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In Memoriam Adelina Georgescu

UNIVERSAL REGULAR AUTONOMOUS ASYNCHRONOUS SYSTEMS: $\omega-$LIMIT SETS, INVARIANCE AND BASINS OF ATTRACTION*

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Abstract

The asynchronous systems are the non-deterministic real time-binary models of the asynchronous circuits from electrical engineering. Autonomy means that the circuits and their models have no input. Regularity means analogies with the dynamical systems, thus such systems may be considered to be the real time dynamical systems with a 'vector field' $\Phi : \{0,1\}^n \to \{0,1\}^n$. Universality refers to the case when the state space of the system is the greatest possible in the sense of the inclusion. The purpose of this paper is that of defining, by analogy with the dynamical systems theory, the $\omega-$limit sets, the invariance and the basins of attraction of the universal regular autonomous asynchronous systems.

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keywords: asynchronous system, $\omega-$limit set, invariance, basin of attraction

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1 Introduction

We denote by $\mathbf{B} = \{0, 1\}$ the binary Boole algebra, endowed with the discrete topology and with the usual algebraical laws:

	$\overline{}$
0	1
1	0

$\cdot$	0	1
0	0	0
1	0	1

$\cup$	0	1
0	0	1
1	1	1

$\oplus$	0	1
0	0	1
1	1	0

Table 1

The real numbers set $\mathbf{R}$ is the time set and $t \in \mathbf{R}$ are the time instants.

The $\mathbf{R} \to \mathbf{B}$ functions give the deterministic[1] real time-binary models of the digital electrical signals and they are not studied in literature. An asynchronous circuit without input, considered as a collection of n signals, should be deterministically modelled by a function $x : \mathbf{R} \to \mathbf{B}^n$ called state. We have however several parameters related with the asynchronous circuit that are either unknown, or perhaps variable or simply ignored in modeling such as the temperature, the tension of the mains and the delays that occur in the computation of the Boolean functions. For this reason, instead of a function x we have in general a set X of functions x, called state space, or non-deterministic[2] autonomous asynchronous system, where each function x represents a possibility of modeling the circuit. When X is constructed by making use of a 'vector field' $\Phi : \mathbf{B}^n \to \mathbf{B}^n$, the system X is called regular. The universal regular autonomous asynchronous systems are the Boolean dynamical systems and they can be identified with Φ.

We give in Figure 1 at a) the example of the NAND gate defined by $\phi : \mathbf{B}^2 \to \mathbf{B}$, $\forall (\mu_1, \mu_2) \in \mathbf{B}^2, \phi(\mu_1, \mu_2) = \overline{\mu_1 \mu_2}$ and at b) the example of an autonomous circuit made with two such devices and characterized by $\Phi : \mathbf{B}^2 \to \mathbf{B}^2$, $\forall (\mu_1, \mu_2) \in \mathbf{B}^2, (\Phi_1(\mu_1, \mu_2), \Phi_2(\mu_1, \mu_2)) = (\overline{\mu_2}, \overline{\mu_1 \mu_2})$.

The dynamics of these asynchronous systems[3] is described by the so called state portraits, see Figure 1 c) where the arrows show the increase of time. For any $i \in \{1, 2\}$, the coordinate μ_i is underlined if $\Phi_i(\mu_1, \mu_2) \neq \mu_i$ and it is called unstable, or enabled, or excited in this case. The coordinates μ_i

[1] 'Deterministic' means that each signal is modeled by exactly one $\mathbf{R} \to \mathbf{B}$ function.

[2] 'Non-deterministic' means that each signal is modeled by several $x_i : \mathbf{R} \to \mathbf{B}$ functions or, equivalently, that each circuit is modeled by several functions $x \in X$.

[3] The systems are (vaguely) the models of the circuits.

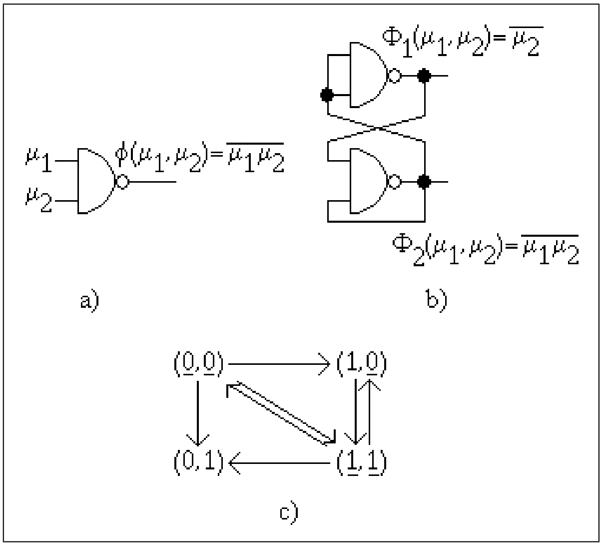

Figure 1: a) The NAND gate, b) Example of system using the NAND gate, c) The state portrait of the system from b)

that are not underlined satisfy by definition $\Phi_i(\mu_1, \mu_2) = \mu_i$ and are called stable, or disabled, or not excited. Three arrows start from the point $(0,0)$ where both coordinates are unstable, showing the fact that $\Phi_1(0,0)$ may be computed first, $\Phi_2(0,0)$ may be computed first or $\Phi_1(0,0), \Phi_2(0,0)$ may be computed simultaneously and similarly for the point $(1,1)$. Note that the two possibilities of defining the system, state portrait and formula, are equivalent. Note also that the system was identified with the function Φ.

The existence of several possibilities of changing the state of the system (three possibilities in $(0,0)$ and $(1,1)$, one possibility in $(1,0)$, no possibility in $(0,1)$) is the key characteristic of asynchronicity, as opposed to synchronicity where the coordinates $\Phi_i(\mu)$ are always computed simultaneously, $i \in \{1,...,n\}$ for all $\mu \in \mathbf{B}^n$ and the system's run is: $\mu, \Phi(\mu), (\Phi \circ \Phi)(\mu), ..., (\Phi \circ ... \circ \Phi)(\mu), ...$

Our present aim is to show how the well-known concepts of $\omega-$limit set, invariance and basin of attraction from the dynamical systems theory, by real to binary translation, may be integrated in the asynchronous systems theory.

2 Preliminaries

Notation 1. *$\chi_A : \mathbf{R} \to \mathbf{B}$ is the notation of the characteristic function of the set $A \subset \mathbf{R}$: $\forall t \in \mathbf{R}$, $\chi_A(t) = \begin{cases} 1, t \in A \\ 0, t \notin A \end{cases}$.*

Notation 2. *We denote by Seq the set of the sequences $t_0 < t_1 < ... < t_k < ...$ of real numbers that are unbounded from above.*

Definition 3. *The sequence $\alpha : \mathbf{N} \to \mathbf{B}^n, \forall k \in \mathbf{N}, \alpha^k = \alpha(k)$ is called **progressive** if the sets $\{k|k \in \mathbf{N}, \alpha_i^k = 1\}$ are infinite for all $i \in \{1, ..., n\}$. We denote the set of the progressive sequences by Π_n.*

Definition 4. *The functions $\rho : \mathbf{R} \to \mathbf{B}^n$ of the form $\forall t \in \mathbf{R}$,*

$$\rho(t) = \alpha^0 \chi_{\{t_0\}}(t) \oplus \alpha^1 \chi_{\{t_1\}}(t) \oplus ... \oplus \alpha^k \chi_{\{t_k\}}(t) \oplus ... \tag{1}$$

*where $\alpha \in \Pi_n$ and $(t_k) \in Seq$ are called **progressive** and their set is denoted by P_n.*

Definition 5. *Let be the function $\Phi : \mathbf{B}^n \to \mathbf{B}^n$. i) For $\nu \in \mathbf{B}^n$ we define $\Phi^\nu : \mathbf{B}^n \to \mathbf{B}^n$ by $\forall \mu \in \mathbf{B}^n$, $\Phi^\nu(\mu) = (\overline{\nu_1}\mu_1 \oplus \nu_1\Phi_1(\mu), ..., \overline{\nu_n}\mu_n \oplus \nu_n\Phi_n(\mu))$.*

ii) The functions $\Phi^{\alpha^0...\alpha^k} : \mathbf{B}^n \to \mathbf{B}^n$ are defined for $k \in \mathbf{N}$ and $\alpha^0, ..., \alpha^k \in \mathbf{B}^n$ iteratively: $\forall \mu \in \mathbf{B}^n$, $\Phi^{\alpha^0...\alpha^k\alpha^{k+1}}(\mu) = \Phi^{\alpha^{k+1}}(\Phi^{\alpha^0...\alpha^k}(\mu))$.

*iii) The function $\Phi^\rho : \mathbf{B}^n \times \mathbf{R} \to \mathbf{B}^n$ that is defined in the following way $\Phi^\rho(\mu, t) = \mu\chi_{(-\infty,t_0)}(t) \oplus \Phi^{\alpha^0}(\mu)\chi_{[t_0,t_1)}(t) \oplus \Phi^{\alpha^0\alpha^1}(\mu)\chi_{[t_1,t_2)}(t) \oplus ... \oplus \Phi^{\alpha^0...\alpha^k}(\mu)\chi_{[t_k,t_{k+1})}(t) \oplus ...$is called **flow**, **motion** or **orbit** (of $\mu \in \mathbf{B}^n$). We have assumed that $\rho \in P_n$ is like at (1).*

*iv) The set $Or_\rho(\mu) = \{\Phi^\rho(\mu, t)|t \in \mathbf{R}\}$ is also called **orbit** (of μ).*

Remark 6. *The function Φ^ν shows how an asynchronous iteration of Φ is made: for any $i \in \{1, ..., n\}$, if $\nu_i = 0$ then Φ_i is not computed, since $\Phi_i^\nu(\mu) = \mu_i$ and if $\nu_i = 1$ then Φ_i is computed, since $\Phi_i^\nu(\mu) = \Phi_i(\mu)$.*

The definition of $\Phi^{\alpha^0...\alpha^k}$ generalizes this idea to an arbitrary number $k+1$ of asynchronous iterations, with the supplementary request that each coordinate Φ_i is computed infinitely many times in the sequence $\mu, \Phi^{\alpha^0}(\mu), \Phi^{\alpha^0\alpha^1}(\mu), ..., \Phi^{\alpha^0...\alpha^k}(\mu), ...$ whenever $\alpha \in \Pi_n$.

The sequences $(t_k) \in Seq$ make the pass from the discrete time $\mathbf{N}$ to the continuous time $\mathbf{R}$ and each $\rho \in P_n$ shows, in addition to $\alpha \in \Pi_n$, the time instants t_k when Φ is computed (asynchronously). Thus $\Phi^\rho(\mu, t), t \in \mathbf{R}$ is *the continuous time computation of the sequence* μ, $\Phi^{\alpha^0}(\mu)$, $\Phi^{\alpha^0\alpha^1}(\mu)$, ..., $\Phi^{\alpha^0...\alpha^k}(\mu)$, ... *made in the following way: if* $t < t_0$ *nothing is computed, if* $t \in [t_0, t_1)$, $\Phi^{\alpha^0}(\mu)$ *is computed, if* $t \in [t_1, t_2)$, $\Phi^{\alpha^0\alpha^1}(\mu)$ *is computed, ..., if* $t \in [t_k, t_{k+1})$, $\Phi^{\alpha^0...\alpha^k}(\mu)$ *is computed, ...*

When α *runs in* Π_n *and* (t_k) *runs in Seq we get the 'unbounded delay model' of computation of the Boolean function* Φ*, represented in discrete time by the sequences* $\mu, \Phi^{\alpha^0}(\mu), \Phi^{\alpha^0\alpha^1}(\mu), ..., \Phi^{\alpha^0...\alpha^k}(\mu), ...$ *and in continuous time by the orbits* $\Phi^{\rho}(\mu, t)$ *respectively. We shall not insist on the non-formalized way that the engineers describe this model; we just mention that the 'unbounded delay model' is a reasonable way of starting the analysis of a circuit in which the delays occurring in the computation of the Boolean functions* Φ *are arbitrary positive numbers. If we restrict suitably the ranges of* α *and* (t_k) *we get the 'bounded delay model' of computation of* Φ *and if both* α, (t_k) *are fixed, then we obtain the 'fixed delay model' of computation of* Φ*, determinism.*

Theorem 7. *Let* $\alpha \in \Pi_n, (t_k) \in Seq$ *be arbitrary and the function* $\rho(t) = \alpha^0\chi_{\{t_0\}}(t) \oplus \alpha^1\chi_{\{t_1\}}(t) \oplus ... \oplus \alpha^k\chi_{\{t_k\}}(t) \oplus ..., \rho \in P_n$*. The following statements are true:*

a) $\{\alpha^k | k \geq k_1\} \in \Pi_n$ *for any* $k_1 \in \mathbf{N}$;

b) $(t_k) \cap (t', \infty) \in Seq$ *for any* $t' \in \mathbf{R}$;

c) $\rho\chi_{(t',\infty)} \in P_n$ *for any* $t' \in \mathbf{R}$;

d) $\forall \mu \in \mathbf{B}^n, \forall \mu' \in \mathbf{B}^n, \forall t' \in \mathbf{R}, \Phi^{\rho}(\mu, t') = \mu' \Longrightarrow \forall t \geq t', \Phi^{\rho}(\mu, t) = \Phi^{\rho\chi_{(t',\infty)}}(\mu', t)$.

Proof. a) If $\{k | k \in \mathbf{N}, \alpha_i^k = 1\}$ is infinite, then $\{k | k \geq k_1, \alpha_i^k = 1\}$ is also infinite, $\forall i \in \{1, ..., n\}$.

b) If $t_0 < t_1 < t_2 < ...$ is unbounded from above, then any sequence of the form $t_{k_1} < t_{k_1+1} < t_{k_1+2} < ...$ is unbounded from above, $k_1 \in \mathbf{N}$.

c) This is a consequence of a) and b).

d) We presume that $t' < t_0$. In this situation $\mu = \mu', \rho = \rho\chi_{(t',\infty)}$ and the statement is obvious, so that we may assume now that $t' \geq t_0$. In this case, some $k_1 \in \mathbf{N}$ exists with $t' \in [t_{k_1}, t_{k_1+1})$ and $\mu' = \Phi^{\alpha^0...\alpha^{k_1}}(\mu)$. Because

$$\rho\chi_{(t',\infty)}(t) = \alpha^{k_1+1}\chi_{\{t_{k_1+1}\}}(t) \oplus \alpha^{k_1+2}\chi_{\{t_{k_1+2}\}}(t) \oplus ...,$$

$$\Phi^{\rho\chi_{(t',\infty)}}(\mu', t) = \mu'\chi_{(-\infty, t_{k_1+1})}(t) \oplus \Phi^{\alpha^{k_1+1}}(\mu')\chi_{[t_{k_1+1}, t_{k_1+2})}(t)$$
$$\oplus \Phi^{\alpha^{k_1+1}\alpha^{k_1+2}}(\mu')\chi_{[t_{k_1+2}, t_{k_1+3})}(t) \oplus ...$$

we get
$\forall t \in [t', t_{k_1+1})$,

$$\Phi^{\rho}(\mu, t) = \Phi^{\alpha^0 \ldots \alpha^{k_1}}(\mu),$$

$$\Phi^{\rho\chi_{(t',\infty)}}(\mu', t) = \mu' = \Phi^{\alpha^0 \ldots \alpha^{k_1}}(\mu);$$

$\forall t \in [t_{k_1+1}, t_{k_1+2})$,

$$\Phi^{\rho}(\mu, t) = \Phi^{\alpha^0 \ldots \alpha^{k_1} \alpha^{k_1+1}}(\mu),$$

$$\Phi^{\rho\chi_{(t',\infty)}}(\mu', t) = \Phi^{\alpha^{k_1+1}}(\mu') = \Phi^{\alpha^{k_1+1}}(\Phi^{\alpha^0 \ldots \alpha^{k_1}}(\mu)) = \Phi^{\alpha^0 \ldots \alpha^{k_1} \alpha^{k_1+1}}(\mu);$$

$$\ldots$$

The statement of the Theorem holds. □

Theorem 8. *Let be $\mu \in \mathbf{B}^n, \rho \in P_n$ and $\tau \in \mathbf{R}$. The function $\rho'(t) = \rho(t-\tau)$ is progressive and we have $\Phi^{\rho'}(\mu, t) = \Phi^{\rho}(\mu, t-\tau)$.*

Proof. We put ρ under the form

$$\rho(t) = \alpha^0 \chi_{\{t_0\}}(t) \oplus \ldots \oplus \alpha^k \chi_{\{t_k\}}(t) \oplus \ldots,$$

$\alpha \in \Pi_n, (t_k) \in Seq$ and we note that

$$\rho'(t) = \rho(t-\tau) = \alpha^0 \chi_{\{t_0+\tau\}}(t) \oplus \ldots \oplus \alpha^k \chi_{\{t_k+\tau\}}(t) \oplus \ldots$$

where $(t_k + \tau) \in Seq$. We infer

$$\Phi^{\rho'}(\mu, t) = \mu \chi_{(-\infty, t_0+\tau)}(t) \oplus \Phi^{\alpha^0}(\mu) \chi_{[t_0+\tau, t_1+\tau)}(t) \oplus \ldots$$

$$\ldots \oplus \Phi^{\alpha^0 \ldots \alpha^k}(\mu) \chi_{[t_k+\tau, t_{k+1}+\tau)}(t) \oplus \ldots = \Phi^{\rho}(\mu, t-\tau).$$

□

Definition 9. *The **universal regular autonomous asynchronous system** that is generated by $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ is by definition $\Xi_\Phi = \{\Phi^{\rho}(\mu, \cdot) | \mu \in \mathbf{B}^n, \rho \in P_n\}$; any $x(t) = \Phi^{\rho}(\mu, t)$ is called **state** (of Ξ_Φ), μ is called **initial value** (of x), or **initial state** (of Ξ_Φ) and Φ is called **generator function** (of Ξ_Φ).*

Remark 10. *The asynchronous systems are non-deterministic in general, due to the uncertainties that occur in the modeling of the asynchronous circuits. Non-determinism is produced, in the case of Ξ_Φ, by the fact that the initial state μ and the way ρ of iterating Φ are not known.*

Definition 11. *Let $v : \mathbf{N} \to \mathbf{B}^n, x : \mathbf{R} \to \mathbf{B}^n$ be some functions. If $\exists k' \in \mathbf{N}, \forall k \geq k', v(k) = v(k')$, we say that **the limit** $\lim\limits_{k\to\infty} v(k)$ **exists** and we use the notation $\lim\limits_{k\to\infty} v(k) = v(k')$. Similarly, if $\exists t' \in \mathbf{R}, \forall t \geq t', x(t) = x(t')$,we say that **the limit** $\lim\limits_{t\to\infty} x(t)$ **exists** and we denote $\lim\limits_{t\to\infty} x(t) = x(t')$. Sometimes $\lim\limits_{k\to\infty} v(k), \lim\limits_{t\to\infty} x(t)$ are called the **final values** of v, x.*

Theorem 12. *[7] $\forall \mu \in \mathbf{B}^n, \forall \mu' \in \mathbf{B}^n, \forall \rho \in P_n, \lim\limits_{t\to\infty} \Phi^\rho(\mu, t) = \mu' \Longrightarrow \Phi(\mu') = \mu'$, if the final value of $\Phi^\rho(\mu, \cdot)$ exists, it is a fixed point of Φ.*

Proof. Let $\mu \in \mathbf{B}^n, \mu' \in \mathbf{B}^n, \rho \in P_n$ be arbitrary and fixed. The hypothesis states the existence of $t' \in \mathbf{R}$ with

$$\forall t \geq t', \Phi^\rho(\mu, t) = \mu'$$

thus, from Theorem 7 d),

$$\forall t \geq t', \Phi^{\rho\chi_{(t',\infty)}}(\mu', t) = \mu'.$$

We infer that $\forall i \in \{1, ..., n\}, \exists t'' > t'$ such that

$$\rho_i(t'') = \rho_i\chi_{(t',\infty)}(t'') = 1,$$

$$\Phi_i^{\rho\chi_{(t',\infty)}}(\mu', t'') = \Phi_i(\mu') = \mu'_i.$$

□

Theorem 13. *[7] $\forall \mu \in \mathbf{B}^n, \forall \mu' \in \mathbf{B}^n, \forall \rho \in P_n, (\Phi(\mu') = \mu'$ and $\exists t' \in \mathbf{R}, \Phi^\rho(\mu, t') = \mu') \Longrightarrow \forall t \geq t', \Phi^\rho(\mu, t) = \mu'$,meaning that if the fixed point μ' of Φ is accessible, then it is the final value of $\Phi^\rho(\mu, \cdot)$.*

Proof. Let $\mu \in \mathbf{B}^n, \mu' \in \mathbf{B}^n, \rho \in P_n$ be arbitrary and fixed. From the hypothesis and Theorem 7 d) we infer

$$\forall t \geq t', \Phi^\rho(\mu, t) = \Phi^{\rho\chi_{(t',\infty)}}(\mu', t)$$

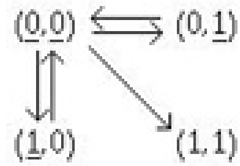

Figure 2: $\exists\rho \in P_2, \omega_\rho((1,0)) = \{(0,0),(0,1)\}$ and $\exists\rho' \in P_2, \omega_{\rho'}((1,0)) = \{(1,1)\}$

thus $\forall i \in \{1,...,n\}, \exists\varepsilon > 0, \forall t \in [t', t'+\varepsilon), \Phi_i^{\rho\chi_{(t',\infty)}}(\mu', t)$ can take one of the values μ'_i and $\Phi_i(\mu')$. But $\mu'_i = \Phi_i(\mu')$, wherefrom the previous property takes place for arbitrary ε and

$$\forall t \geq t', \Phi^\rho(\mu, t) = \mu'.$$

□

Corollary 14. $\forall\mu \in \mathbf{B}^n, \forall\rho \in P_n, \Phi(\mu) = \mu \Longrightarrow \forall t \in \mathbf{R}, \Phi^\rho(\mu, t) = \mu.$

Proof. From Theorem 13, with $\mu = \mu'$, where t' may be chosen such that $\forall t < t', \rho(t) = 0$. □

3 $\omega-$limit sets

Definition 15. *For $\mu \in \mathbf{B}^n$ and $\rho \in P_n$, the set $\omega_\rho(\mu) = \{\mu' | \mu' \in \mathbf{B}^n, \exists(t_k) \in Seq, \lim\limits_{k\to\infty} \Phi^\rho(\mu, t_k) = \mu'\}$ is called the $\omega-$**limit set** of the orbit $\Phi^\rho(\mu, \cdot)$.*

Remark 16. *The previous definition agrees with the usual definitions of the $\omega-$limit sets of the real time or discrete time dynamical systems see [2] page 5, [5] page 26, [1] page 20.*

Example 17. *In Figure 2, we consider*

$$\rho(t) = (1,1)\chi_{\{0\}}(t) \oplus (0,1)\chi_{\{1\}}(t) \oplus (1,1)\chi_{\{2\}}(t) \oplus (0,1)\chi_{\{3\}}(t) \oplus ...,$$

$$\rho'(t) = (1,1)\chi_{\{0\}}(t) \oplus (1,1)\chi_{\{1\}}(t) \oplus (1,1)\chi_{\{2\}}(t) \oplus ...$$

and we have

$$\Phi^\rho((1,0), t) = (1,0)\chi_{(-\infty,0)}(t) \oplus (0,0)\chi_{[0,1)}(t) \oplus (0,1)\chi_{[1,2)}(t)$$

$$\oplus(0,0)\chi_{[2,3)}(t)\oplus(0,1)\chi_{[3,4)}(t)\oplus...,$$

$$\Phi^{\rho'}((1,0),t)=(1,0)\chi_{(-\infty,0)}(t)\oplus(0,0)\chi_{[0,1)}(t)\oplus(1,1)\chi_{[1,\infty)}(t),$$

thus $\omega_\rho((1,0))=\{(0,0),(0,1)\},\omega_{\rho'}((1,0))=\{(1,1)\}$.

Theorem 18. *For any* $\mu\in\mathbf{B}^n$ *and any* $\rho\in P_n$*, we have:*

a) $\omega_\rho(\mu)\neq\emptyset$;

b) $\forall t'\in\mathbf{R}$, $\omega_\rho(\mu)\subset\{\Phi^\rho(\mu,t)|t\geq t'\}\subset Or_\rho(\mu)$;

c) $\exists t'\in\mathbf{R},\omega_\rho(\mu)=\{\Phi^\rho(\mu,t)|t\geq t'\}$ *and any* $t''\geq t'$ *fulfills* $\omega_\rho(\mu)=\{\Phi^\rho(\mu,t)|t\geq t''\}$;

d) $\forall t'\in\mathbf{R},\forall t''\geq t',\{\Phi^\rho(\mu,t)|t\geq t'\}=\{\Phi^\rho(\mu,t)|t\geq t''\}$ *implies* $\omega_\rho(\mu)=\{\Phi^\rho(\mu,t)|t\geq t'\}$;

e) we presume that $\omega_\rho(\mu)=\{\Phi^\rho(\mu,t)|t\geq t'\},t'\in\mathbf{R}$. *Then* $\forall\mu'\in\omega_\rho(\mu),\forall t''\geq t'$, *if* $\Phi^\rho(\mu,t'')=\mu'$ *we get* $\omega_\rho(\mu)=\{\Phi^{\rho\chi_{(t'',\infty)}}(\mu',t)|t\geq t''\}=Or_{\rho\chi_{(t'',\infty)}}(\mu')=\omega_{\rho\chi_{(t'',\infty)}}(\mu')$.

Proof. We put $\rho\in P_n$ under the form

$$\rho(t)=\alpha^0\chi_{\{t_0\}}(t)\oplus...\oplus\alpha^k\chi_{\{t_k\}}(t)\oplus...$$

where $\alpha\in\Pi_n$ and $(t_k)\in Seq$. We ask, without loosing the generality, that $\alpha^0=(0,...,0)\in\mathbf{B}^n$, hence $\Phi^\rho(\mu,t_0)=\mu$ and $Or_\rho(\mu)=\{\Phi^\rho(\mu,t_k)|k\in\mathbf{N}\}$.

a) If $Or_\rho(\mu)=\{\mu^1,...,\mu^p\},p\in\{1,...,2^n\}$, we denote with $I_1,...,I_p\subset\mathbf{N}$ the sets

$$I_j=\{k|k\in\mathbf{N},\Phi^\rho(\mu,t_k)=\mu^j\},j=\overline{1,p}.$$

Because $I_1\cup...\cup I_p=\mathbf{N}$, some of these sets are infinite, let them be without loosing the generality $I_1,...,I_{p'},p'\leq p$. We infer $\omega_\rho(\mu)=\{\mu^1,...,\mu^{p'}\}$.

b) For $t'\in\mathbf{R}$, we define

$$k_1=\begin{cases}0,t'<t_0\\k,t'\in[t_k,t_{k+1})\end{cases}$$

and we obtain

$$\omega_\rho(\mu)=\{\mu^1,...,\mu^{p'}\}=\{\Phi^\rho(\mu,t_k)|k\in I_1\cup...\cup I_{p'}\}$$

$$=\{\Phi^\rho(\mu,t_k)|k\in(I_1\cup...\cup I_{p'})\cap[k_1,\infty)\}$$

$$\subset\{\Phi^\rho(\mu,t_k)|k\in(I_1\cup...\cup I_p)\cap[k_1,\infty)\}=\{\Phi^\rho(\mu,t)|t\geq t'\}$$

$$\subset \{\Phi^\rho(\mu,t_k)|k\in I_1\cup...\cup I_p\}=\{\mu^1,...,\mu^p\}=Or_\rho(\mu).$$

c) If $p'=p$, then $\forall t'\in\mathbf{R}$, $\omega_\rho(\mu)=\{\Phi^\rho(\mu,t)|t\geq t'\}=Or_\rho(\mu)$ from b) and the property holds, thus we can assume that $p'<p$. In this case we define

$$\begin{aligned}k''&=\min\{k|k\in\mathbf{N},\forall k'\geq k,k'\in I_1\cup...\cup I_{p'}\}\\&=1+\max(I_{p'+1}\cup...\cup I_p)\end{aligned}$$

for which we have

$$(I_{p'+1}\cup...\cup I_p)\cap[k'',\infty)=\emptyset$$

and $t'=t_{k''}$ fulfills

$$\omega_\rho(\mu)=\{\mu^1,...,\mu^{p'}\}=\{\Phi^\rho(\mu,t_k)|k\in I_1\cup...\cup I_{p'}\}$$

$$=\{\Phi^\rho(\mu,t_k)|k\in(I_1\cup...\cup I_{p'})\cap[k'',\infty)\}$$

$$=\{\Phi^\rho(\mu,t_k)|k\in(I_1\cup...\cup I_p)\cap[k'',\infty)\}=\{\Phi^\rho(\mu,t)|t\geq t'\};$$

any $t''\geq t'$ gives

$$\omega_\rho(\mu)\overset{b)}{\subset}\{\Phi^\rho(\mu,t)|t\geq t''\}\subset\{\Phi^\rho(\mu,t)|t\geq t'\}=\omega_\rho(\mu).$$

d) Let be $t'\in\mathbf{R}$ such that $\forall t''\geq t'$,

$$\{\Phi^\rho(\mu,t)|t\geq t'\}=\{\Phi^\rho(\mu,t)|t\geq t''\} \tag{2}$$

and we claim that in this case we have

$$\forall\mu'\in\{\Phi^\rho(\mu,t)|t\geq t'\},\exists(t'_k)\in Seq,\forall k\in\mathbf{N},\Phi^\rho(\mu,t'_k)=\mu'. \tag{3}$$

We assume against all reason that (3) is false, meaning that

$$\exists\mu'\in\{\Phi^\rho(\mu,t)|t\geq t'\},\text{ the set }\{t_k|k\in\mathbf{N},\Phi^\rho(\mu,t_k)=\mu'\}\text{ is finite.}$$

Then $\exists t''>\max\{\max\{t_k|k\in\mathbf{N},\Phi^\rho(\mu,t_k)=\mu'\},t'\}$ that fulfills $\mu'\in\{\Phi^\rho(\mu,t)|t\geq t'\}\setminus\{\Phi^\rho(\mu,t)|t\geq t''\}$, contradiction with (2). The truth of (3) shows that $\mu'\in\omega_\rho(\mu)$, i.e. $\{\Phi^\rho(\mu,t)|t\geq t'\}\subset\omega_\rho(\mu)$. For all $t''\geq t'$ we have then

$$\omega_\rho(\mu)\overset{b)}{\subset}\{\Phi^\rho(\mu,t)|t\geq t''\}=\{\Phi^\rho(\mu,t)|t\geq t'\}\subset\omega_\rho(\mu).$$

e) We note that for $t'' \geq t'$ and $\Phi^\rho(\mu, t'') = \mu'$ we can write

$$\omega_\rho(\mu) = \{\Phi^\rho(\mu, t)|t \geq t'\} \overset{c)}{=} \{\Phi^\rho(\mu, t)|t \geq t''\}$$

$$\overset{Theorem\ 7\ d)}{=} \{\Phi^{\rho\chi_{(t'',\infty)}}(\mu', t)|t \geq t''\} = \{\Phi^{\rho\chi_{(t'',\infty)}}(\mu', t)|t \in \mathbf{R}\}$$

$$= Or_{\rho\chi_{(t'',\infty)}}(\mu').$$

The fact that $\forall t''' \geq t''$,

$$\{\Phi^{\rho\chi_{(t'',\infty)}}(\mu', t)|t \geq t''\} \overset{Theorem\ 7\ d)}{=} \{\Phi^\rho(\mu, t)|t \geq t''\} \overset{c)}{=} \{\Phi^\rho(\mu, t)|t \geq t'\}$$

$$\overset{c)}{=} \{\Phi^\rho(\mu, t)|t \geq t'''\} \overset{Theorem\ 7\ d)}{=} \{\Phi^{\rho\chi_{(t'',\infty)}}(\mu', t)|t \geq t'''\}$$

shows, by taking into account d), that

$$\{\Phi^{\rho\chi_{(t'',\infty)}}(\mu', t)|t \geq t''\} = \omega_{\rho\chi_{(t'',\infty)}}(\mu').$$

□

Remark 19. *If in Theorem 18 e) we take $t'' \in \mathbf{R}$ arbitrarily, the equation*

$$\omega_\rho(\mu) = \omega_{\rho\chi_{(t'',\infty)}}(\Phi^\rho(\mu, t'')) \tag{4}$$

is still true. Indeed, for sufficiently great t''', the terms in (4) are equal with

$$\{\Phi^\rho(\mu, t)|t \geq t'''\} = \{\Phi^{\rho\chi_{(t'',\infty)}}(\Phi^\rho(\mu, t''), t)|t \geq t'''\}.$$

Theorem 20. *For arbitrary $\mu \in \mathbf{B}^n$,$\rho \in P_n$ the following statements are true:*

a) $\lim\limits_{t\to\infty} \Phi^\rho(\mu, t)$ exists $\Longleftrightarrow card(\omega_\rho(\mu)) = 1$;

b) if $\exists \mu' \in \mathbf{B}^n, \omega_\rho(\mu) = \{\mu'\}$, then $\lim\limits_{t\to\infty} \Phi^\rho(\mu, t) = \mu'$ and $\Phi(\mu') = \mu'$;

c) if $\exists \mu' \in \mathbf{B}^n, \Phi(\mu') = \mu'$ and $\mu' \in Or_\rho(\mu)$, then $\omega_\rho(\mu) = \{\mu'\}$.

Proof. a) Let $\mu \in \mathbf{B}^n$,$\rho \in P_n$ be arbitrary. We get

$$\lim_{t\to\infty} \Phi^\rho(\mu, t) \text{ exists} \Longleftrightarrow \exists \mu' \in \mathbf{B}^n, \exists t' \in \mathbf{R}, \forall t \geq t', \Phi^\rho(\mu, t) = \mu'$$

$$\Longleftrightarrow \exists \mu' \in \mathbf{B}^n, \exists t' \in \mathbf{R}, \{\Phi^\rho(\mu, t)|t \geq t'\} = \{\mu'\}$$

$$\Longleftrightarrow \exists \mu' \in \mathbf{B}^n, \omega_\rho(\mu) = \{\mu'\} \Longleftrightarrow card(\omega_\rho(\mu)) = 1.$$

b) We assume that $\exists \mu' \in \mathbf{B}^n, \omega_\rho(\mu) = \{\mu'\}$, i.e. $\exists \mu' \in \mathbf{B}^n, \exists t' \in \mathbf{R}, \{\Phi^\rho(\mu, t)|t \geq t'\} = \{\mu'\}$ in other words $\lim\limits_{t\to\infty} \Phi^\rho(\mu, t) = \mu'$. The fact that $\Phi(\mu') = \mu'$ results from Theorem 12.

c) This is a consequence of Theorem 13. □

Theorem 21. *Let be $\mu \in \mathbf{B}^n, \rho \in P_n, \tau \in \mathbf{R}$. The function $\rho' \in P_n, \rho'(t) = \rho(t-\tau)$ fulfills $\omega_\rho(\mu) = \omega_{\rho'}(\mu)$.*

Proof. We use Theorem 8 and we infer the existence of $t' \in \mathbf{R}$ such that

$$\omega_\rho(\mu) = \{\Phi^\rho(\mu,t)|t \geq t'\} = \{\Phi^\rho(\mu,t-\tau)|t-\tau \geq t'\}$$

$$= \{\Phi^{\rho'}(\mu,t)|t \geq t'+\tau\} = \omega_{\rho'}(\mu).$$

□

4 P-invariant and n-invariant sets

Theorem 22. *We consider the function $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ and let be the set $A \in P^*(\mathbf{B}^n)$. For any $\mu \in A$, the following properties are equivalent*

$$\exists\rho \in P_n, Or_\rho(\mu) \subset A, \tag{5}$$

$$\exists\rho \in P_n, \forall t \in \mathbf{R}, \Phi^\rho(\mu,t) \in A, \tag{6}$$

$$\exists\alpha \in \Pi_n, \forall k \in \mathbf{N}, \Phi^{\alpha^0...\alpha^k}(\mu) \in A \tag{7}$$

and the following properties are also equivalent

$$\forall\rho \in P_n, Or_\rho(\mu) \subset A, \tag{8}$$

$$\forall\rho \in P_n, \forall t \in \mathbf{R}, \Phi^\rho(\mu,t) \in A, \tag{9}$$

$$\forall\alpha \in \Pi_n, \forall k \in \mathbf{N}, \Phi^{\alpha^0...\alpha^k}(\mu) \in A, \tag{10}$$

$$\forall\lambda \in \mathbf{B}^n, \Phi^\lambda(\mu) \in A. \tag{11}$$

Proof. (9)$\Longrightarrow$(11) Let $\mu \in A, \lambda \in \mathbf{B}^n$ and the function $\rho \in P_n$ be arbitrary,

$$\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ... \tag{12}$$

with $\alpha \in \Pi_n$ and $(t_k) \in Seq$. We define

$$\rho'(t) = \lambda \cdot \chi_{\{t'\}}(t) \oplus \alpha^0 \cdot \chi_{\{t'+t_0\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t'+t_k\}}(t) \oplus ...$$

where $t' \in \mathbf{R}$ is arbitrary and we can see that $\rho' \in P_n$. (9) implies $\Phi^\lambda(\mu) = \Phi^{\rho'}(\mu,t') \in A$.

(11)$\Longrightarrow$(9) Let $\mu \in A$ and $\rho \in P_n$ be arbitrary, given by (12), with $\alpha \in \Pi_n, (t_k) \in Seq$. We get by induction on k :

$t < t_0 :$ $\Phi^\rho(\mu, t) = \mu \in A,$

$t \in [t_0, t_1) :$ $\Phi^\rho(\mu, t) = \Phi^{\alpha^0}(\mu) \in A$ from (11),

...

$t \in [t_{k-1}, t_k) :$ $\Phi^{\alpha^0 \ldots \alpha^{k-1}}(\mu) \in A$ due to the hypothesis of the induction,

$t \in [t_k, t_{k+1}) :$ $\Phi^\rho(\mu, t) = \Phi^{\alpha^k}(\Phi^{\alpha^0 \ldots \alpha^{k-1}}(\mu)) \in A$ from (11),

...

The rest of the implications are obvious. □

Definition 23. *The set $A \in P^*(\mathbf{B}^n)$ is called a **p-invariant** (or **p-stable**) **set** of the system Ξ_Φ if it fulfills for any $\mu \in A$ one of (5),..., (7) and it is called an **n-invariant** (or n-**stable**) **set** of Ξ_Φ if it fulfills $\forall \mu \in A$ one of (8),..., (11).*

Remark 24. *In the previous terminology, the letter 'p' comes from 'possibly' and the letter 'n' comes from 'necessarily'. Both 'p' and 'n' refer to the quantification of ρ. Such kind of p-definitions and n-definitions recalling logic are caused by the fact that we translate 'real' concepts into 'binary' concepts and the former have no ρ parameters, thus after translation ρ may appear quantified in two ways. The obvious implication is n-invariance $\Longrightarrow$ p-invariance.*

Example 25. *Let $\Phi : \mathbf{B}^2 \to \mathbf{B}^2$ be defined by $\forall \mu \in \mathbf{B}^2, \Phi(\mu_1, \mu_2) = (\overline{\mu_1}, \overline{\mu_2})$ and $\rho(t) = (1,1) \cdot \chi_{\{0,1,2,\ldots\}}(t)$. The set $A = \{(0,1),(1,0)\}$ fulfills $\forall \mu \in A, \forall t \in \mathbf{R}, \Phi^\rho(\mu, t) \in A$ i.e. it satisfies (6):*

$$\Phi^\rho((0,1), t) = (0,1) \cdot \chi_{(-\infty,0)}(t) \oplus (1,0) \cdot \chi_{[0,1)}(t) \oplus$$

$$\oplus (0,1) \cdot \chi_{[1,2)}(t) \oplus (1,0) \cdot \chi_{[2,3)}(t) \oplus \ldots$$

$$\Phi^\rho((1,0), t) = (1,0) \cdot \chi_{(-\infty,0)}(t) \oplus (0,1) \cdot \chi_{[0,1)}(t) \oplus$$

$$\oplus (1,0) \cdot \chi_{[1,2)}(t) \oplus (0,1) \cdot \chi_{[2,3)}(t) \oplus \ldots$$

see Figure 3; $A = \{(0,0),(1,1)\}$ satisfies the same invariance property.

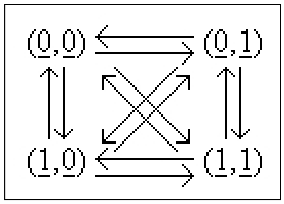

Figure 3: The sets $\{(0,1),(1,0)\}$ and $\{(0,0),(1,1)\}$ are p-invariant

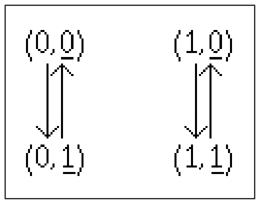

Figure 4: The sets $\{(0,0),(0,1)\}$ and $\{(1,0),(1,1)\}$ are n-invariant

Example 26. *We define the function* $\Phi:\mathbf{B}^2\rightarrow\mathbf{B}^2$ *by* $\forall\mu\in\mathbf{B}^2$, $\Phi(\mu_1,\mu_2) = (\mu_1,\overline{\mu_2})$, *see Figure 4. We notice that the sets* $A=\{(0,0),(0,1)\}$ *and* $A=\{(1,0),(1,1)\}$ *are n-invariant, as they fulfill* $\forall\mu\in A,\forall\rho\in P_2, Or_\rho(\mu)=A$.

Theorem 27. *Let be* $\mu\in\mathbf{B}^n$ *and* $\rho'\in P_n$.

a) If $\Phi(\mu)=\mu$, *then* $\{\mu\}$ *is an n-invariant set and the set* Eq *of the fixed points of* Φ *is also n-invariant;*

b) the set $Or_{\rho'}(\mu)$ *is p-invariant and* $\bigcup\limits_{\rho\in P_n} Or_\rho(\mu)$[4] *is n-invariant;*

c) the set $\omega_{\rho'}(\mu)$ *is p-invariant.*

Proof. a) From Corollary 14 we have that

$$\forall\rho\in P_n,\forall t\in\mathbf{R},\Phi^\rho(\mu,t)=\mu\in\{\mu\}.$$

Furthermore, we infer $\forall\mu'\in Eq$, $\forall\rho\in P_n$, $\forall t\in\mathbf{R}$,

$$\Phi^\rho(\mu',t)=\mu'\in Eq.$$

[4] $\bigcup\limits_{\rho\in P_n} Or_\rho(\mu)=\{\mu'|\exists\rho\in P_n,\mu'\in Or_\rho(\mu)\}$.

b) Let be $\mu' \in Or_{\rho'}(\mu)$, thus $t' \in \mathbf{R}$ exists such that $\mu' = \Phi^{\rho'}(\mu, t')$. Then $\forall t \in \mathbf{R}$,

$$\Phi^{\rho' \cdot \chi_{(t',\infty)}}(\mu', t) = \begin{cases} \Phi^{\rho'}(\mu, t), t > t' \\ \mu', t \leq t' \end{cases} \in Or_{\rho'}(\mu).$$

We have proved that $Or_{\rho'}(\mu)$ is p-invariant.

We remark the equality

$$\bigcup_{\rho \in P_n} Or_\rho(\mu) = \bigcup_{\alpha \in \Pi_n} \{\Phi^{\alpha^0 \ldots \alpha^k}(\mu) | k \in \mathbf{N}\}$$

and let us take an arbitrary $\mu' \in \bigcup_{\rho \in P_n} Or_\rho(\mu)$. If $\mu' = \mu$ then the statement of the theorem is proved, thus we can assume that $\mu' \neq \mu, \mu' = \Phi^{\alpha^0 \ldots \alpha^k}(\mu)$, $\alpha^0, ..., \alpha^k \in \mathbf{B}^n$. For any $\rho'' \in P_n$,

$$\rho'' = \beta^0 \cdot \chi_{\{t'_0\}} \oplus ... \oplus \beta^k \cdot \chi_{\{t'_k\}} \oplus ...$$

$\beta \in \Pi_n, (t'_k) \in Seq$ and any $t \in \mathbf{R}$, we have that $\Phi^{\rho''}(\mu', t)$ is an element of the sequence $\Phi^{\alpha^0 \ldots \alpha^k}(\mu)$, $\Phi^{\alpha^0 \ldots \alpha^k \beta^0}(\mu)$, ..., $\Phi^{\alpha^0 \ldots \alpha^k \beta^0 \ldots \beta^{k'}}(\mu)$, ... where $\alpha^0, ..., \alpha^k, \beta^0, ..., \beta^{k'}, ... \in \Pi_n$. The conclusion is that $\Phi^{\rho''}(\mu', t) \in \bigcup_{\rho \in P_n} Or_\rho(\mu)$.

c) This is a consequence of Theorem 18 e). □

5 The basin of p-attraction and the basin of n-attraction

Theorem 28. *We consider the set $A \in P^*(\mathbf{B}^n)$. For any $\mu \in \mathbf{B}^n$, the following statements are equivalent*

$$\exists \rho \in P_n, \omega_\rho(\mu) \subset A, \tag{13}$$

$$\exists \rho \in P_n, \exists t' \in R, \forall t \geq t', \Phi^\rho(\mu, t) \in A, \tag{14}$$

$$\exists \alpha \in \Pi_n, \exists k' \in \mathbf{N}, \forall k \geq k', \Phi^{\alpha^0 \ldots \alpha^k}(\mu) \in A \tag{15}$$

and the following statements are equivalent, too

$$\forall \rho \in P_n, \omega_\rho(\mu) \subset A, \tag{16}$$

$$\forall \rho \in P_n, \exists t' \in R, \forall t \geq t', \Phi^\rho(\mu, t) \in A, \tag{17}$$

$$\forall \alpha \in \Pi_n, \exists k' \in \mathbf{N}, \forall k \geq k', \Phi^{\alpha^0 \ldots \alpha^k}(\mu) \in A. \tag{18}$$

Proof. (13)$\Longrightarrow$(14) We presume that (13) is true. Some t' exists with

$$\omega_\rho(\mu) = \{\Phi^\rho(\mu, t)|t \geq t'\}$$

and we conclude that $\forall t \geq t'$,

$$\Phi^\rho(\mu, t) \in \omega_\rho(\mu) \subset A.$$

(14)$\Longrightarrow$(13) As $t'' \in \mathbf{R}$ exists with

$$\omega_\rho(\mu) = \{\Phi^\rho(\mu, t)|t \geq t''\},$$

from the truth of (14) we have that

$$\omega_\rho(\mu) \subset \{\Phi^\rho(\mu, t)|t \geq \max\{t', t''\}\} \subset A.$$

□

Definition 29. *The **basin** (or **kingdom**, or **domain**) **of p-attraction** or the **p-stable set** of the set $A \in P^*(\mathbf{B}^n)$ is given by $\overline{W}(A) = \{\mu|\mu \in \mathbf{B}^n, \exists\rho \in P_n, \omega_\rho(\mu) \subset A\}$; the **basin** (or **kingdom**, or **domain**) **of n-attraction** or the **n-stable set** of the set A is given by $\underline{W}(A) = \{\mu|\mu \in \mathbf{B}^n, \forall\rho \in P_n, \omega_\rho(\mu) \subset A\}$.*

Remark 30. *Definition 29 makes use of the properties (13) and (16). We can make use also in this Definition of the other equivalent properties from Theorem 28.*

In Definition 29, one or both basins of attraction $\overline{W}(A), \underline{W}(A)$ may be empty.

Theorem 31. *We have:*

i) $\overline{W}(\mathbf{B}^n) = \underline{W}(\mathbf{B}^n) = \mathbf{B}^n$;

ii) if $A \subset A'$, then $\overline{W}(A) \subset \overline{W}(A')$ and $\underline{W}(A) \subset \underline{W}(A')$ hold.

Definition 32. *When $\overline{W}(A) \neq \emptyset$, A is said to be **p-attractive** and for any non-empty set $B \subset \overline{W}(A)$, we say that A is **p-attractive** for B and that B is **p-attracted** by A; A is by definition **partially p-attractive** if $\overline{W}(A) \notin \{\emptyset, \mathbf{B}^n\}$ and **totally p-attractive** whenever $\overline{W}(A) = \mathbf{B}^n$.*

*The fact that $\underline{W}(A) \neq \emptyset$ makes us say that A is **n-attractive** and in this situation for any non-empty $B \subset \underline{W}(A)$, A is called **n-attractive** for B and B is called to be **n-attracted** by A; we use to say that A is **partially n-attractive** if $\underline{W}(A) \notin \{\emptyset, \mathbf{B}^n\}$ and **totally n-attractive** if $\underline{W}(A) = \mathbf{B}^n$.*

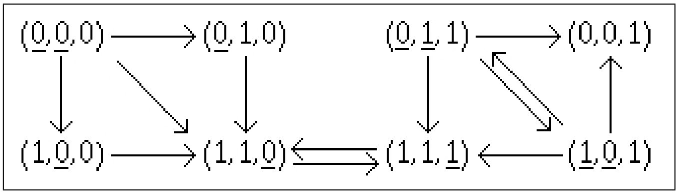

Figure 5: Invariant sets and basins of attraction

Example 33. *We consider the system from Figure 5. The set* $A = \{(0,0,0)\}$ *is neither p-invariant, nor n-invariant:* $\overline{W}(A) = \underline{W}(A) = \emptyset$.

The set $A = \{(0,0,0),(1,1,0),(1,1,1)\}$ *is p-invariant but not n-invariant:* $\overline{W}(A) = \mathbf{B}^3 \setminus \{(0,0,1)\}$, $\underline{W}(A) = \{(0,0,0),(0,1,0),(1,0,0),(1,1,0),(1,1,1)\}$.

We take $A = \{(1,1,0),(1,1,1),\ (0,0,1)\}$ *which is both p-invariant and n-invariant.* A *is totally p-attractive,* $\overline{W}(A) = \mathbf{B}^3$ *and it is not totally n-attractive, since* $\underline{W}(A) = \mathbf{B}^3 \setminus \{(0,1,1),(1,0,1)\}$.

The set $A = \{(1,1,0),\ (1,1,1),(0,1,1),(0,0,1),(1,0,1)\}$ *is p-invariant, n-invariant, totally p-attractive and totally n-attractive because* $\overline{W}(A) = \underline{W}(A) = \mathbf{B}^3$.

Example 34. *The set* $\mathbf{B}^n$ *is totally p-attractive and totally n-attractive (Theorem 31 i)).*

Theorem 35. *Let* $A \in P^*(\mathbf{B}^n)$ *be some set. If* A *is p-invariant, then* $A \subset \overline{W}(A)$ *and* A *is also p-attractive; if* A *is n-invariant, then* $A \subset \underline{W}(A)$ *and* A *is also n-attractive.*

Proof. Let $\mu \in A$ be arbitrary. The existence of $\rho \in P_n$ such that $Or_\rho(\mu) \subset A$ (from the p-invariance of A) and the inclusion $\omega_\rho(\mu) \subset Or_\rho(\mu)$ show that $\omega_\rho(\mu) \subset A$, thus $\mu \in \overline{W}(A)$. As μ was arbitrary, we get that $A \subset \overline{W}(A)$ and finally that $\overline{W}(A) \neq \emptyset$. A is p-attractive. $\square$

Remark 36. *The previous Theorem shows the connection that exists between invariance and attractiveness. If* A *is p-attractive, then* $\overline{W}(A)$ *is the greatest set that is p-attracted by* A *and the point is that this really happens when* A *is p-invariant. The other situation is dual.*

Theorem 37. *Let be* $A \in P^*(\mathbf{B}^n)$. *If* A *is p-attractive, then* $\overline{W}(A)$ *is p-invariant and if* A *is n-attractive, then* $\underline{W}(A)$ *is n-invariant.*

Proof. If A is p-attractive then $\overline{W}(A)\neq\emptyset$ and we prove that $\overline{W}(A)$ is p-invariant. Let $\mu\in\overline{W}(A)$ be arbitrary and fixed. From the definition of $\overline{W}(A)$ some $\rho\in P_n$ exists with the property that $\omega_\rho(\mu)\subset A$. We show that

$$\forall t'\in\mathbf{R},\Phi^\rho(\mu,t')\in\overline{W}(A),$$

i.e.

$$\forall t'\in\mathbf{R},\exists\rho'\in P_n,\omega_{\rho'}(\Phi^\rho(\mu,t'))\subset A.$$

Indeed, we fix arbitrarily some $t'\in\mathbf{R}$. With

$$\rho'=\rho\chi_{(t',\infty)}$$

we can write, from Remark 19, equation (4) that

$$\omega_{\rho'}(\Phi^\rho(\mu,t'))=\omega_{\rho\chi_{(t',\infty)}}(\Phi^\rho(\mu,t'))=\omega_\rho(\mu)\subset A.$$

We prove now that $\underline{W}(A)$, which is non-empty from the n-attractiveness of A, is also n-invariant. The property

$$\forall\mu'\in\underline{W}(A),\forall\rho'\in P_n,Or_{\rho'}(\mu')\subset\underline{W}(A),$$

that is equivalent with

$$\forall\mu'\in\underline{W}(A),\forall\rho'\in P_n,\forall\mu''\in Or_{\rho'}(\mu'),\mu''\in\underline{W}(A)$$

and with

$$\forall\mu'\in\mathbf{B}^n,\forall\rho\in P_n,\omega_\rho(\mu')\subset A\Longrightarrow$$
$$\Longrightarrow\forall\rho'\in P_n,\forall\mu''\in Or_{\rho'}(\mu'),\forall\rho''\in P_n,\omega_{\rho''}(\mu'')\subset A,$$

means the following. Let $\mu'\in\mathbf{B}^n$ and $\rho''\in P_n$ be arbitrary and fixed. The hypothesis states that for any

$$\rho=\alpha^0\cdot\chi_{\{t_0\}}\oplus...\oplus\alpha^k\cdot\chi_{\{t_k\}}\oplus...$$

$\alpha\in\Pi_n,(t_k)\in Seq$ we have

$$\exists k_1\in\mathbf{N},\{\Phi^{\alpha^0...\alpha^k}(\mu')|k\geq k_1\}(=\omega_\rho(\mu'))\subset A. \tag{19}$$

We consider arbitrarily the function $\rho'\in P_n$,

$$\rho'=\alpha'^0\cdot\chi_{\{t'_0\}}\oplus...\oplus\alpha'^k\cdot\chi_{\{t'_k\}}\oplus...$$

$\alpha' \in \Pi_n, (t'_k) \in Seq$ and the point $\mu'' \in Or_{\rho'}(\mu')$, thus $k' \in \mathbf{N}$ exists with the property

$$\mu'' = \Phi^{\alpha'^0 \dots \alpha'^{k'}}(\mu').$$

We put ρ'' under the form

$$\rho'' = \alpha''^0 \cdot \chi_{\{t''_0\}} \oplus \dots \oplus \alpha''^k \cdot \chi_{\{t''_k\}} \oplus \dots$$

$\alpha'' \in \Pi_n, (t''_k) \in Seq$. The sequence

$$\Phi^{\alpha''^0 \dots \alpha''^k}(\mu'') = \Phi^{\alpha''^0 \dots \alpha''^k}(\Phi^{\alpha'^0 \dots \alpha'^{k'}}(\mu')) = \Phi^{\alpha'^0 \dots \alpha'^{k'} \alpha''^0 \dots \alpha''^k}(\mu'),$$

$k \in \mathbf{N}$ fulfills the property (19), thus

$$\exists k_2 \in \mathbf{N}, \{\Phi^{\alpha''^0 \dots \alpha''^k}(\mu'') | k \geq k_2\} (= \omega_{\rho''}(\mu'')) \subset A.$$

□

Corollary 38. *If the set $A \in P^*(\mathbf{B}^n)$ is p-invariant, then $\overline{W}(A)$ is p-invariant and if A is n-invariant, then the basin of n-attraction $\underline{W}(A)$ is n-invariant.*

Proof. These result from Theorem 35 and Theorem 37. □

6 Discussion

Some notes on the terminology:

- universality means the greatest in the sense of inclusion. Any $X \subset \Xi_\Phi$ is a system, but we did not study such systems in the present paper;

- regularity means the existence of a generator function Φ, i.e. analogies with the dynamical systems theory;

- autonomy means here that no input exists. We mention the fact that autonomy has another non-equivalent definition also, a system is called autonomous if its input set has exactly one element;

- asynchronicity refers (vaguely) to the fact that we work with real time and binary values. Its antonym synchronicity means that 'discrete time' (and binary values) in which the iterates of Φ are: $\Phi, \Phi \circ \Phi, \dots, \Phi \circ \dots \circ \Phi, \dots$ i.e. in the sequence $\Phi^{\alpha^0}, \Phi^{\alpha^0 \alpha^1}, \dots, \Phi^{\alpha^0 \dots \alpha^k}, \dots$ all α^k are $(1, \dots, 1), k \in \mathbf{N}$. That is the discrete time of the dynamical systems.

Our concept of invariance from Definition 23 reproduces the point of view expressed in [4], page 11, where the dynamical system $S = (T, X, \Phi)$ is given, with $T = \mathbf{R}$ the time set, X the state space and $\Phi : T \times X \to X$ the flow: the set $A \subset X$ is said to be invariant for the system S if $\forall x \in A, \forall t \in T, \Phi_t(x) \in A$. This idea coincides with the one from [5], page 27 where the state space X is a differentiable manifold M.

In [3], page 92 the set $A \subset X$ is called globally invariant via Φ if $\forall t \in T, \Phi_t(A) = A$, recalling the situation of Example 26 and Figure 4. In [6], page 3, the global invariance and the invariance of $A \subset X$ are defined like at [3] and [4].

We mention also the definition of invariance from [1], page 19. Let $P = (T, X, \Phi)$ be a process, where $T = \mathbf{R}$, X is the state space and $\Phi : \overline{T} \times X \to X$ is the flow of P; we have denoted $\overline{T} = \{(t', t) | t', t \in T, t \leq t'\}$. Then $A \subset X$ is invariant relative to Φ if $\Phi_{t',t}(A) \subset A$ for any $(t', t) \in \overline{T}$. This last definition agrees itself with ours in the special case when $t' = 0$ but it is more general since it addresses systems which are not time invariant.

Stability is defined in [5], page 27 where M is a differentiable manifold and the evolution operator $\Phi_t : M \to M, t \in T$ is given. The subset $A \subset M$ is stable for Φ if for any sufficiently small neighborhood U of A a neighborhood V of A exists such that $\forall x \in V, \forall t \geq 0, \Phi_t(x) \in U$. In our case when $M = \mathbf{B}^n$ has the discrete topology, $A \subset \mathbf{B}^n$ and $U = V = A$, this comes to the invariance of A.

In [4], page 16 the closed invariant set $A \subset X$ is called stable for (T, X, Φ) if i) for any sufficiently small neighborhood $U \supset A$ there exists a neighborhood $V \supset A$ such that $\forall t > 0, \forall x \in V, \Phi_t(x) \in U$ and ii) there exists a neighborhood $W \supset A$ such that $\forall x \in W, \Phi_t(x) \to A$ as $t \to \infty$. We see that i) is the same request like at [5] and ii) brings nothing new (item i) means $Or_\rho(\mu) \subset A$, thus a stronger request than item ii) which is $\omega_\rho(\mu) \subset A$ in our case).

In a series of works ([5], page 27), either the set $A \subset M$ is called asymptotically stable if it is stable and attractive, where M is a differentiable manifold, or ([3], page 112, [6], page 5) the fixed point $x_0 \in X$ is called asymptotically stable if it is stable and attractive. We interpret stability as invariance and stating that A or x_0 is stable and attractive means that it is invariant and a weaker property than invariance takes place (see Theorem 35) and finally asymptotic stability means invariance too.

In [2], page 132 the statement is made that many times, in applications, by stability is understood attractiveness. This would mean, in the conditions of Theorem 35, weakening of the invariance request and we cannot accept this point of view.

In literature, [2] defines at page 6 the basin of attraction of a chaotic attractor $A \subset X$ as the set of the points whose $\omega-$limit set is contained in A. This was reproduced at (13) and (16), where $A \in P^*(\mathbf{B}^n)$ was considered however arbitrary.

The work [3] defines at page 124 the kingdom of attraction of an attractive set $A \subset X$ as the greatest set of points of X whose dynamic ends (for $t \to \infty$) in A; when the kingdom of attraction is an open set, it is called basin of attraction. For us, all the subsets $A \subset \mathbf{B}^n$ are open in the discrete topology of $\mathbf{B}^n$.

In [3], page 123 the invariant set $A \subset X$ is called attractive set for $B \subset X$ if the distance between A and $\Phi_t(B)$ tends to 0 for $t \to \infty$; a set A is attractive if $B \neq \emptyset$ exists that is attracted by A. A slightly different idea is expressed in [6], page 4 where the invariant set A is called attractive for B if $\lim\limits_{t\to\infty} \Phi_t(B) = A$. Unlike these definitions, in Definition 32 the set $A \subset \mathbf{B}^n$ is not required to be invariant and the statement $B \subset \overline{W}(A)$ showing that B is p-attracted by A, i.e. $\forall \mu \in B, \exists \rho \in P_n, \omega_\rho(\mu) \subset A$, reproduces the fact that the distance between A and $\Phi_t(B)$ tends to 0 for $t \to \infty$.

In [5], page 27 M is a differentiable manifold and the subset $A \subset M$ is called attractive for Φ if a neighborhood U of A exists such that $\forall x \in U, \lim\limits_{t\to\infty} \Phi_t(x) \in A$; in this case we say that U is attracted by A. We have reached (13), (16) and the requests of attractiveness $\overline{W}(A) \neq \emptyset, \underline{W}(A) \neq \emptyset$ from Definition 32.

In [2], page 5 a closed invariant set $A \subset X$ is called attractive if a neighborhood U of A exists such that $\forall x \in U, \forall t \geq 0, \Phi_t(x) \in U$ and $\Phi_t(x) \to A$ when $t \to \infty$. Then the set $\bigcup\limits_{t\leq 0} \Phi_t(U)$ is called the basin (the domain) of attraction of the set A.

In [6], page 4 the open set $W(A) \subset X$ representing the greatest set of points of X which is attracted by the attractive set A is called basin of attraction. This definition represents exactly $\overline{W}(A), \underline{W}(A)$ from Definition 29 in the circumstances that (Definition 32) the attractiveness of A means that the previous sets are non-empty.

We have the definition of the basin of attraction from [5], page 27: the maximal set attracted by an attractor $A \subset X$ (invariant set, attractive for one of its neighborhoods) is called the kingdom of attraction of A; when the kingdom of attraction is an open set, it is called basin of attraction. We conclude, related with the real to binary translation of this definition, that if $A \in P^*(\mathbf{B}^n)$ is p-invariant, then it is p-attractive for itself and thus an 'attractor'; its basin of attraction $\overline{W}(A)$ is non-empty in this case and it is the maximal set attracted by A.

We note that the stable manifold of the equilibrium point $x_0 \in X$ is defined in [6], page 4 and [3], page 93 for the dynamical system (T, X, Φ) by $W(x_0) = \{x \in X | \lim_{t \to \infty} \Phi_t(x) = x_0\}$. In [4], page 46 the terminology of stable set is used for this concept and [6] mentions this terminology too. Thus, by replacing $x_0 \in X$ with $A \subset \mathbf{B}^n$ and $\lim_{t \to \infty} \Phi_t(x) = x_0$ with $\omega_\rho(\mu) \subset A$ we get for $\overline{W}(A), \underline{W}(A)$ the alternative terminology of stable sets (i.e. invariant sets) of A.

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