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In Memoriam Adelina Georgescu

FINITE SINGULARITIES OF TOTAL MULTIPLICITY FOUR FOR A PARTICULAR SYSTEM WITH TWO PARAMETERS*

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Abstract

A particular Lotka-Volterra system with two parameters describing the dynamics of two competing species is analyzed from the algebraic viewpoint. This study involves the invariants and the comitants of the system determinated by the application of the affine transformations group. First, the conditions for the existence of four (different or equal) finite singularities for the general system are proofed, then is studied the particular case.

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1 Introduction

In this paper we study a particular family of planar vector fields with two parameters modeling the dynamics of two competing populations.

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We consider the general form of a Lotka -Volterra system as [4], [8]

$$\begin{cases} \dot{x} = x(c + gx + hy), \\ \dot{y} = y(f + mx + ny), \end{cases} \tag{1}$$

where x, y represent the number of the populations of the two species, c, f represent the growth rates of the species, and g, h, m, n represent the competitive impacts of one specie to another. The equilibrium points of (1) are: $M_1(0,0)$, $M_2(-c/g, 0)$, $M_3(0, -f/n)$ and $M_4((fh - cn)/(gn - hm), (cm - fg)/(gn - hm))$. All these points are in the finite part of the phase plane if and only if $gn(gn - hm) \neq 0$. On the other hand, for the system (1), we have $\mu_0 = gn(gn - hm)$, where μ_0 is defined in the Appendix.

Therefore, for $\mu_0 \neq 0$ the system (1) has four different or equal equilibrium points.

The following two theorems holds, and their proofs can be found in [3].

Theorem 1. *[3]. For $\mu_0 \neq 0$ the number of the four finite singularities of the system (1) are determinated by the following conditions:*

$$\begin{array}{lcl} 4 \textit{ simple} & \Leftrightarrow & \mathbf{D} \neq 0; \\ 2 \textit{ simple}, \ 1 \textit{ double} & \Leftrightarrow & \mathbf{D} = 0, \ \mathbf{S} \neq 0; \\ 2 \textit{ double} & \Leftrightarrow & \mathbf{D} = \mathbf{S} = 0, \ \mathbf{P} \neq 0; \\ 1 \textit{ of multiplicity } 4 & \Leftrightarrow & \mathbf{D} = \mathbf{S} = \mathbf{P} = 0, \end{array}$$

where $\mathbf{D}, \mathbf{S}, \mathbf{P}$ are defined in the Appendix.

Since $\mu_0 \neq 0$, due to the transformation $(x, y) \mapsto (x/g, y/n)$, we can consider $g = n = 1$. Therefore, the system (1) becomes

$$\begin{cases} \dot{x} = x(c + x + hy), \\ \dot{y} = y(f + mx + y), \end{cases} \tag{2}$$

for which $\mu_0 = 1 - hm$, $\mathbf{D} = -c^2 f^2 (c - fh)^2 (f - cm)^2$ and

$$\begin{aligned} &\mathbf{S} = 3c^4 m^2 (x + hy)^2 [3m^2 x^2 - 2m(hm - 4)xy + (3h^2 m^2 - 8hm + 8)y^2], \\ &\mathbf{P} = c^4 y^2 (mx + y)^2 (\text{ if } cf = 0), \end{aligned}$$

or

$$\begin{aligned} &\mathbf{S} = 3c^4 m^2 (hm - 1)^4 x^2 (3m^2 x^2 + 8mxy + 8y^2), \\ &\mathbf{P} = c^4 (hm - 1)^2 y^2 (mx + y)^2 (\text{ if } (c - fh)(f - cm) = 0). \end{aligned}$$

We use the following abbreviations: S=saddle, N=node, F=focus, C=center, SN=saddle-node.
In addition, K, W_3, W_4 are defined in the Appendix.

Theorem 2. *[3]. Let us consider the system (1) with $\mu_0 \neq 0$. Then the type of the finite singularities of this system is determinated by the following affine-invariant conditions:*

1) $S,\ S,\ S,\ N \quad \Leftrightarrow \quad \mathbf{D} \neq 0,\ \mu_0 < 0,\ K < 0,\ W_4 \geq 0;$

2) $S,\ S,\ S,\ F \quad \Leftrightarrow \quad \mathbf{D} \neq 0,\ \mu_0 < 0,\ K < 0,\ W_4 < 0,\ B_3 \neq 0;$

3) $S,\ S,\ S,\ C \quad \Leftrightarrow \quad \mathbf{D} \neq 0,\ \mu_0 < 0,\ K < 0,\ W_4 < 0,\ B_3 = 0;$

4) $S,\ N,\ N,\ N \quad \Leftrightarrow \quad \mathbf{D} \neq 0,\ \mu_0 < 0,\ K > 0 \ \text{ and } \begin{cases} W_4 > 0 \text{ or} \\ W_4 = 0,\ W_3 \geq 0; \end{cases}$

5) $S,\ N,\ N,\ F \quad \Leftrightarrow \quad \mathbf{D} \neq 0,\ \mu_0 < 0,\ K > 0 \text{ and } \begin{cases} W_4 < 0,\ B_3 \neq 0 \text{ or} \\ W_4 = 0,\ W_3 < 0 \end{cases} ;$

6) $S,\ N,\ N,\ C \quad \Leftrightarrow \quad \mathbf{D} \neq 0,\ \mu_0 < 0,\ K > 0 \text{ and } W_4 < 0, B_3 = 0;$

7) $S,\ S,\ N,\ N \quad \Leftrightarrow \quad \mathbf{D} \neq 0,\ \mu_0 > 0 \text{ and } \begin{cases} W_4 > 0 \text{ or} \\ W_4 = 0,\ W_3 \geq 0; \end{cases}$

8) $S,\ S,\ N,\ F \quad \Leftrightarrow \quad \mathbf{D} \neq 0,\ \mu_0 > 0 \text{ and } \begin{cases} W_4 < 0 \text{ or} \\ W_4 = 0,\ W_3 < 0; \end{cases}$

9) $SN,\ S,\ S \quad \Leftrightarrow \quad \mathbf{D} = 0,\ \mathbf{S} \neq 0, \mu_0 < 0,\ K < 0;$

10) $SN,\ N,\ N \quad \Leftrightarrow \quad \mathbf{D} = 0,\ \mathbf{S} \neq 0,\ \mu_0 < 0,\ K > 0$

$\quad \text{and } \begin{cases} W_4 > 0 \text{ or} \\ W_4 = 0,\ W_3 \geq 0; \end{cases}$

11) $SN,\ N,\ F \quad \Leftrightarrow \quad \mathbf{D} = 0,\ \mathbf{S} \neq 0,\ \mu_0 < 0,\ K > 0,\ W_4 < 0;$

12) $SN,\ N,\ C \quad \Leftrightarrow \quad \mathbf{D} = 0,\ \mathbf{S} \neq 0,\ \mu_0 < 0,\ K > 0,\ W_4 = 0,\ W_3 < 0;$

13) $SN,\ S,\ N \quad \Leftrightarrow \quad \mathbf{D} = 0,\ \mathbf{S} \neq 0,\ \mu_0 > 0,\ W_4 \geq 0;$

14) $SN,\ S,\ F \quad \Leftrightarrow \quad \mathbf{D} = 0,\ \mathbf{S} \neq 0,\ \mu_0 > 0,\ W_4 < 0;$

15) $SN,\ SN \quad \Leftrightarrow \quad \mathbf{D} = \mathbf{S} = 0,\ \mathbf{P} \neq 0;$

16) a degenerated nonhyperbolic point of the multiplicity 4

$(a) \quad \Leftrightarrow \quad \mathbf{D} = \mathbf{S} = \mathbf{P} = 0,\ \mu_0 < 0,\ \eta > 0,\ \chi > 0;$

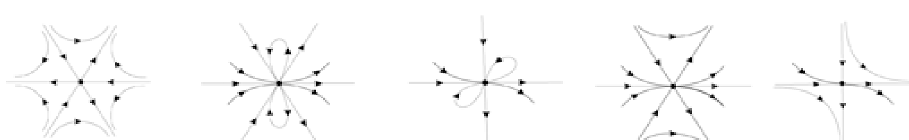
$(b) \quad \Leftrightarrow \quad \mathbf{D} = \mathbf{S} = \mathbf{P} = 0,\ \mu_0 < 0,\ \eta > 0,\ \chi < 0;$

$(c) \quad \Leftrightarrow \quad \mathbf{D} = \mathbf{S} = \mathbf{P} = 0,\ \mu_0 < 0,\ \eta = 0;$

$(d) \quad \Leftrightarrow \quad \mathbf{D} = \mathbf{S} = \mathbf{P} = 0,\ \mu_0 > 0,\ \eta > 0;$

$(e) \quad \Leftrightarrow \quad \mathbf{D} = \mathbf{S} = \mathbf{P} = 0, \mu_0 > 0,\ \eta = 0,$

where (a)-(e) have the representations:



(a) (b) (c) (d) (e)

2 The particular competing species model

The model we study in this paper is proposed as an application by M.W. Hirsch, S. Smale and R. L. Devaney in [5] and has the form

$$\left\{\begin{array}{lcl} \dot{x} & = & x(a-x-ay), \\ \dot{y} & = & y(b-bx-y), \end{array}\right. \tag{3}$$

where $x,\ y$ represent the number of the populations of the two species, a and b are positive parameters.

In order to apply the Theorems 1 and 2, we transform the system (3) into the system (1).

The system (3) is equivalent with

$$\left\{\begin{array}{lcl} \dot{x} & = & -x(-a+x+ay), \\ \dot{y} & = & -y(-b+bx+y), \end{array}\right.$$

and, by the change of the sense of the time $t \mapsto -t$ we obtain the system

$$\left\{\begin{array}{lcl} \dot{x} & = & x(-a+x+ay), \\ \dot{y} & = & y(-b+bx+y), \end{array}\right. \tag{4}$$

which is the system we are concerned herein.

Due to physical reasons, the phase space must be the first quadrant (without axes of coordinates). However, for mathematical (namely bifurcation) reasons we consider, in addition, the origin and the half-axes.

Remark 1. *The system (4) has the same equilibrium points as (3), but the attractive properties of the equilibria of the system (4) are opposite of those of the system (3).*

3 The equilibrium points

By convention, we say that an equilibrium exists if its coordinates are finite and positive. Therefore, this is a biological, not a mathematical existence.

The equilibrium points of the system (4) are $M_1(0,0)$, $M_2(a,0)$, $M_3(0,b)$, $M_4(a(1-b))/(1-ab), b(1-a)/(1-ab)$. For these points we compute Δ_i,

ρ_i, δ_i, $(i = 1, 2, 3, 4)$ given in the Appendix.

$$\begin{gathered}\Delta_1 = ab,\ \rho_1 = -a - b,\ \delta_1 = (a-b)^2,\\ \Delta_2 = ab(a-1),\ \rho_2 = a - b + ab,\ \delta_2 = (a+b-ab)^2,\\ \Delta_3 = ab(b-1),\ \rho_3 = -a + b + ab,\ \delta_3 = (a+b-ab)^2,\\ \Delta_4 = ab(a-1)(b-1)/(1-ab),\ \rho_4 = (a+b-2ab)/(1-ab),\\ \delta_4 = [(a-b)^2 + 4a^2b^2(a-1)(b-1)]/(1-ab)^2\end{gathered}\tag{5}$$

For the system (4) we have

$$\begin{gathered}\mu_0 = 1 - ab,\ \mu_1 = (-2b + ab + ab^2)x + (2a - ab - a^2b)y,\\ \mathbf{D} = -a^4b^4(a-1)^2(b-1)^2,\ K = 2(bx^2 + 2xy + ay^2),\\ W_4 = (a-b)^2(ab-a-b)^2[(a-b)^2 + 4a^2b^2(a-1)(b-1)].\end{gathered}\tag{6}$$

In the following, we study the nature of the finite singularities of the system (4) for the case $\mu_0 \neq 0$ (i.e. $ab \neq 1$).

Case $\mathbf{D} \neq 0$. This case is equivalent with $1 - ab \neq 0$, $a \notin \{0, 1\}$, $b \notin \{0, 1\}$. From Theorem 1, it follows that the system (4) has four simple equilibrium points.

• If $\mu_0 < 0$ then $1 - ab < 0$. Since a and b are positive parameters, it follows that $K > 0$. If $W_4 > 0$, then we have $a > 1,\ b > 1$ and we are in the case 4 from the Theorem 2 (i.e. the system (4) has three nodes and a saddle) or $(a > 1,\ b < 1)$, $(a < 1,\ b > 1)$, where the point M_4 is not in the first quadrant, therefore it does not exist from biological viewpoint. In this case there are only three points from biological viewpoint (two nodes and a saddle). On the other hand, W_4 can not be negative. Indeed, if $W_4 < 0$ then $(a-1)(b-1) < 0$, therefore $a > 1,\ b < 1$ or $a < 1,\ b > 1$. It follows that M_4 is not in the first quadrant, therefore it does not exist from biological viewpoint. Again there are only three points from biological viewpoint (two nodes and a saddle).

Thus, the finite singularities of total multiplicity four of the system (3) which exist from biological viewpoint are as follows: if $a > 1,\ b > 1$, then M_1 is a repulsive node, $M_2,\ M_3$ are attractive nodes and M_4 is a saddle; if $a > 1,\ b < 1$, then M_1 is a repulsive node, M_2 is an attractive node and M_3 is a saddle; if $a < 1,\ b > 1$, then M_1 is a repulsive node, M_2 is a saddle and M_3 is an attractive node.

• If $\mu_0 > 0$ then $1 - ab > 0$. If $W_4 > 0$, then we have $a < 1,\ b < 1$ and we are in the case 7 from the Theorem 2 (i.e. the system (4) has two nodes and two saddles), or $(a > 1,\ b < 1)$, $(a < 1,\ b > 1)$, when M_4 is not in the first quadrant, therefore it does not exist from biological viewpoint. In this case there are only three points (two nodes and a saddle). On the other hand, W_4 can not be negative. Indeed, if $W_4 < 0$ then $(a-1)(b-1) < 0$, equivalently with $a > 1,\ b < 1$ or $a < 1,\ b > 1$ therefore, the point M_4 is not in the first quadrant, so it does not exist from biological viewpoint. Again there are only three points from biological viewpoint (two nodes and a saddle).

Thus, the finite singularities of total multiplicity four of the system (3) that exist from biological viewpoint are as follows: if $a < 1,\ b < 1$, then M_1 is a repulsive node, $M_2,\ M_3$ are saddles and M_4 is an attractive node; if $a > 1,\ b < 1$, then M_1 is a repulsive node, M_2 is an attractive node and M_3 is a saddle; if $a < 1,\ b > 1$, then M_1 is a repulsive node, M_2 is a saddle and M_3 is an attractive node.

Case $\mathbf{D} = 0$. We have two subcases: $ab = 0$ or $(a-1)(b-1) = 0$.

• For $ab = 0$, without loss of generality, due to the change $x \leftrightarrow y$, $a \leftrightarrow b$, which keeps the system (4) unchanged, we can consider only $a = 0$. In this case $\mathbf{S} = 0$ and $\mathbf{P} = b^4x^4$.

If $b \neq 0$, then $\mathbf{P} \neq 0$ and we are in the case 15 from the Theorem 2 (i.e. the system (4) has two saddle-nodes).

If $b = 0$, then $\mathbf{P} = 0$, $\mu_0 = 1 > 0$ and $\eta = 1 > 0$, therefore we are in the case 16 (d) from the Theorem 2 (i.e. the system (4) has a point of multiplicity 4).

Thus, in the plane, the type of the finite singularities for the system (3) are as follows: if $a = 0,\ b \neq 0$ $(a \neq 0,\ b = 0)$, then $M_1 = M_2$, $M_3 = M_4$ $(M_1 = M_3,\ M_2 = M_4)$ are saddle-nodes ; if $a = 0,\ b = 0$, then $M_1 = M_2 = M_3 = M_4$, i.e. we have a nonhyperbolic point of multiplicity 4.

• For $(a-1)(b-1) = 0$, without loss of generality, due to the change $x \leftrightarrow y$, $a \leftrightarrow b$, which keeps the system (4) unchanged, we can consider only $a = 1$. In this case $\mathbf{S} = 3b^2(b-1)^4x^2(3b^2x^2 + 8bxy + 8y^2)$.

If $\mathbf{S} \neq 0$, then $b \notin \{0, 1\}$ and $\mu_0 = 1 - b$, $\eta = 0$, $W_4 = (b-1)^2$.

For $\mu_0 < 0$ (i.e. $b > 1$), we have $K > 0$, $W_4 > 0$, therefore we are in the case 10 from the Theorem 2 (i.e. the system (4) two nodes and a saddle-node).

For $\mu_0 > 0$ (i.e. $b < 1$), we have $K > 0$, $W_4 > 0$ therefore we are in the case 13 from the Theorem 2 (i.e. the system (4) has a node, a saddle and a saddle-node).

Thus, the finite singularities of total multiplicity four of the system (3) are as follows: if $a = 1,\ b < 1$ $(b > 1)$, then M_1 is a repulsive node, M_3 is a saddle, $M_2 = M_4$ is a saddle-node (M_1 is a repulsive node, M_3 is an attractive node, $M_2 = M_4$ is a saddle-node); if $b = 1,\ a < 1$ $(a > 1)$ then M_1 is a repulsive node, M_2 is a saddle, $M_3 = M_4$ is a saddle-node (M_1 is a repulsive node, M_2 is an attractive node, $M_3 = M_4$ is a saddle-node).

If $\mathbf{S} = 0$, then $b = 0$ (if $b = 1$ we obtain a contradiction: $\mu_0 = 0$). For $b = 0$ we have two saddle-nodes $M_1 = M_3$ and $M_2 = M_4$.

4 The phase portraits

In [2] the system (3) was studied by the topological methods and the dynamic bifurcation diagram was representing. Here we represent only the phase portraits that have a biological significance (i.e the equilibria are in the first quadrant) and where the equilibrium points have total multiplicity four (fig.2). The parametric portrait (fig.1) is representing by the strata 0-10 without the curve T (corresponding to $ab = 1$).

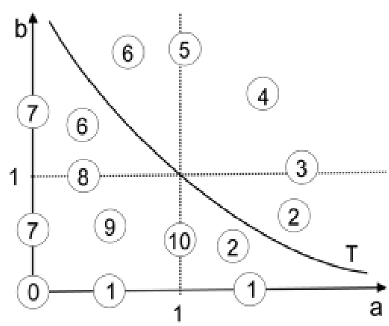


Fig. 1. The parametric portrait.

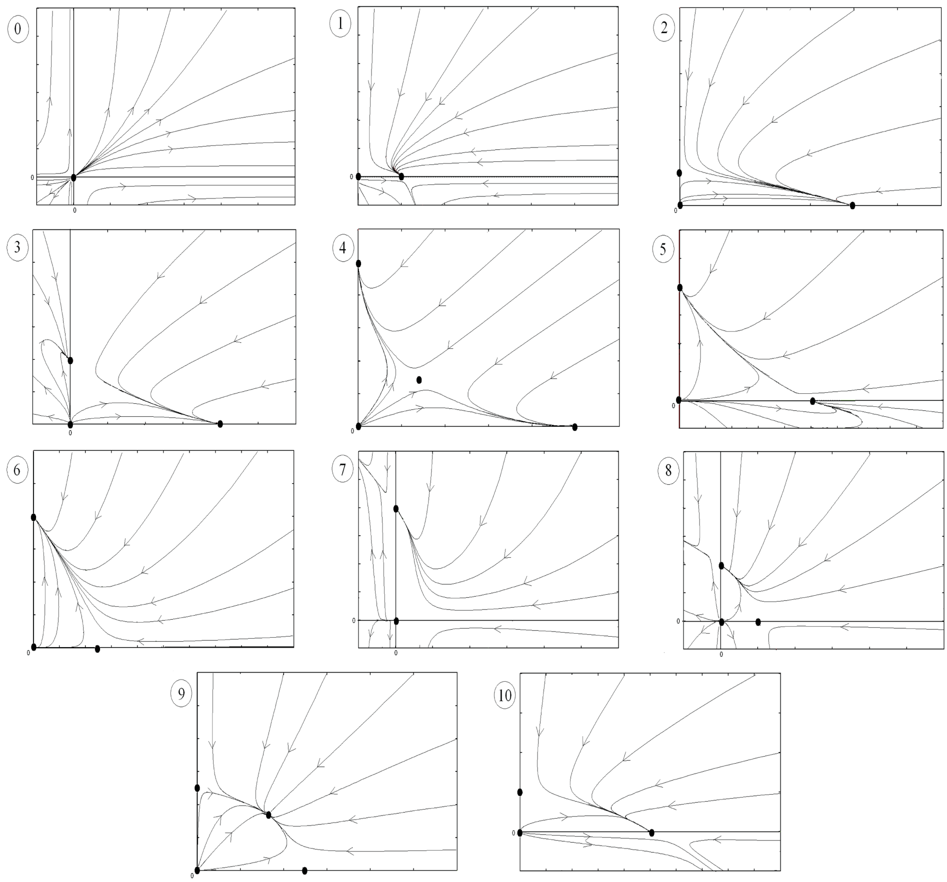


Fig. 2. The phase portraits for the various strata from Fig. 1

5 Appendix

Consider the two-dimensional nonlinear system of ordinary differential equations

$$\begin{cases} \dot{x} = p_0(x,y) + p_1(x,y) + p_2(x,y) \equiv p(x,y), \\ \dot{y} = q_0(x,y) + q_1(x,y) + q_2(x,y) \equiv q(x,y), \end{cases} \tag{7}$$

where p_i and q_i, i=0,1,2 homogenous polynomials of i degree.

For a singular point $M_i(x_i, y_i)$ we use the notations:

$$\rho_i = (p'_x(x,y) + q'_y(x,y))|_{(x_i,y_i)} = \mathrm{tr}\mathbf{A}_i,$$

$$\Delta_i = \left| \begin{array}{cc} p'_x(x,y) & p'_y(x,y) \\ q'_y(x,y) & q'_y(x,y) \end{array} \right|_{(x_i,y_i)} = \det \mathbf{A}_i,$$

$$\delta_i = \rho_i^2 - 4\Delta_i = \mathrm{tr}^2\ \mathbf{A}_i - 4 \det \mathbf{A}_i,$$

where $\mathbf{A}_i$ is the matrix of the linear terms from the linearized system around the point (x_i, y_i).

The following polynomials are the GL-comitants and T-comitants of the system (7) [1], [6], [7]:

$$\begin{array}{l}
C_i(\mathbf{a},x,y) = yp_i(\mathbf{a},x,y) - xq_i(\mathbf{a},x,y), \quad i = 0,1,2; \\
\eta(\mathbf{a}) = Discrim(C_2(\mathbf{a},x,y)); \\
K(\mathbf{a},x,y) = Jacob(p_2(\mathbf{a},x,y), q_2(\mathbf{a},x,y)); \\
\mu_0(\mathbf{a}) = Res_x(p_2,q_2)/y^4 = Discrim(K(\mathbf{a},x,y))/16; \\
\mathbf{D}(\mathbf{a}) = -\left(\left((D,D)^{(2)},D\right)^{(1)},D\right)^{(3)}/576 \equiv -Discrim(D); \\
\mathbf{P}(\mathbf{a},x,y) = \mu_2^2 - 3\mu_1\mu_3 + 12\mu_0\mu_4; \\
\mathbf{S}(\mathbf{a},x,y) = \left[3\mu_1^2 - 8\mu_0\mu_2\right]^2 - 16\mu_0^2\mathbf{P}; \\
B_3(\mathbf{a},x,y) = (C_2,D)^{(1)} = Jacob(C_2,D), \\
W_3 = \mu_0^2 \sum\limits_{1\le i<j<l\le 4} \delta_i\delta_j\delta_l, \\
W_4 = \mu_0^2\delta_1\delta_2\delta_3\delta_4.
\end{array}$$

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