME SCALES*

Saroj Panigrahi†

Abstract

In this paper, Liapunov-type integral inequalities has been obtained for an even order dynamic equations on time scales. As an applications, an estimate for the number of zeros of an oscillatory solution and a criterion for disconjugacy of an even order dynamic equation is obtained in an interval $[a, \sigma(b)]_{\mathbb{T}}$.

MSC: 34 C 10, 34 N 05

keywords: Liapunov-type inequality, disconjugacy, number of zeros, even order dynamic equations.

1 Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger [12] in his Ph. D. thesis in 1988 in order to unify

*Accepted for publication in revised form on January 3, 2012.

†panigrahi2008@gmail.com, spsm@uohyd.ernet.in, Department of Mathematics, University of Tennessee, Chattanooga TN 37403-2598 USA; Indo-US fellow 2011 in Mathematical and Computational Sciences. Research supported by Indo-US Science and Technology Forum (IUSSTF), Fullbright House, New Delhi, India.

continuous and discrete analysis. Several authors have expounded on various aspects of this new theory; see the survey paper of Agarwal et. al. [1] and references cited therein and a book on the subject of time scales by Bohner and Peterson [2]. A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represents the classical theories of differential equations and difference equations respectively.

In [13], Russian mathematician Liapunov proved that If $y(t)$ is a nontrivial solution of

$$y'' + p(t)y = 0 \tag{1.1}$$

with $y(a) = 0 = y(b)$, where $a, b \in \mathbb{R}$ with $a < b$ and $y(t) \neq 0$ for $t \in (a, b)$, then

$$\int_a^b |p(t)|dt > \frac{4}{b-a} \tag{1.2}$$

holds, where $p \in L^1_{loc}$.

This result has found applications in differential and difference equations in the study of various properties of solutions of (1.1) and it is useful tools in oscillation theory, disconjugacy and eigenvalue problems (see [4 - 14]).

Bohner et al. [2] extended the Liapunov inequality (1.2) on time scale $\mathbb{T}$ for the dynamic equation

$$y^{\Delta\Delta}(t) + p(t)y^{\sigma}(t) = 0, \tag{1.3}$$

where $p(t)$ is a positive rd-continious function defined on $\mathbb{T}$. They proved, by using the quadratic functional equation

$$F(y) = \int_a^b [(y^{\Delta}(t))^2 - p(t)(y^{\sigma})^2]\Delta t = 0,$$

that if $y(t)$ is a nontrivial solution of (1.3) with $y(a) = 0 = y(b)(a < b)$, then

$$\int_a^b p(t)\Delta t > \frac{(b-a)}{f(d)},$$

where $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = (t-a)(t-b)$ and $d \in \mathbb{T}$ such that $f(d) = \max\{f(t) : t \in [a, b]\}$. In particular, using the fact that, $a < c < b$ and

$$\frac{1}{c-a} + \frac{1}{b-c} = \frac{(a+b-2c)^2}{(b-a)(c-a)(b-c)} + \frac{4}{b-a} > \frac{4}{b-a},$$

they obtained

$$\int_a^b p(t)\Delta t > \frac{4}{b-a}.$$

Consider the $2n$-order dynamic equation

$$y^{\Delta^{2n}} + p(t)y^{\sigma} = 0, \tag{1.4}$$

on an arbitrary time scales $\mathbb{T}$, where p is a real rd-continuous function defined on $[0,\infty)_{\mathbb{T}} = [0,\infty) \cap \mathbb{T}$ and $\sigma(t)$ is the forward jump operator defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$.

The main objective of this paper is to determine (i) the lower bound for the distance between consecutive zeros of the solutions, (ii) the number of zeros of solutions of (1.4) over an interval $[0,T]_{\mathbb{T}}$, and (ii) establish some sufficient condition for the disconjugacy of (1.4) on an interval $[a,\sigma(b)]_{\mathbb{T}}$.

Note that (1.4) in its general form involves some different types of differential and difference equations depending on the choice of time scales $\mathbb{T}$. For example, when $\mathbb{T} = \mathbb{R}$, (1.4) becomes a even order differential equation. When $\mathbb{T} = \mathbb{Z}$, (1.4) is an even order difference equation. When $\mathbb{T} = h\mathbb{Z}$, then (1.4) becomes a generalized difference equation and when $\mathbb{T} = q^{\mathbb{N}}$, then (1.4) becomes a quantum difference equation. Note also that results in this paper can be applied on the time scales $\mathbb{T} = \mathbb{N}^2 = \{t^2 : t \in \mathbb{N}\}$, $\mathbb{T}_2 = \{\sqrt{n} : n \in \mathbb{N}_0\}$, $\mathbb{T}_3 = \{\sqrt[3]{n} : n \in \mathbb{N}_0\}$ and when $\mathbb{T} = \mathbb{T}_n = \{t_n : n \in \mathbb{N}_0\}$, where $\{t_n\}$ is a set of harmonic numbers.

Let $\mathbb{T}$ is bounded below and $t_0 = \min \mathbb{T}$. We say that a solution y of (1.4) has a zero at t in case $y(t) = 0$. We say that $y(t)$ has a generalized zero in $(t, \sigma(t))$, if t is right-scattered and $y(t)y(\sigma(t)) < 0$. We say that $t = t_0$ is a generalized zero (GZ) of order greater than k of y if

$$y^{\Delta^j}(t_0) = 0, j = 0, 1, ..., k-1.$$

We say (1.4) is disconjugate on $\mathbb{I}_{\mathbb{T}} = [a,\sigma(b)]_{\mathbb{T}} = [a,\sigma(b)] \cap \mathbb{T}$, if there is no nontrivial solution of (1.4) with $2n$ (or more) generalized zero in $\mathbb{I}_{\mathbb{T}}$.

A nontrivial solution of (1.4) is called oscillatory if it has infinitely many (isolated) generalized zeros in $[t_0,\infty)_{\mathbb{T}}$; otherwise it is called nonoscillatory.

The organizations of the paper is as follows. Section 2 will give some preliminaries on time scales. In Section 3, Liapunov- type integral inequality has been derived for even order dynamic equations. As an application, a criterion for disconjugacy is obtained in an interval $[a, \sigma(b)]_{\mathbb{T}}$ and an estimate for the number of zeros of an oscillatory solutions of (1.4) on an interval $[0, T]_{\mathbb{T}}$.

2 Preliminaries on Time Scales

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$. On any time scale we define the "forward and backward jump operators" by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

We make the convention:

$$\inf \phi = sup\mathbb{T}, \quad \sup \phi = \inf \mathbb{T}.$$

A point $t \in \mathbb{T}$ is said to be left dense if $\rho(t) = t$, right dense if $\sigma(t) = t$, left scattered if $\rho(t) < t$, right scattered if $\sigma(t) > t$. The points that are simultaneously right-dense and left-dense are called dense.

The mappings $\mu, \nu : \mathbb{T} \to [0, +\infty)$ defined by

$$\mu(t) = \sigma(t) - t$$

and

$$\nu(t) = t - \rho(t)$$

are called, respectively, the forward and backward graininess functions.

If $\mathbb{T}$ has a right- scattered minimum m, then define $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has left-scattered maximum M, then define $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. Finally, put $\mathbb{T}_k^k = \mathbb{T}_k \cap \mathbb{T}^k$. For a function $f : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^k$ the delta derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if f is continuous at t and t is right-scattered. If t is right-dense, then derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t^+} \frac{f(\sigma(t)) - f(s)}{t - s} = \lim_{s \to t^+} \frac{f(t) - f(s)}{t - s},$$

provided this limit exists.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right dense point and if there exists a finite left limit at all left dense points. The set of rd-continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$. The derivative and the shift operator σ are related by the formula

$$f^\sigma = f + \mu f^\Delta, \quad where \quad f^\sigma = f \circ \sigma.$$

Let f be a real-valued function defined on an interval $[a, b]$. We say that f is increasing, decreasing, nonincreasing, and nondecreasing on $[a, b]$ if $t_1, t_2 \in [a, b]$ and $t_2 > t_1$ imply $f(t_2) > f(t_1)$, $f(t_2) < f(t_1)$, $f(t_2) \leq f(t_1)$, $f(t_2) \geq f(t_1)$, respectively. Let f be a differentiable function on $[a, b]$. Then f is increasing, decreasing, nonincreasing, and nondecreasing on $[a, b]$ if $f^\Delta(t) > 0$, $f^\Delta(t) < 0$, $f^\Delta(t) \leq 0$, $f^\Delta(t) \geq 0$, for all $t \in [a, b)$, respectively.

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g of two differentiable functions f and g:

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)),$$

and

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}.$$

For $a, b \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^Δ is defined by

$$\int_a^b f^\Delta(t)\Delta t = f(b) - f(a).$$

The integration by parts formula read as

$$\int_a^b f^\Delta(t)g(t)\Delta t = f(b)g(b) - f(a)g(a) + \int_a^b f^\sigma(t)g^\Delta(t)\Delta t,$$

and infinite integrals are defined as

$$\int_a^\infty f(s)\Delta s = \lim_{t\to\infty} \int_a^t f(s)\Delta s.$$

A function $f \to \mathbb{R}$ is called convex on $\mathbb{I}_{\mathbb{T}}$, if

$$f(\lambda t + (1-\lambda)s) \leq \lambda f(t) + (1-\lambda)f(s), \tag{2.1}$$

for all $t, s \in \mathbb{I}_{\mathbb{T}}$ and $\lambda \in [0,1]$ such that $\lambda t + (1-\lambda)s \in \mathbb{I}_{\mathbb{T}}$. The function f is strictly convex on $\mathbb{I}_T$ if the inequality (2.1) is strict for distinct $t, s \in \mathbb{I}_{\mathbb{T}}$ and $\lambda \in (0,1)$.

The function f is concave (respectively, strictly concave) on $\mathbb{I}_{\mathbb{T}}$, if $-f$ is convex (respectively, strictly convex).

A function that is both convex and concave on $\mathbb{I}_{\mathbb{T}}$ is called affine on $\mathbb{I}_{\mathbb{T}}$.

Theorem 2.1. *Let $f : \mathbb{I}_{\mathbb{T}} \to \mathbb{R}$ be a delta differentiable function on $\mathbb{I}_{\mathbb{T}}^k$. If f^{Δ} is nondecreasing (nonincreasing) on $\mathbb{I}_{\mathbb{T}}^k$, then f is convex (concave) on $\mathbb{I}_{\mathbb{T}}$.*

Theorem 2.2. *(Rolle's Theorem [2]) Let $y(t)$ be a continuous on $[t_1, t_2]$, and assume that y^{Δ} is continuous on (t_1, t_2). If $y(t_1) = 0$ and y has a GZ at t_2, then there exists $c \in (t_1, t_2)$ such that y^{Δ} has GZ at c.*

Theorem 2.3. *(Holder's Inequality) Let $a, b \in \mathbb{T}$. For rd- continuous $f, g : [a, b] \to \mathbb{R}$ we have*

$$\int_a^b |f(t)g(t)|\Delta t \leq \left\{ \int_a^b |f(t)|^p \Delta t \right\}^{\frac{1}{p}} \left\{ \int_a^b |g(t)|^q \Delta t \right\}^{\frac{1}{q}},$$

where $p > 1$ and $q = p/(p-1)$.

The special case $p = q = 2$ reduces to the Cauchy-Schwarz Inequality.

Theorem 2.4. *Let $a, b \in \mathbb{T}$. For rd- continuous $f, g : [a, b] \to \mathbb{R}$, we have*

$$\int_a^b |f(t)g(t)|\Delta t \leq \left\{ \int_a^b |f(t)|^2 \Delta t \right\}^{\frac{1}{2}} \left\{ \int_a^b |g(t)|^2 \Delta t \right\}^{\frac{1}{2}}.$$

3 Main Results

In this work, we establish the Liapunov-type inequality for an even order dynamic equation of the form

$$y^{\Delta^{2n}} + p(t)y^{\sigma} = 0, \tag{3.1}$$

where $p \in C_{rd}([0, \infty)_{\mathbb{T}}, \mathbb{R})$.

Theorem 3.1. *Let $y(t)$ be a solution of (3.1) on $\mathbb{I}_{\mathbb{T}}$ satisfying $y^{\Delta^{2i}}(a) = 0 = y^{\Delta^{2i}}(\sigma(b)), i = 0, 1, 2, ..., n-1$ and $y(t) \neq 0$ for $t \in (a, \sigma(b))$, then*

$$\int_a^{\sigma(b)} |p(t)|\Delta t > \frac{2^{2n}}{(\sigma(b) - a)^{2n-1}}. \tag{3.2}$$

Proof. Since $y(t)$ is a nontrivial solution of (3.1), we deduce that M is defined (note that $y(t)$ is continuous by Theorem 1.16(i) in [2]) and $M = |y(\tau)| = \max\{y(t)| : t \in \mathbb{I}_{\mathbb{T}}\}$.

First we prove for $i = 0, 1, ..., n-1$,

$$|y^{\Delta^{2i}}(t)| \leq \left(\frac{\sigma(b) - a}{4}\right) \int_a^{\sigma(b)} |y^{\Delta^{2i+2}}(s)|\Delta s. \tag{3.3}$$

Infact,

$$|y^{\Delta^{2i}}(t)| = \left| \int_a^{\sigma(t)} y^{\Delta^{2i+1}}(s)\Delta s \right| \leq \int_a^t |y^{\Delta^{2i+1}}(s)|\Delta s$$

and

$$|y^{\Delta^{2i}}(t)| = |-y^{\Delta^{2i}}(t)| \leq \int_t^{\sigma(b)} |y^{\Delta^{2i+1}}(s)|\Delta s.$$

Therefore

$$|y^{\Delta^{2i}}(t)| \leq \frac{1}{2} \int_a^t |y^{\Delta^{2i+1(s)}}(s)|\Delta s. \tag{3.4}$$

Since $y^{\Delta^{2i}}(a) = y^{\Delta^{2i}}(\sigma(b)) = 0$, then there exists $\tau_i \in (a, \sigma(b))_{\mathbb{T}}$ such that $y^{\Delta^{2i+1}}(\tau_i) = 0$, for $i = 0, 1, ..., n-1$ and hence

$$|y^{\Delta^{2i+1}}(t)| = \left| \int_{\tau_i}^t y^{\Delta^{2i+2}}(s)\Delta s \right| \leq \int_{\tau_i}^t |y^{\Delta^{2i+2}}(s)|\Delta s \leq \int_{\tau_i}^{\sigma(b)} |y^{\Delta^{2i+2}}(s)|\Delta s$$

and

$$|y^{\Delta^{2i+1}}(t)| = \left| -y^{\Delta^{2i+1}}(t) \right| \leq \int_t^{\tau_i} |y^{\Delta^{2i+2}}(s)|\Delta s \quad \leq \quad \int_a^{\tau_i} |y^{\Delta^{2i+2}}(s)|\Delta s.$$

Therefore again summing up these last two inequalities, we obtain

$$|y^{\Delta^{2i+1}}(t)| \leq \frac{1}{2} \int_a^{\sigma(b)} |y^{\Delta^{2i+2}}(s)|\Delta s. \tag{3.5}$$

Thus substituting (3.5) in (3.4), we obtain

$$
\begin{aligned}
|y^{\Delta^{2i}}(t)| \leq \frac{1}{2}\int_a^{\sigma(b)} |y^{\Delta^{2i+1}}(s)|\Delta s \quad &\leq \quad \frac{1}{2}\int_a^{\sigma(b)} \left(\frac{1}{2}\int_a^{\sigma(b)} |y^{\Delta^{2i+2}}(\xi)|\Delta\xi\right)\Delta s \\
&= \quad \left(\frac{\sigma(b)-a}{4}\right)\int_a^{\sigma(b)} |y^{\Delta^{2i+2}}(s)|\Delta s.
\end{aligned}
$$

Hence Eq.(3.3) is proved.

From (3.3),

$$
\begin{aligned}
0 < |y(\tau)| \quad &\leq \quad \left(\frac{\sigma(b)-a}{4}\right)\int_a^{\sigma(b)} |y^{\Delta^2}(s)|\Delta s \\
&= \quad \left(\frac{\sigma(b)-a}{4}\right)\int_a^{\sigma(b)} \left[\left(\frac{\sigma(b)-a}{4}\right)\int_a^{\sigma(b)} |y^{\Delta^6}(\xi)|\Delta\xi\right]\Delta s \\
&= \quad \frac{(\sigma(b)-a)^3}{2^4}\int_a^{\sigma(b)} |y^{\Delta^4}(s)|\Delta s \\
&\leq \quad \frac{(\sigma(b)-a)^3}{2^4}\int_a^{\sigma(b)} \left[\left(\frac{\sigma(b)-a}{4}\right)\int_a^{\sigma(b)} |y^{\Delta^6}(\xi)|\Delta\xi\right]\Delta s \\
&= \quad \frac{(\sigma(b)-a)^5}{2^6}\int_a^{\sigma(b)} |y^{\Delta^6}(s)|\Delta s \\
&\leq \quad ... \leq \frac{(\sigma(b)-a)^{2n-1}}{2^{2n}}\int_a^{\sigma(b)} |y^{\Delta^{2n}}(s)|\Delta s \\
&\leq \quad \frac{(\sigma(b)-a)^{2n-1}}{2^{2n}}\int_a^{\sigma(b)} |-p(s)y^{\sigma}(s)|\Delta s \\
&\leq \quad \frac{(\sigma(b)-a)^{2n-1}}{2^{2n}}|y(\tau)|\left(\int_a^{\sigma(b)} |p(s)|\Delta s\right),
\end{aligned}
$$

which yields (3.2). Hence proof of the Theorem 3.1 is complete.

Remark 3.2. It is easy to see that the Theorem 3.1 holds for the dynamic equation

$$
y^{\Delta^{2n}} + (-1)^k p(t) y^{\sigma} = 0,
$$

where $k \in \mathbb{Z}$.

Remark 3.3. If $n = 1$, then the above equation (3.1) reduces to

$$y^{\Delta^2} + p(t)y^{\sigma} = 0. \tag{3.6}$$

If $y(t)$ is a solution of (3.6) satisfying $y(a) = 0 = y(\sigma(b))$ $(a < \sigma(b))$ and $y(t) \neq 0$ for $t \in (a, \sigma(b))$, then

$$\int_a^{\sigma(b)} |p(t)|\Delta t > \frac{4}{(\sigma(b) - a)}.$$

This is same as obtained by [2].

Remark 3.4. If $n = 1$ and $\mathbb{T} = \mathbb{R}$, then the inequality (3.2) reduces to the Liapunov inequality (1.2).

In the following we obtain an estimate for the number of zeros of an oscillatory solution of (3.1) on an interval $[0, T]_{\mathbb{T}}$.

Theorem 3.5. *If $y(t)$ is a solution of (3.1), which has N zeros $\{t_k\}_{k=1}^N$ in the interval $[0, T]$, where $0 < a \leq t_1 < t_2 < .... < t_N \leq \sigma(b) \leq T$, then*

$$T^{2n-1} \int_o^T |p(t)|\Delta t > 2^{2n}(N - 1^{2n}). \tag{3.7}$$

Proof. From Theorem 3.1 it follows that

$$\int_{t_k}^{t_{k+1}} |p(t)|\Delta t > \frac{2^{2n}}{(t_{k+1} - t_k)^{2n-1}}$$

for $k = 1, 2, ..., N - 1$. Hence,

$$\int_0^T |p(t)|\Delta t \leq \sum_{k=1}^{N-1} \int_{t_k}^{t_{k+1}} |p(t)|\Delta t > 2^{2n} \sum_{k=1}^{N-1} \frac{1}{(t_{k+1} - t_k)^{2n-1}}. \tag{3.8}$$

Since $f(u) = u^{-2n+1}$ is convex for $u > 0$, we have for $x_k = t_{k+1} - t_k > 0, k = 1, 2, ..., N - 1$,

$$\sum_{k=1}^{N-1} f(x_k) > (N - 1)f\left(\frac{\sum_{k=1}^{N-1} x_k}{N - 1}\right),$$

that is,

$$\sum_{k=1}^{N-1} \frac{1}{(t_{k+1} - t_k)^{2n-1}} > (N - 1)f\left(\frac{t_N - t_1}{N - 1}\right) = \frac{(N - 1)^{2n}}{(t_N - t_1)^{2n-1}}$$

$$\geq \frac{(N-1)^{2n}}{T^{2n-1}}. \tag{3.9}$$

Hence (3.7) follows from (3.8) and (3.9).

Theorem 3.6. *If*

$$\int_a^{\sigma(b)} |p(t)|\Delta t < \frac{2^{2n}}{(\sigma(b)-a)^{2n-1}},$$

then Eq.(3.1) is disconjugate on $[a, \sigma(b)]_{\mathbb{T}}$.

Proof. Suppose, on the contrary, that Eq.(3.1) is not disconjugate on $[a, \sigma(b)]_{\mathbb{T}}$. By defination, there exists a nontrivial solution of Eq.(3.1), which has at least $2n$ - generalized zeros (counting multiplicities) in $[a, \sigma(b)]_{\mathbb{T}}$.

Case I. One of the generalized zeros (counting multiplicities of order n) is at the left end point a, that is,

$$y^{\Delta^{2i}}(a) = 0 : i = 0, 1, ..., n-1,$$

the other is at $\sigma(b_0) \in (a, \sigma(b))$, that is

$$y^{\Delta^{2i}}(\sigma(b_0)) = 0 : i = 0, 1, ..., n-1.$$

Therefore, by using Theorem 3.1, we obtain

$$\int_a^{\sigma(b_0)} |p(t)|\Delta t > \frac{2^{2n}}{(\sigma(b_0)-a)^{2n-1}},$$

which is a contradiction to (3.1).

Case II. None of the generalized zero at the left end point a. Then y has two generalized zeros (counting multiplicities of order n) both at $\sigma(a_0)$ and $\sigma(b_0)$ with $\sigma(a_0) < \sigma(b_0)$ in $(a, \sigma(b))$, then

$$\int_{\sigma(a_0)}^{\sigma(b_0)} |p(t)|\Delta t > \frac{2^{2n}}{(\sigma(b_0)-\sigma(a_0))^{2n-1}},$$

that is,

$$\int_a^{\sigma(b)} |p(t)|\Delta t > \frac{2^{2n}}{(\sigma(b)-a)^{2n-1}},$$

which is a contradiction to (3.1). Hence the proof of the theorem is complete.

Theorem 3.7. *If $y(t)$ is a solution of*

$$y^{\Delta^{2n}} \pm \lambda p(t) y = 0,$$

with $y^{\Delta^{2i}}(a) = 0 = y^{\Delta^{2i}}(\sigma(b)); i = 0, 1, ..., n-1,$ and $y(t) \neq 0$ for $t \in [a, \sigma(t)]_{\mathbb{T}}$, where $p \in C_{rd}([0, \infty)_{\mathbb{T}}, \mathbb{R})$ and $\lambda \in \mathbb{R}$ be an eigenvalue, then

$$|\lambda| \geq \frac{2^{2n}}{\left(\int_a^{\sigma(b)} |p(t)| \Delta t\right)(\sigma(t) - a)^{2n-1}}.$$

The proof of the Theorem 3.7 follows from the Theorem 3.1.

References

[1] R. P. Agarwal, M. Bohner and A. Peterson, Inequalities on time scales: A Survey, *Math. Ineq. Appl.* 4(4) (2001), 535-557.

[2] M. Bohner, A. Peterson, *Advances in Dynamic Equations on Time Scale,* Birkhaüser, 2002.

[3] M. Bohner, S. Clark, J. Ridenhour, Liapunov inequalities for time scales, *J. of Inequal. Appl.* 7(1) (2002), 61-77.

[4] R. C. Brown, D. B. Hinton, Opial's inequality and oscillation of 2nd order equations, *Proc. Amer. Math. Soc.* 125 (1997), 1123-1129.

[5] S. S. Cheng, A discrete analogue of the inequality of Liapunov, *Hokkaido Math J.* 12 (1983), 105-112.

[6] S. S. Cheng, Liapunov inequalities for differential and difference equations, *Fasc. Math.* 23 (1991), 25-41.

[7] R. S. Dahiya, B. Singh, A Liapunov inequality and nonoscillation theorem for a second order nonlinear differential-difference equations, *J. Math. Phys. Sci.* 7 (1973), 163-170.

[8] S. B. Eliason, A Liapunov inequality, *J. Math. Anal. Appl.* 32 (1972), 461-466.

[9] S. B. Eliason, A Liapunov inequality for a certain nonlinear differential equation, *J. London Math. Soc.* 2 (1970), 461-466.

[10] S. B. Eliason, A Liapunov inequalities for certain second order functional differential equations, *SIAM J. Appl. Math.* 27 (1) (1974), 180-199.

[11] P. Hartman, *Ordinary Differential Equations,* Wiley, New york, 1964 an Birkhaüser, Boston, 1982 MR 30 1270.

[12] S. Hilger, *Ein MaβKettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten, Ph. D. Thesis*, Universität Würzburg, 1988.

[13] A. M. Liapunov, Probleme général de la stabilité du mouvment (French translation of Russian paper dated 1893), *Ann. Fac. Sci. Univ. Toulouse,* 2 (1907) 27 - 247, *Reprinted as Ann. Math. Studies,* vol. 17, Prineton, 1947.

[14] N. Parhi, S. Panigrahi, On Liapunov-type inequality for third order differential equations, *J. Math. Anal. Appl.* 233 (2) (1999) 445-460.