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In Memoriam Adelina Georgescu

AN APPLICATION OF DOUBLE-SCALE METHOD TO THE STUDY OF NON-LINEAR DISSIPATIVE WAVES IN JEFFREYS MEDIA*

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Abstract

In previous papers we sketched out the general use of the double-scale method to nonlinear hyperbolic partial differential equations (PDEs) in order to study the asymptotic waves and as an example the model governing the motion of a rheological medium (Maxwell medium) with one mechanical internal variable was studied. In this paper the double scale method is applied to investigate non-linear dissipative waves in viscoanelastic media without memory of order one (Jeffreys media), that were studied by one of the authors (L. R.) in more classical way. For these media the equations of motion include second order derivative terms multiplied by a very small parameter. We give a physical interpretation of the new (fast) variable, related to the surfaces across which the solutions or/and some of their derivatives vary steeply. The paper concludes with one-dimensional application containing original results.

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Introduction

The mathematical aspects involved into the study of asymptotic waves belong to singular perturbation theory, namely the double-scale method ([1, 10, 14, 16, 24, 30, 31, 35, 36, 40, 42, 43]. The multiple-scale method, and, in particular, the double-scale approach, is appropriate to phenomena which possess qualitatively distinct aspects at various scales. For instance, at some well-determined times or space coordinates, the characteristics of the motion vary steeply, while at larger scale the characteristics are slow and describe another type of motion. In addition, the scales are defined by some small parameters.

The theoretical interest in nonlinear waves was manifest as early as the years '50 and '60 of the last century and a lot of applications to various branches of physics were worked out [2, 3, 4, 12, 13, 19, 20, 21, 22, 23, 32, 33].

In the context of rheological media studies on non-linear waves were carried out in [7]-[9]. In previous papers (see [17] and [39]) we sketched out the general use of the double-scale method to nonlinear hyperbolic partial differential equations (PDEs) in order to study the asymptotic waves and as an application the model governing the motion of anelastic media without shape and bulk memory (Maxwell media) was studied.

In this paper the double scale method (see [16]) is applied to investigate non-linear dissipative waves in isotropic viscoanelastic media without memory of order one in which a viscous flow phenomenon occurs (Jeffreys media), that were studied in [8] by one of the authors (L. R.) in more classical way, following the methodologies developed in [3] and generalized in [15]. Only shear phenomena are taken into consideration and the hydrostatic pressure is assumed constant and uniform. Furthermore, the isothermal case is considered. For these media the equations of motion include second order derivative terms multiplied by a very small parameter, that play a very important role because they usually have a balancing effect on the non-linear steepening of waves. In Section 1 the various steps in applying the double scale method are introduced and the asymptotic approximations of first and second order are obtained. In Section 2 the propagation into an uniform unperturbed state is discussed and in Section 3 the first approximation of wavefront and of **U** are derived. In Section 4 the equations governing the motion of Jeffreys media are treated and the mechanical relaxation equation valid for these media is described in the framework of classical irreversible thermodynamics (TIP) with internal variables [5, 11, 25, 26, 27, 28, 29, 34, 37, 38]. In Section 5 an one-dimensional application is carried out containing original results.

1 Asymptotic dissipative waves from the point of view of double-scale method

Let E^{3+1} be an Euclidean space and let $P \in E^{3+1}$ be a current point. Let $\mathbf{U} = \mathbf{U}(P)$ be the unknown vector function, solution of a system of PDEs written in the following matrix form

$$\mathbf{A}^{\alpha}(\mathbf{U})\mathbf{U}_{\alpha} + \omega^{-1}\left[\mathbf{H}^{k}\frac{\partial^2 \mathbf{U}}{\partial t \partial x^k} + \mathbf{H}^{ik}\frac{\partial^2 \mathbf{U}}{\partial x^i \partial x^k}\right] = \mathbf{B}(\mathbf{U}),$$

$$(\alpha = 0, 1, 2, 3);\ (i, k = 1, 2, 3), \tag{1}$$

where $x^0 = t$ (time), x^1, x^2, x^3 are the space coordinates, $\mathbf{U}$ depends on x^{α}, $\mathbf{U}_{\alpha} = \frac{\partial \mathbf{U}}{\partial x^{\alpha}}$, $\mathbf{A}^{\alpha}$, $\mathbf{H}^{k}$, $\mathbf{H}^{ik}$ are appropriate matrices 9×9 and

$$\mathbf{A}^{\alpha}(\mathbf{U})\mathbf{U}_{\alpha} = \mathbf{B}(\mathbf{U}) \tag{2}$$

is the associated system of nonlinear hyperbolic PDEs.

In [8] it was shown that the motion of viscoanelastic media without memory, in the isothermal case, where only shear phenomena are taken into consideration and the hydrostatic pressure is constant and uniform, is described by a system of nonlinear PDEs having the form (1). The system of PDEs (1) includes terms containing second order derivatives multiplied by a very small parameter. These terms play a very important role because they usually have a balancing effect on the non-linear steepening of the waves. In [41], using (1), the propagation of linear acoustic waves was considered and the velocity and attenuation of the waves were investigated. In [8] the non-linear dissipative waves were worked out (see [2, 3, 4, 12, 13, 19, 20, 21, 22, 23, 32, 33]) and, in particular, a method, developed by G. Boillat [3] and generalized by D. Fusco [15], was applied to construct asymptotic approximations of order 1 of solutions of the system of equations (1).

In this Section we study these non-linear dissipative waves from the point of view of double scale-method.

Following A. Jeffrey in [23], let us introduce for systems of PDEs of type (1) (or type (2)) the concepts of waves (called *dissipative waves* only for the system (1)) and associated wavefronts. Precisely, the solution hypersurfaces of systems of type (1) (or type (2)) are referred to as waves, because they may be interpreted as representing propagating wavefronts. When physical problems are associated with such interpretation the solution on the side of the wavefront towards which the propagation takes place may then be regarded as being the *undisturbed solution* ahead of the wavefront, whilst the solution on the other side may be regarded as a propagating *disturbance wave* which is entering a region occupied by the undisturbed solution. This is because the solution at a point in the undisturbed region characterises the state of the physical system at that time and place before the advancing wave has reached it.

The smooth solutions of systems of type (1) (or type(2)) that present a steep variation in the normal direction to the associated wavefront are called *asymptotic waves*. Then, there exists a family of hypersurfaces S (defined by the equation $\varphi(x^\alpha) = 0$) moving in the Euclidean space E^{3+1} (consisting of points of coordinates x^α, $\alpha = 0, 1, 2, 3$, or, equivalently of the time $t = x^0$ and the space coordinates x^i, i=1, 2, 3, having equation

$$\varphi(t, x^i) = \bar{\xi} = const, \tag{3}$$

such that the solutions $\mathbf{U}$ or/and some of their derivatives vary steeply across S while along S their variation is slow [1]. From the double scale method point of view this means that around S there exist *asymptotic internal layers* (see [16]) such that *the order of magnitude (i.e. the scale)* of the solutions or/and of some of their derivatives inside these layers and far away from them differs very much. In systems of equations of type (1) the coefficient ω^{-1} is the small parameter, that is associated with the order of magnitude of the interior layer. Therefore, it is natural to introduce a new independent variable ξ, related to the hypersurfaces S,

$$\xi = \omega\bar{\xi} = \omega\varphi(t, x^i), \tag{4}$$

with $\xi = \frac{\varphi(t,x^i)}{\omega^{-1}}$ asymptotically fixed, i.e. $\xi = Ord(1)$ as $\omega^{-1} \to 0$, and $\omega \gg 1$ a very large parameter, to assume that the solution depends on the old as well as the new variable, i.e. $\mathbf{U} = \mathbf{U}(x^\alpha, \xi)$, and to consider that x^α and ξ are independent.

Taking into account that $\mathbf{U}$ is sufficiently smooth, hence it has sufficiently many bounded derivatives, it follows that, except for the terms containing ω, all the other terms are asymptotically fixed and the computation can proceed formally. In this way, if $x^\alpha = x^\alpha(s)$ are the parametric equations of a curve C in E^{3+1}, we have

$$\frac{d\mathbf{U}}{ds} = \omega \frac{\partial \mathbf{U}}{\partial \xi}\frac{\partial \varphi}{\partial s} + \frac{\partial U^\alpha}{\partial x^\alpha}\frac{dx^\alpha}{ds}$$

(where the dummy index convention is understood). This relation shows that, indeed, along C, $\mathbf{U}$ does not vary too much if C belongs to the hypersurface S (in this case $\frac{d\varphi}{ds} = 0$) but has a large variation if C is not situated on S. For these reasons ξ is referred to as the fast variable.

Let us sketch the various steps in applying the double-scale method.

First, we look for the solution of the equations as an asymptotic series of powers of the small parameter, say ϵ, namely with respect to the asymptotic sequence $\left\{1, \varepsilon^{a+1}, \varepsilon^{a+2}, ...,\right\}$ or $\left\{1, \varepsilon^{\frac{1}{p}}, \varepsilon^{\frac{2}{p}}, ...,\right\}$, as $\varepsilon \to 0$. In [7] - [9] it is considered $p = 1$ and $\varepsilon = \omega^{-1}$, such that $\mathbf{U}(x^\alpha, \xi)$ is written as an asymptotic power series of the small parameter ω^{-1}, i.e. with respect to the asymptotic sequence 1, ω^{-1}, $\omega^{-2}, ...,$ as $\omega^{-1} \to 0$, the $\mathbf{U}^i$ $(i = 1, 2, ...)$ being functions of x^α and ξ,

$$\mathbf{U}(x^\alpha, \xi) \sim \mathbf{U}^0(x^\alpha, \xi) + \omega^{-1}\mathbf{U}^1(x^\alpha, \xi) + O(\omega^{-2}), \text{ as } \omega^{-1} \to 0. \quad (5)$$

In (5) $\mathbf{U}^0(x^\alpha, \xi)$ is a known solution [15] of

$$\mathbf{A}^\alpha(\mathbf{U}^0)\mathbf{U}_\alpha(\mathbf{U}^0) = \mathbf{B}(\mathbf{U}^0), \quad (6)$$

where $\mathbf{U}^0$ is taken as the initial unperturbed state.

The next step of the double-scale method consists in expressing the derivatives with respect to x^α, $\frac{\partial}{\partial x^\alpha}$, in terms of the derivatives with respect to x^α and ξ, i.e. $\frac{\partial}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha} + \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha} + \omega\frac{\partial}{\partial \xi}\frac{\partial \varphi}{\partial x^\alpha}$, so that the derivative $\mathbf{U}_\alpha = \frac{\partial \mathbf{U}}{\partial x^\alpha}$ has the form

$$\frac{\partial \mathbf{U}}{\partial x^\alpha} \sim \omega^{-1}\left(\frac{\partial \mathbf{U}^1}{\partial x^\alpha} + \omega\frac{\partial \mathbf{U}^1}{\partial \xi}\frac{\partial \varphi}{\partial x^\alpha}\right) + \omega^{-1}\frac{\partial \mathbf{U}^2}{\partial \xi}\frac{\partial \varphi}{\partial x^\alpha} + O(\omega^{-2}), \text{ as } \omega^{-1} \to 0, \quad (7)$$

where we have assumed that the first approximation $\mathbf{U}^0$ is constant.

Then, taking into account the form of $\mathbf{A}^\alpha$, $\mathbf{H}^k$, $\mathbf{H}^{ik}$ and $\mathbf{B}$, the following asymptotic expansions are deduced:

$$\mathbf{A}^\alpha(\mathbf{U}) \sim \mathbf{A}^\alpha(\mathbf{U}^0) + \frac{1}{\omega}\nabla\mathbf{A}^\alpha(\mathbf{U}^0)\mathbf{U}^1 + O\left(\frac{1}{\omega^2}\right), \quad \text{as} \quad \omega^{-1} \to 0, \tag{8}$$

$$\mathbf{H}^k(\mathbf{U}) \sim \mathbf{H}^k(\mathbf{U}^0) + \frac{1}{\omega}\nabla\mathbf{H}^k(\mathbf{U}^0)\mathbf{U}^1 + O\left(\frac{1}{\omega^2}\right), \text{ as } \omega^{-1} \to 0, \ (k = 1,2,3), \tag{9}$$

$$\mathbf{H}^{ik}(\mathbf{U}) \sim \mathbf{H}^{ik}(\mathbf{U}^0) + \frac{1}{\omega}\nabla\mathbf{H}^{ik}(\mathbf{U}^0)\mathbf{U}^1 + O\left(\frac{1}{\omega^2}\right), \text{ as } \omega^{-1} \to 0, \ (i,k = 1,2,3), \tag{10}$$

$$\mathbf{B}(\mathbf{U}) \sim \mathbf{B}(\mathbf{U}^0) + \frac{1}{\omega}\nabla\mathbf{B}(\mathbf{U}^0)\mathbf{U}^1 + O\left(\frac{1}{\omega^2}\right), \quad \text{as} \quad \omega^{-1} \to 0, \tag{11}$$

where $\nabla = \frac{\partial}{\partial \mathbf{U}}$.

The last point of the method consists in introducing the asymptotic expansions (7) - (11) into (1) and matching the obtained series.

It follows

$$(\mathbf{A}^\alpha)_0 \Phi_\alpha \frac{\partial \mathbf{U}^1}{\partial \xi} = \mathbf{0} \quad (\alpha = 0,1,2,3), \tag{12}$$

$$(\mathbf{A}^\alpha)_0 \left(\Phi_\alpha \frac{\partial \mathbf{U}^2}{\partial \xi}\right) = -\left[(\mathbf{A}^\alpha)_0 \frac{\partial \mathbf{U}^1}{\partial x^\alpha} + (\nabla\mathbf{A}^\alpha)_0 \mathbf{U}^1 \left(\Phi_\alpha \frac{\partial \mathbf{U}^1}{\partial \xi}\right)\right.$$
$$\left. + (\mathbf{H}^k)_0 \Phi_0 \Phi_k \frac{\partial^2 \mathbf{U}^1}{\partial \xi^2} + (\mathbf{H}^{ik})_0 \Phi_i \Phi_k \frac{\partial^2 \mathbf{U}^1}{\partial \xi^2} - (\nabla \mathbf{B})_0 \mathbf{U}^1\right], \tag{13}$$

where $\Phi_\alpha = \frac{\partial \varphi}{\partial x^\alpha}$ ($\Phi_k = \frac{\partial \varphi}{\partial x^k}, \ k = 1,2,3$) and the symbol $(...)_0$ indicates that the quantities are calculated in $\mathbf{U}^0$. Equation (12) is linear in $\mathbf{U}^1$, while (13) is affine in $\mathbf{U}^2$.

Remind that the wavefront φ is still an unknown function. In order to determine it, we recall its equation is $\varphi(t, x^1, x^2, x^3) = 0)$. This implies that along the wavefront we have $\frac{d\varphi}{dt} = 0$, implying $\frac{\partial \varphi}{\partial t} + \mathbf{v} \cdot grad\varphi = 0$, or equivalently, $\frac{\frac{\partial \varphi}{\partial t}}{|grad\varphi|} + \mathbf{v} \cdot \frac{grad\varphi}{|grad\varphi|} = 0$. Obviously,

$$\frac{grad\varphi}{|grad\varphi|} = \mathbf{n}, \tag{14}$$

such that the previous equality reads

$$\frac{\frac{\partial\varphi}{\partial t}}{|grad\varphi|} + \mathbf{v}\cdot\mathbf{n} = 0. \tag{15}$$

Introduce the notation

$$\lambda = -\frac{\frac{\partial\varphi}{\partial t}}{|grad\varphi|}, \tag{16}$$

so that

$$\lambda(\mathbf{U},\mathbf{n}) = \mathbf{v}\cdot\mathbf{n}, \tag{17}$$

where λ is called the *velocity normal to the progressive wave*, being $\mathbf{n}$ the unit vector normal to the wave front.

Following the general theory [3] we introduce the quantity

$$\Psi(\mathbf{U},\Phi_\alpha) = \varphi_t + |grad\varphi|\lambda(\mathbf{U},\mathbf{n}). \tag{18}$$

The characteristic equations for (18) are

$$\frac{dx^\alpha}{d\sigma} = \frac{\partial\Psi}{\partial\Phi_\alpha}, \qquad \frac{d\Phi_\alpha}{d\sigma} = -\frac{\partial\Psi}{\partial x^\alpha} \quad (\alpha = 0,1,2,3), \tag{19}$$

where σ is the time along the rays.
The i-th component of the radial velocity $\mathbf{\Lambda}$ is defined by

$$\Lambda_i(\mathbf{U},\mathbf{n}) \equiv \frac{dx^i}{d\sigma} = \frac{\partial\Psi}{\partial\phi_i} = \lambda n_i + \frac{\partial\lambda}{\partial n_i} - \left(\mathbf{n}\cdot\frac{\partial\lambda}{\partial\mathbf{n}}\right) n_i = \lambda n_i + v_i - (n_k v_k) n_i, \tag{20}$$

$$(i = 1,2,3).$$

Hence,

$$\mathbf{\Lambda}(\mathbf{U},\mathbf{n}) = \mathbf{v} - (v_n - \lambda)\mathbf{n}. \tag{21}$$

The theory in [3] enables us to deduce the equation for φ by using (15). Of course, equations of asymptotic approximations of higher order can be written and they are affine, but their solutions are very difficult. Just to solve the linear equation (12), a method was developed by G. Boillat [3] and generalized by D. Fusco [15] (see Sections 2, 3).

2 Propagation into an uniform unperturbed state

Consider an uniform unperturbed state $\mathbf{U}^0$, solution of (6). If the quantities (14) and (16) are introduced in the expression (12) we obtain

$$(A_{0n} - \lambda I)\frac{\partial \mathbf{U}^1}{\partial \xi} = 0, \tag{22}$$

where $(\mathbf{A}_n)_0 = A_{0n}$ and $\mathbf{A}_n(\mathbf{U}) = \mathbf{A}^i n_i$. In the case where the eigenvalues are real and the eigenvectors of the matrix $\mathbf{A}_n$ are linearly independent, the system of PDEs (2) is hyperbolic (see [17] for the definition of hyperbolicity). Furthermore, in [3] it was shown that only for the waves propagating with a velocity λ such that $\nabla\lambda \cdot \ \mathbf{r} \neq 0$ (with $\mathbf{r}$ the right eigenvector of $(\mathbf{A}_n)_0$ corresponding to the eigenvalue λ), i.e. with a velocity that does not satisfy the Lax - Boillat exceptionality condition $\nabla\lambda \cdot \ \mathbf{r} = 0$, our results are valid. Eq. (22) shows that $\mathbf{U}^1(x^\alpha, \xi)$, by integration, has the form

$$\mathbf{U}^1(x^\alpha, \xi) = u(x^\alpha, \xi)\mathbf{r}(\mathbf{U}^0, \mathbf{n}) + \mathbf{v}^1(x^\alpha), \tag{23}$$

where u is a scalar function to be determined and $\mathbf{v}^1$ is an arbitrary function of integration which can be taken as zero, without loss of generality. It may be observed that in (23) u gives rise to the phenomenon of the distortion of the signals and this term governs the first-order perturbation obeying a non-linear partial differential equation (see Section 3). We conclude this section by showing how the wave front $\varphi(t, x^1, x^2, x^3) = 0$ can be determined (see [8]). Since we are considering the propagation into an uniform unperturbed state, it is known [3] that the wave front φ satisfies the partial differential equation

$$\Psi(\mathbf{U}^0, \Phi_\alpha) = \varphi_t + |grad\varphi|\lambda(\mathbf{U}^0, \mathbf{n}^0) = \Psi^0 = 0, \tag{24}$$

where $\mathbf{n}^0$ is a constant value of $\mathbf{n}$, and so

$$\Lambda_i(\mathbf{U}^0, \mathbf{n}^0) = \frac{\partial \Psi^0}{\partial \Phi_i} \quad (i = 1, 2, 3). \tag{25}$$

The characteristic equations for (24) are

$$\frac{dx^\alpha}{d\sigma} = \frac{\partial \Psi^0}{\partial \Phi_\alpha}, \qquad \frac{d\Phi_\alpha}{d\sigma} = -\frac{\partial \Psi^0}{\partial x^\alpha} \quad (\alpha = 0, 1, 2, 3), \tag{26}$$

where σ is the time along the rays.

By integration of (26) one obtains

$$x^0 = t = \sigma,\ x^i = (x^i)^0 + \Lambda_i^0(\mathbf{U}^0, \mathbf{n}^0)t,\ \text{with } (x^i)^0 = (x^i)_{t=0}\ (i=1,2,3). \quad (27)$$

If we denote by φ^0 the given initial surface, we have $(\varphi)_{t=0} = \varphi^0\left[(x^i)^0\right]$ and $\mathbf{n}^0$ represents the normal vector at the point $(x^i)^0$ defined by $\mathbf{n}^0 = \left(\frac{grad\varphi}{|grad\varphi|}\right)_{t=0} = \frac{grad^0\varphi^0}{|grad^0\varphi^0|}$, where $(grad^0)_i \equiv \frac{\partial}{\partial(x^i)^0}$ $(i = 1, 2, 3)$. Then, $\mathbf{x} = \mathbf{x}|_{t=0} + \mathbf{\Lambda}^0 t$ and since the Jacobian J of the transformation $\mathbf{x} \to \mathbf{x}|_{t=0}$ is nonvanishing, i.e. $J = det|\delta_{ik} + \frac{\partial\Lambda_k^0}{\partial(x^i)^0}t| \neq 0$ $(i, k = 1, 2, 3)$, $(x^i)^0$ can be deduced from equations $(27)_2$ and φ in the first approximation takes the following form

$$\varphi(t, x^i) = \varphi^0(x^i - \Lambda_i^0 t). \quad (28)$$

3 First approximation of the wavefront and of U

In [8] it is shown that, by utilizing (13) and (23) (see [3] and [15]), the following equation for $u(x^\alpha, \xi)$ can be obtained:

$$\frac{\partial u}{\partial\sigma} + (\nabla\Psi \cdot \mathbf{r})_0\, u\frac{\partial u}{\partial\xi} + \frac{1}{\sqrt{J}}\frac{\partial\sqrt{J}}{\partial\sigma}u + \mu^0\frac{\partial^2 u}{\partial\xi^2} = \nu^0 u, \quad (29)$$

where

$$(\nabla\Psi \cdot \mathbf{r})_0 = (|grad\varphi|)_0\,(\nabla\lambda \cdot \mathbf{r})_0\,, \quad (30)$$

$$\mu^0 = \frac{\left[\mathbf{l}\cdot\left(\mathbf{H}^k\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial x^k} + \mathbf{H}^{ik}\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^k}\right)\mathbf{r}\right]_0}{(\mathbf{l}\cdot\mathbf{r})_0}, \quad (31)$$

$$\nu^0 = \frac{(\mathbf{l}\cdot\nabla\mathbf{B}\,\mathbf{r})_0}{(\mathbf{l}\cdot\mathbf{r})_0}, \quad (32)$$

with $\mathbf{l}$ the left eigenvector and $\mathbf{r}$ the right eigenvector corresponding to the eigenvalue λ, that does not satisfy the Lax- Boillat condition.

By using the transformation of variables (see [15])

$$u = \frac{v}{\sqrt{J}}e^w, \qquad \kappa = \int_0^\sigma \frac{e^w}{\sqrt{J}}\,(\nabla\Psi \cdot \mathbf{r})_0\, d\sigma, \quad \text{with} \quad w = \int_0^\sigma \nu^0 d\sigma, \quad (33)$$

equation (29) can be reduced to an equation of the type

$$\frac{\partial v}{\partial \kappa} + v\frac{\partial v}{\partial \xi} + \hat{\mu}^0 \frac{\partial^2 v}{\partial \xi^2} = 0, \quad \text{with} \quad \hat{\mu}^0 = \frac{\mu^0 \sqrt{J} e^{-w}}{(\nabla \Psi \cdot \mathbf{r})_0}, \tag{34}$$

which is similar to Burger's equation and is valid along the characteristic rays. Equation $(34)_1$ can be reduced to the semilinear heat equation [18]

$$\frac{\partial h}{\partial \kappa} = \hat{\mu}^0 \frac{\partial^2 h}{\partial \xi^2} - h log \frac{h}{\hat{\mu}^0} \frac{d\hat{\mu}^0}{d\kappa}, \tag{35}$$

for which the solution is known, using the following Hopf transformation

$$v(\xi, \kappa) = \hat{\mu}_0 \frac{\partial}{\partial \xi} log h(\xi, \kappa). \tag{36}$$

4 Equations governing the motion of Jeffreys media and their matrix form

In [27] a theory for mechanical relaxation phenomena, based on thermodynamics of irreversible processes [11, 29, 34, 37] with internal variables [29], was developed by G. A. Kluitenberg. It was assumed that several microscopic phenomena occur, which give rise to inelastic deformation, such that the tensor of the total strain $\varepsilon_{\alpha\beta}$ can be split in two parts: $\varepsilon_{\alpha\beta} = \varepsilon^{el}_{\alpha\beta} + \varepsilon^{in}_{\alpha\beta}$, where the tensors $\varepsilon^{el}_{\alpha\beta}$ and $\varepsilon^{in}_{\alpha\beta}$ describe the elastic and inelastic strains, respectively. Contrary to the elastic strain, the inelastic deformation is due to the effects of lattice defects (slip, dislocations,...) and to the influence of microscopic stress fields, surrounding imperfections in the medium, that can give rise to memory effects on the mechanical and thermodynamic behavior of rheological media. Experiments show that there exist several types of such independent and simultaneous contributions to the inelastic strain, so that, assuming that they are of n different types, then $\varepsilon^{in}_{\alpha\beta}$ can be split in n contributions $\varepsilon^{(k)}_{\alpha\beta} (k = 1, 2, ..., n)$: $\varepsilon^{in}_{\alpha\beta} = \sum_{k=1}^{n} \varepsilon^{(k)}_{\alpha\beta}$ (with n arbitrary), that are introduced as *internal variables* in the thermodynamical state vector.

In the theory of Kluitenberg Eulerian formalism is used and it is assumed that the gradient of the displacement field is small. This implies that the deformations are supposed to be small from a kinematical (or geometrical) point of view. However the translations and the velocity of the medium may be large [29]. Then, the strain tensor ε_{ik} is assumed to be small, i.e. $\varepsilon_{ik} = \frac{1}{2}\left(\frac{\partial}{\partial x^k}u_i + \frac{\partial}{\partial x^i}u_k\right)(i,k = 1,2,...,n)$, where u_i is the i-th component of the displacement field $\mathbf{u}$ and x^i is the i-th component of the position vector $\mathbf{x}$ in Eulerian coordinates in a Cartesian reference frame. It should be emphasized, however, that the same physical ideas which are developed in this theory can also be reformulated for the case where the deformations are large from a kinematical point of view [29].

In [27], eliminating the internal tensorial variables, for shear phenomena in the isotropic case, the following mechanical relaxation equation between the deviators $\tilde{\tau}_{ik}$ of the mechanical stress tensor (which occurs in the equation of motion and in the first law of thermodynamics) and $\tilde{\epsilon}_{ik}$ of the strain tensor was derived

$$R^{(\tau)}_{(d)0}\tilde{\tau}_{ik} + \sum_{m=1}^{n-1} R^{(\tau)}_{(d)m}\frac{d^m}{dt^m}\tilde{\tau}_{ik} + \frac{d^n}{dt^n}\tilde{\tau}_{ik} = R^{(\epsilon)}_{(d)0}\tilde{\epsilon}_{ik} + \sum_{m=1}^{n+1} R^{(\epsilon)}_{(d)m}\frac{d^m}{dt^m}\tilde{\epsilon}_{ik}$$

$$(i,k = 1,2,3). \tag{37}$$

In the above equations $\frac{d}{dt}$ is the material derivative with respect to time [29] and $R^{(\tau)}_{(d)m}$ $(m = 0,1,...,n-1)$ and $R^{(\epsilon)}_{(d)m}$ $(m = 0,1,...,n+1)$ are algebraic functions of the coefficients occurring in the phenomenological equations and in the equations of state. The rheological relations for ordinary viscous fluids, for thermoelastic media and for Maxwell, Kelvin (Voigt), Jeffreys, Burgers, Poynting-Thomson, Prandtl-Reuss, Bingham, Saint Venant and Hooke media are special cases of this more general mentioned above relation (see also [5, 11, 25, 26, 27, 28, 29, 34, 37, 38]). Assuming that only one microscopic phenomenon gives rise to inelastic strain (n = 1), in the isothermal and isotropic case, for shear phenomena, when the hydrostatic pressure is assumed constant and uniform, the mechanical relaxation equation (37) describing the behaviour of viscoanelastic media without memory (Jeffreys media) can be written in the following form [8]

$$R^{(\tau)}_{(d)0}\tilde{P}_{ik} + \frac{d}{dt}\tilde{P}_{ik} + R^{(\epsilon)}_{(d)1}\frac{d}{dt}\tilde{\epsilon}_{ik} + R^{(\epsilon)}_{(d)2}\frac{d^2}{dt^2}\tilde{\epsilon}_{ik} = 0, \tag{38}$$

where $\tilde{P}_{ik}$ and $\tilde{\epsilon}_{ik}$ are the deviators of the mechanical pressure tensor P_{ik} and of the strain tensor ϵ_{ik}, respectively, and $\frac{d\varepsilon_{ik}}{dt} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x^k} + \frac{\partial v_k}{\partial x^i}\right)$. We define P_{ik} in terms of the symmetric Cauchy stress tensor $P_{ik} = -\tau_{ik}$ $(i,k = 1,2,3)$

and the following quantities

$$\tilde{P}_{ik} = P_{ik} - \frac{1}{3}P_{ss}\delta_{ik}, \quad P = \frac{1}{3}P_{ss}, \quad P_{ss} = trP,$$

$$P_{ik} = \tilde{P}_{ik} + P\delta_{ik}, \quad \tilde{P}_{ss} = 0,$$

where the hydrostatic pressure P is the scalar part of the tensor P_{ik}. Analogous definitions are valid for the deviator $\tilde{\epsilon}_{ik}$ and the scalar part ϵ of the strain tensor. In eq. (38) the coefficients satisfy the relations

$$R^{(\tau)}_{(d)0} = a^{(0,0)}\eta_s^{(1,1)} \geq 0, \tag{39}$$

$$R^{(\epsilon)}_{(d)1} = a^{(0,0)}\left[\left(1 + \eta_s^{(0,1)}\right)^2 + \eta_s^{(0,0)}\eta_s^{(1,1)}\right] \geq 0, \tag{40}$$

$$R^{(\epsilon)}_{(d)2} = \eta_s^{(0,0)} \geq 0, \tag{41}$$

where $a^{(0,0)}$ is a scalar constant which occurs in the equations of state, while the coefficients $\eta_s^{(0,0)}$, $\eta_s^{(0,1)}$ and $\eta_s^{(1,1)}$ are called *phenomenological coefficients* and represent fluidities.

The balance equations for the mass density and momentum in the case of Jeffreys media read

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0, \qquad (i = 1, 2, 3) \tag{42}$$

$$\rho\left(\frac{\partial}{\partial t}v_i + v_k\frac{\partial}{\partial x^k}v_i\right) + \frac{\partial}{\partial x^k}\tilde{P}_{ik} = 0, \tag{43}$$

where $v_i = \frac{du_i}{dt}$ is the i-th component of the velocity field and the force per unit mass is neglected.

5 One-dimensional case

In this Section the one-dimensional case is studied, containing original results. Consider the system of equations (38), (42) and (43). Assume that $v_2 = v_3 = 0$, $x_2 = x_3 = 0$ and that the involved physical quantities depend only on x_1, denoted by x. Denote $v_1(x, t)$ by v and the components of the deviator of the mechanical pressure tensor P_{ik}, $\tilde{P}_{ik}$, by D_{ik}. Then, the system of equations (38), (42) and (43) read

$$\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}+\rho\frac{\partial v}{\partial x}=0, \tag{44}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{1}{\rho}\frac{\partial D_{11}}{\partial x}=0, \tag{45}$$

$$\frac{\partial D_{21}}{\partial x}=0, \tag{46}$$

$$\frac{\partial D_{31}}{\partial x}=0, \tag{47}$$

$$\frac{\partial D_{11}}{\partial t}+v\frac{\partial D_{11}}{\partial x}+\frac{2}{3}R^{(\epsilon)}_{(d)1}\frac{\partial v}{\partial x}+\frac{2}{3}R^{(\epsilon)}_{(d)2}\frac{\partial^2 v}{\partial t\partial x}+\frac{2}{3}R^{(\epsilon)}_{(d)2}\frac{\partial^2 v}{\partial x^2}v+R^{(\tau)}_{(d)0}D_{11}=0, \tag{48}$$

$$\frac{\partial D_{12}}{\partial t}+v\frac{\partial D_{12}}{\partial x}+R^{(\tau)}_{(d)0}D_{12}=0, \tag{49}$$

$$\frac{\partial D_{13}}{\partial t}+v\frac{\partial D_{13}}{\partial x}+R^{(\tau)}_{(d)0}D_{13}=0, \tag{50}$$

$$\frac{\partial D_{22}}{\partial t}+v\frac{\partial D_{22}}{\partial x}-\frac{1}{3}R^{(\epsilon)}_{(d)1}\frac{\partial v}{\partial x}-\frac{1}{3}R^{(\epsilon)}_{(d)2}\frac{\partial^2 v}{\partial t\partial x}-\frac{1}{3}R^{(\epsilon)}_{(d)2}\frac{\partial^2 v}{\partial x^2}v+R^{(\tau)}_{(d)0}D_{22}=0, \tag{51}$$

$$\frac{\partial D_{23}}{\partial t}+v\frac{\partial D_{23}}{\partial x}+R^{(\tau)}_{(d)0}D_{23}=0, \tag{52}$$

where $D_{ik}=D_{ki}$.
Thus, equs. (46) and (47) show that $D_{21}=f(t)$, $D_{31}=f_1(t)$, where f and f_1 are functions of t. Therefore, from eqs. (49) and (50) we have

$$D_{12}=e^{-R^{(\tau)}_{(d)0}t}+D^0_{12},\qquad D_{13}=e^{-R^{(\tau)}_{(d)0}t}+D^0_{13}.$$

Remark that, due to the presence of a tensorial internal variable, there is a response time of the medium possessing mechanical relaxation properties.

Then, the remained system of equations (44), (45), (48), (51) and (52) takes the matrix form (1)

$$\mathbf{U}_t+\mathbf{A}\mathbf{U}_x+\omega^{-1}\left[\mathbf{H}^1\frac{\partial^2\mathbf{U}}{\partial t\partial x}+\mathbf{H}^{11}\frac{\partial^2\mathbf{U}}{\partial x^2}\right]=\mathbf{B}(\mathbf{U}), \tag{53}$$

having the following associated system of nonlinear hyperbolic PDEs

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x = \mathbf{B}(\mathbf{U}), \tag{54}$$

where $\mathbf{A}^0(\mathbf{U}) = \mathbf{I}$ is the identity matrix,

$$\mathbf{U} = (\rho, v, D_{11}, D_{22}, D_{23})^T, \qquad \mathbf{B} = (0, 0, -R^{(\tau)}_{(d)0}D_{11}, -R^{(\tau)}_{(d)0}D_{22}, -R^{(\tau)}_{(d)0}D_{23})^T,$$

$$\mathbf{A} = \begin{pmatrix} v & \rho & 0 & 0 & 0 \\ 0 & v & \frac{1}{\rho} & 0 & 0 \\ 0 & \frac{2}{3}R^{(\epsilon)}_{(d)1} & v & 0 & 0 \\ 0 & -\frac{1}{3}R^{(\epsilon)}_{(d)1} & 0 & v & 0 \\ 0 & 0 & 0 & 0 & v \end{pmatrix}, \tag{55}$$

$$\mathbf{H}^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3}R'^{(\epsilon)}_{(d)2} & 0 & 0 & 0 \\ 0 & -\frac{1}{3}R'^{(\epsilon)}_{(d)2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{H}^{11} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3}R'^{(\epsilon)}_{(d)2}v & 0 & 0 & 0 \\ 0 & -\frac{1}{3}R'^{(\epsilon)}_{(d)2}v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{56}$$

with $R^{(\epsilon)}_{(d)2} = \omega^{-1}R'^{(\epsilon)}_{(d)2}$. The symbol $(...)^T$ means that $\mathbf{U}$ and $\mathbf{B}$ are column vectors of 5 components.

The eigenvalues of the matrix $\mathbf{A}$ are:

- $\lambda_1 = v$ (of multiplicity equal to 3);
- the simple eigenvalues $\lambda_2^{(\pm)} = v \pm \gamma$, with $\gamma = \sqrt{\frac{2R^{(\epsilon)}_{(d)1}}{3\rho}}$.

The right eigenvectors corresponding to $\lambda_2^{(\pm)}$ can be taken as

$$\mathbf{r}_2^{(\pm)} = \left(\rho, -(v - \lambda_2^{(\pm)}),\ \frac{2}{3}R^{(\epsilon)}_{(d)1}, -\frac{R^{(\epsilon)}_{(d)1}}{3},\ 0\right)^T. \tag{57}$$

The left eigenvectors are taken as

$$\mathbf{l}_2^{(\pm)} = \left(0, -\left(v - \lambda_2^{(\pm)}\right),\ \frac{1}{\rho},\ 0,\ 0\right). \tag{58}$$

Only, the eigenvalues $\lambda_2^{(\pm)}$ do not satisfy the Lax - Boillat exceptionality condition because $\nabla\lambda_2^{(\pm)} \cdot \mathbf{r}_\mathbf{2}^{(\pm)} \neq 0$. Thus, our results are valid for $\lambda_2^{(\pm)}$. Now, let us consider only the longitudinal wave traveling in the right direction and the case where the propagation is in a constant state $\mathbf{U}^0$, i. e.

$$\lambda_2^{(+)} = v + \gamma \quad \text{and} \quad \mathbf{U}^0 = (\rho^0,\ 0,\ 0,\ 0,\ 0),$$

with ρ^0 constant. The characteristic rays are

$$x^0 = \sigma = t, \quad x = (x)^0 + \lambda_2^{(+)}(\mathbf{U}^0)\sigma = (x)^0 + \gamma^0 t, \tag{59}$$

whence the wave front is

$$\varphi(t,x) = \varphi^0\left(x(t) - \gamma^0 t\right), \quad \text{where} \quad \gamma^0 = \gamma(\mathbf{U}^0), \quad \gamma^0 = \sqrt{\frac{2R_{(d)1}^{(\epsilon)}}{3\rho^0}}, \tag{60}$$

implying $\varphi_x = 1$.
In order to compute the terms in (29) we start with

$$\nabla\Psi \cdot r_2^{(+)} = \varphi_x(\nabla\lambda_2^{(+)} \cdot r_2^{(+)}), \text{ with } \nabla \equiv \left(\frac{\partial}{\partial\rho}, \frac{\partial}{\partial v}, \frac{\partial}{\partial D_{11}}, \frac{\partial}{\partial D_{22}}, \frac{\partial}{\partial D_{23}}\right). \tag{61}$$

Hence, $\left(\nabla\lambda_2^{(+)} \cdot \mathbf{r}_2^{(+)}\right)_0 = \frac{1}{2}\gamma^0$, being $(\nabla\lambda_2^{(+)})_0 = \left(-\frac{\gamma^0}{2\rho^0}, 1, 0, 0, 0, 0\right)$. (62)

Furthermore, a direct easy computation gives

$$\left(\mathbf{l}_2^{(+)} \cdot \nabla\mathbf{B}\, \mathbf{r}_2^{(+)}\right)_0 = -\frac{2R_{(d)0}^{(\tau)} R_{(d)1}^{(\epsilon)}}{3\rho^0}, \quad \left(\mathbf{l}_2^{(+)} \cdot \mathbf{r}_2^{(+)}\right)_0 = 2(\gamma_0)^2 = \frac{4R_{(d)1}^{(\epsilon)}}{3\rho^0}, \tag{63}$$

and so from (32) we have

$$\nu^0 = -\frac{R_{(d)0}^{(\tau)}}{2}. \tag{64}$$

Finally, we have

$$\mu^0 = \frac{\left[\mathbf{l}_2^{(+)} \cdot \left(\mathbf{H}^1 \frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial x} + \mathbf{H}^{11}\frac{\partial^2\varphi}{\partial x^2}\right)\left(\mathbf{r}_2^{(+)}\right)\right]_0}{\left(\mathbf{l}_2^{(+)} \cdot \mathbf{r}_2^{(+)}\right)_0} = \frac{\left(\frac{\partial\varphi}{\partial t}\right)_0 R_{(d)2}^{\prime(\epsilon)}}{\sqrt{6\rho^0 R_{(d)1}^{(\epsilon)}}}. \tag{65}$$

This example, in spite of its simplicity, shows the influence of a tensorial internal variable on the motion of Jeffreys media.

Note of L. R. *The present paper is one of a series of works started and planned in 2004 during a visit of Adelina Georgescu at the Department of Mathematics of the University of Messina in occasion of study days on "Asymptotic Methods with Applications to Waves and Shocks". These papers contain a systematic formulation of previous studies on nonlinear dissipative waves on rheological media, performed in a classical way by the second author L. R., following the modern point of view of double scale method as in the book [16] on asymptotic treatments of A.G. These works were continued during the scientific collaboration of the two authors, in particular at Messina in 2005 and 2007, during meetings dedicated to series of lectures of A. G. on "Applied Mathematics" and next visits of L. R. at Bucharest in 2007 and 2009. They were finished in 2009 and written in final version in 2010. These studies come from enlightening discussions on some mathematical tools, together with their physical interpretations. The author L. R. is very grateful to A. G. for her precious encouragement to this joint study regarding a revision of previous studies and the derivation of original results on the same subject.*

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