SUFFICIENT OPTIMALITY CONDITIONS FOR THE MOREAU-YOSIDA-TYPE REGULARIZATION CONCEPT APPLIED TO SEMILINEAR ELLIPTIC OPTIMAL CONTROL PROBLEMS WITH POINTWISE STATE CONSTRAINTS[*]

Klaus Krumbiegel[†]   Ira Neitzel [‡]   Arnd Rösch[§]

Abstract

We develop sufficient optimality conditions for a Moreau-Yosida regularized optimal control problem governed by a semilinear elliptic PDE with pointwise constraints on the state and the control. We make use of the equivalence of a setting of Moreau-Yosida regularization to a special setting of the virtual control concept, for which standard second order sufficient conditions have been shown. Moreover, we present a numerical example, solving a Moreau-Yosida regularized model problem with an SQP method.

MSC: 49K20, 49M25, 49M29

keywords: Optimal control, semilinear elliptic equation, state constraints, regularization, Moreau-Yosida approximation, virtual control, sufficient optimality conditions

1   Introduction

In this paper we consider the following class of semilinear optimal control problems with pointwise state and control constraints

$$
\left.
\begin{aligned}
\min \quad & J(y,u) := \frac{1}{2}\|y - y_d\|_{L^2(\Omega)}^2 + \frac{\nu}{2}\|u\|_{L^2(\Omega)}^2 \\
& Ay + d(x,y) = u \quad \text{in } \Omega \\
& \partial_{n_A} y = 0 \quad \text{on } \Gamma \\
u_a \leq & \, u(x) \leq u_b \quad \text{a.e. in } \Omega \\
& y(x) \geq y_c(x) \quad \text{a.e. in } \bar{\Omega}.
\end{aligned}
\right\} \tag{P}
$$

The precise assumptions on the given setting are stated in Assumption 1. Due to the nonlinearity of the state equation the above model problem is of nonconvex type, which makes it necessary to consider sufficient optimality conditions ensuring local optimality of stationary points. We point out the results in [7, 8, 9] where second order sufficient conditions were established for semilinear elliptic control problems with pointwise state constraints. However, it is well known that Lagrange multipliers with respect to pointwise state constraints are in general only regular Borel measures, cf. [1, 4, 5]. The presence of these measures in the optimality system complicates the numerical treatment of such problems significantly, since a pointwise evaluation of the complementary slackness conditions is not possible. For that reason, several regularization concepts to overcome this lack of regularity have been developed in the recent past. We mention for example the penalization method by Ito and Kunisch, [16], Lavrentiev regularization by Meyer, Rösch, and Tröltzsch, [20], as well as interior point methods, cf. [28] and the references therein. Special methods have been developed for boundary control problems, such as an extension of Lavrentiev regularization by a source term representation of the control, see [31] and [24], and the virtual control approach [17]. This approach has been extended to distributed controls in [10] and turned out to be suitable for problems were control and state constraints are active simultaneously. Efficient optimization algorithms are

available for all these regularized problems, see section 6 for detailed information. Concerning second order sufficient conditions for Lavrentiev regularized problems, we point out the results in [26]. For the Moreau-Yosida regularization concept, one can easily see that a classical second order analysis is not possible due to the fact that the regularized objective function is not twice differentiable.

However, interpreting a specific setting of the virtual control concept, i.e. $\phi = 0$, as a Moreau-Yosida regularization, we are able to derive a sufficient optimality condition for the Moreau-Yosida regularization making use of classical second order sufficient conditions for the virtual control concept. This condition ensures local optimality of controls satisfying the first order optimality conditions of Moreau-Yosida regularized problems. These results are not strictly limited to problem (P). In section 5, we therefore give examples of problem classes to which the theory can be extended, including boundary control problems as well as problems governed by parabolic PDEs.

2　Assumptions and properties of the state equation

We begin by briefly laying out the setting of the optimal control problem and stating some properties of the problem and the underlying PDE. Throughout the paper, we will use the following notation: By $\| \cdot \|$ we denote the usual norm in $L^2(\Omega)$, and $(\cdot, \cdot)$ is the associated inner product. The $L^\infty(\Omega)$-norm is specified by $\| \cdot \|_\infty$.

Assumption 1.

- *The function $y_d \in L^2(\Omega)$ and $y_c \in L^\infty(\Omega)$ are given functions and $u_a < u_b$, $\nu > 0$ are real numbers.*

- *Ω denotes a bounded domain in $\mathbb{R}^N$, $N = \{2,3\}$, which is convex or has a $C^{1,1}$-boundary $\partial\Omega$.*

- *A denotes a second order elliptic operator of the form*

$$Ay(x) = -\sum_{i,j=1}^{N} \partial_{x_j}(a_{ij}(x)\partial_{x_i}y(x)),$$

where the coefficients a_{ij} belong to $C^{0,1}(\bar{\Omega})$ with the ellipticity condition

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \qquad \forall(x,\xi) \in \Omega \times \mathbb{R}^N, \quad \theta > 0.$$

Moreover, ∂_{n_A} denotes the conormal-derivative associated with A.

- *The function $d = d(x,y)\colon \Omega \times \mathbb{R}$ is measurable with respect to $x \in \Omega$ for all fixed $y \in \mathbb{R}$, and twice continuously differentiable with respect to y, for almost all $x \in \Omega$. Moreover, d_{yy} is assumed to be a locally bounded and locally Lipschitz continuous function with respect to y, i.e. the following Carathéodory type conditions hold true: there exists $K > 0$ such that*

$$\|d(\cdot,0)\|_\infty + \|d_y(\cdot,0)\|_\infty + \|d_{yy}(\cdot,0)\|_\infty \leq K$$

and for any $M > 0$ there exists $L_M > 0$ such that

$$\|d_{yy}(\cdot,y_1) - d_{yy}(\cdot,y_2)\|_\infty \leq L_M|y_1 - y_2|$$

for all $y_i \in \mathbb{R}$ with $|y_i| \leq M$, $i = 1, 2$.
Additionally, we assume that $d_y(x,y)$ is nonnegative for almost all $x \in \Omega$ and $y \in \mathbb{R}$ and positive on a set $E_\Omega \times \mathbb{R}$, where $E_\Omega \subset \Omega$ is of positive measure.

Under the previous assumptions, we can deduce the following standard result for the state equation in problem (P):

Theorem 1. *Under Assumption 1 the semilinear elliptic boundary value problem*

$$\begin{aligned} Ay + d(x,y) &= u \quad in \ \Omega \\ \partial_{n_A}y &= 0 \quad on \ \Gamma \end{aligned} \tag{1}$$

admits for every right hand side $u \in L^2(\Omega)$ a unique solution $y \in H^1(\Omega) \cap C(\bar{\Omega})$.

The proof can be found e.g. in [6]. Based on this theorem, we introduce the control-to-state operator

$$G\colon L^2(\Omega) \to H^1(\Omega) \cap C(\bar{\Omega}), \ u \mapsto y, \tag{2}$$

that assigns to each $u \in L^2(\Omega)$ the weak solution $y \in H^1(\Omega) \cap C(\bar{\Omega})$ of (1). For future reference, we will provide results concerning differentiability of the control-to-state operator, that can be found in, e.g., [30].

Theorem 2. *Let Assumption 1 be fulfilled. Then the mapping $G\colon L^2(\Omega) \to H^1(\Omega) \cap C(\bar{\Omega})$, defined by $G(u) = y$ is of class C^2. Moreover, for all $u, h \in L^2(\Omega)$, $y_h = G'(u)h$ is defined as the solution of*

$$
\begin{aligned}
Ay_h + d_y(x, y)y_h &= h \quad in \ \Omega \\
\partial_{n_A} y_h &= 0 \quad on \ \Gamma.
\end{aligned}
\tag{3}
$$

Furthermore, for every $h_1, h_2 \in L^2(\Omega)$, $y_{h_1,h_2} = G''(u)[h_1, h_2]$ is the solution of

$$
\begin{aligned}
Ay_{h_1,h_2} + d_y(x, y)y_{h_1,h_2} &= -d_{yy}(x, y)y_{h_1}y_{h_2} \quad in \ \Omega \\
\partial_{n_A} y_{h_1,h_2} &= 0 \quad on \ \Gamma,
\end{aligned}
\tag{4}
$$

where $y_{h_i} = G'(u)h_i$, $i = 1, 2$.

Due to the convexity of the cost functional with respect to the control u and the associated state $y = G(u)$, the existence of at least one solution of problem (P) can be obtained by standard arguments, assuming that the set of feasible controls is nonempty. For future references, we define the set of admissible controls handling the box constraints

$$
U_{ad} = \{u \in L^2(\Omega)\colon u_a \leq u \leq u_b \text{ a.e. in } \Omega\}.
\tag{5}
$$

Relying on the standard assumption of a Mangasarian-Fromovitz constraint qualification, sometimes called linearized Slater condition, of the existence of a control $u_0 \in U_{ad}$ and a constant $\tau > 0$ such that

$$
G(\bar{u}) + G'(\bar{u})(u_0 - \bar{u}) \geq y_c + \tau
\tag{6}
$$

for the pure state constraints, we obtain the following first order necessary optimality conditions for a locally optimal control $\bar{u}$:

Theorem 3. *Assume that condition (6) is satisfied, and let $\bar{u}$ be a solution of problem (P) and let $\bar{y} = G\bar{u}$ be the associated state. Then, a regular Borel measure $\bar{\mu} := \bar{\mu}_\Omega + \bar{\mu}_\Gamma \in \mathcal{M}(\bar{\Omega})$ and an adjoint state $\bar{p} \in W^{1,s}(\Omega)$, $s < N/(N-1)$ exist, such that the following optimality system is satisfied:*

$$
\begin{aligned}
A\bar{y} + d(x, \bar{y}) &= \bar{u} & A^*\bar{p} + d_y(x, \bar{y})\bar{p} &= \bar{y} - y_d - \bar{\mu}_\Omega \\
\partial_{n_A}\bar{y} &= 0 & \partial_{n_{A^*}}\bar{p} &= -\bar{\mu}_\Gamma
\end{aligned}
\tag{7}
$$

$$(\bar{p} + \nu\bar{u}, \, u - \bar{u}) \geq 0, \quad \forall u \in U_{ad} \tag{8}$$

$$\int_{\bar{\Omega}} (y_c - \bar{y}) d\bar{\mu} = 0, \quad \bar{y}(x) \geq y_c(x) \quad \text{for all } x \in \bar{\Omega}$$

$$\int_{\bar{\Omega}} \varphi d\bar{\mu} \geq 0 \quad \forall \varphi \in C(\bar{\Omega})^+, \tag{9}$$

where $C(\bar{\Omega})^+$ is defined by $C(\bar{\Omega})^+ := \{y \in C(\bar{\Omega}) \, | \, y(x) \geq 0 \, \forall \, x \in \bar{\Omega}\}$.

Here and in the following, A^* denotes the formally adjoint operator to the differential operator A. This result can be obtained adapting the theory of Casas, cf. [6].

With the help of the classical reduced Lagrange functional

$$\mathcal{L}(u, \mu) = J(G(u), u) + \int_{\bar{\Omega}} (y_c - G(u)) \, d\mu,$$

the second order sufficient condition

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\mu}) h^2 \geq \alpha \|h\|^2, \quad \alpha > 0, \, \forall h \in L^2(\Omega) \tag{10}$$

guarantees $\bar{u}$ to be a local minimum of (P) since the quadratic growth condition

$$J(G(u), u) \geq J(G(\bar{u}), \bar{u}) + \beta \|u - \bar{u}\|^2$$

is satisfied for a constant $\beta > 0$ for all $u \in U_{ad}$ in a sufficiently small L^2-neighborhood of $\bar{u}$.

Remark 1. *Condition (10) is a strong second order sufficient condition. A weaker formulation is possible along the lines of, e.g., [7], but also rather technical. Moreover, while weaker conditions are important for theoretical investigations, they are more difficult to verify in numerical computations. A nice presentation of general results can be found in the book of Bonnans and Shapiro, [3, Section 2.3], that also explicitly takes into account possibly non-unique Lagrange multipliers.*

3 Regularization approaches

The main focus of this paper is on regularized versions of problem (P). In this section we present the two regularization approaches we will examine in this

paper, the Moreau-Yosida approximation on the one hand and the virtual control concept on the other. We will elaborate that the simple version of Moreau-Yosida regularization is equivalent to a special setting of the virtual control concept.

3.1   Moreau-Yosida regularization

The penalization technique by Ito and Kunisch, [16], based on a Moreau-Yosida approximation of the Lagrange multipliers with respect to the state constraints, applied to problem (P), leads to the following family of regularized problems

$$
\left.
\begin{aligned}
\min \quad & J^{MY}(y_\gamma, u_\gamma) := J(y_\gamma, u_\gamma) + \frac{\gamma}{2} \int_\Omega ((y_c - y_\gamma)_+)^2 dx \\
& Ay_\gamma + d(x, y_\gamma) = u_\gamma \qquad \text{in } \Omega \\
& \quad\qquad \partial_{n_A} y_\gamma = 0 \qquad\quad \text{on } \Gamma \\
& u_a \le u_\gamma(x) \le u_b \quad \text{a.e. in } \Omega,
\end{aligned}
\right\}
\qquad (\text{P}^{MY})
$$

where $\gamma > 0$ is a regularization parameter that is taken large. Note, that the mapping $(\cdot)_+$ denotes the positive part of a measurable function, i.e. $(f)_+ := \max\{0, f\}$.

Introducing a reduced formulation of problem (P^{MY}) by the control-to-state mapping G in (2) for the state equation, the following existence theorem can be proven since the set of admissible controls is nonempty. We refer to, e.g., [30] for details.

Theorem 4. *Under Assumption 1, the regularized optimal control* (P^{MY}) *admits at least one (globally) optimal control* $\bar{u}_\gamma$ *with associated optimal state* $\bar{y}_\gamma = G(\bar{u}_\gamma)$.

Due to the nonlinearity of the state equation, the optimal control problem is nonconvex and one has to take into account the existence of multiple locally optimal controls. Forthcoming, let $\bar{u}_\gamma$ be a locally optimal control of problem (P^{MY}) with associated state $\bar{y}_\gamma = G(\bar{u}_\gamma)$. Using the classical Lagrange formulation, straight forward computations yield the following first order necessary optimality conditions, cf. [16] for the linear-quadratic setting.

Proposition 1. *Let $(\bar{y}_\gamma, \bar{u}_\gamma)$ be a locally optimal solution of problem (P^{MY}). Then, there exists a unique adjoint state $\bar{p}_\gamma \in H^1(\Omega) \cap C(\bar{\Omega})$ such that the following optimality system is satisfied*

$$
\begin{aligned}
A\bar{y}_\gamma + d(x, y_\gamma) = \bar{u}_\gamma \qquad & A^*\bar{p}_\gamma + d_y(x, \bar{y}_\gamma)\bar{p}_\gamma = \bar{y}_\gamma - y_d - \bar{\lambda}_\gamma \\
\partial_{n_A}\bar{y}_\gamma = 0 \qquad & \partial_{n_{A^*}}\bar{p}_\gamma = 0
\end{aligned}
\tag{11}
$$

$$
(\bar{p}_\gamma + \nu\bar{u}_\gamma, \, u - \bar{u}_\gamma) \geq 0 \quad \forall u \in U_{ad} \tag{12}
$$

$$
\bar{\lambda}_\gamma = \gamma(y_c - \bar{y}_\gamma)_+ \in L^2(\Omega) \tag{13}
$$

Convergence analysis as γ tends to infinity is discussed in [22]. Convergence results of the Moreau-Yosida approximation applied to control and state constrained optimal control problems governed by semilinear parabolic PDEs are derived in [23].

3.2 Virtual control concept

In this section, we will apply the so called virtual control concept, first introduced in [17]. Instead of problem (P), we will investigate a family of regularized optimal control problems with mixed control-state constraints:

$$
\left.
\begin{aligned}
\min \quad & J^{VC}(y_\varepsilon, u_\varepsilon, v_\varepsilon) := J(y_\varepsilon, u_\varepsilon) + \frac{\psi(\varepsilon)}{2}\|v_\varepsilon\|_{L^2(\Omega)}^2 \\
& Ay_\varepsilon + d(x, y) = u_\varepsilon + \phi(\varepsilon)v_\varepsilon \quad \text{in } \Omega \\
& \partial_{n_A}y_\varepsilon = 0 \quad \text{on } \Gamma \\
& u_a \leq u_\varepsilon(x) \leq u_b \quad \text{a.e. in } \Omega \\
& y_\varepsilon(x) \geq y_c(x) - \xi(\varepsilon)v_\varepsilon \quad \text{a.e. in } \Omega,
\end{aligned}
\right\}
\quad (\mathrm{P}^{VC})
$$

with a regularization parameter $\varepsilon > 0$ and positive and real valued parameter functions $\psi(\varepsilon)$, $\phi(\varepsilon)$ and $\xi(\varepsilon)$. The remaining given quantities are defined as for problem (P), see Assumption 1.

Denoting a local optimal control of (P) by $\bar{u}$, we point out that the pair $(\bar{u}, 0)$ is feasible for all problems (P^{VC}). Then, using a continuous control-to-state mapping, the existence of at least one pair of optimal controls $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ can be proven by standard arguments.

The existence of regular Lagrange multipliers with respect to mixed control-state constraints is known from e.g. [25] and [27], assuming that a constraint

qualification is satisfied. For (P^{VC}), constraint qualifications are not necessary since the problem can be transformed into a purely control constrained problem with $u_a \leq u_\varepsilon \leq u_b$ and $w := \xi(\varepsilon)v_\varepsilon + y_\varepsilon \geq y_c$, cf. [21] for a Lavrentiev regularized problem without constraints on the control u. Based on this, the following first order necessary optimality conditions for (P^{VC}) are obtained in a straight forward manner.

Proposition 2. *Let $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ be an optimal solution of (P^{VC}) and let $\bar{y}_\varepsilon$ be the associated state. Then, there exist a unique adjoint state $\bar{p}_\varepsilon \in H^1(\Omega) \cap C(\bar{\Omega})$ and a unique Lagrange multiplier $\bar{\mu}_\varepsilon \in L^2(\Omega)$ so that the following optimality system is satisfied*

$$A\bar{y}_\varepsilon + d(x, \bar{y}_\varepsilon) = \bar{u}_\varepsilon + \phi(\varepsilon)\bar{v}_\varepsilon \qquad A^*\bar{p}_\varepsilon + d_y(x, \bar{y}_\varepsilon)\bar{p}_\varepsilon = \bar{y}_\varepsilon - y_d - \bar{\mu}_\varepsilon \tag{14}$$
$$\partial_{n_A}\bar{y}_\varepsilon = 0 \qquad\qquad\qquad \partial_{n_{A^*}}\bar{p}_\varepsilon = 0$$

$$(\bar{p}_\varepsilon + \nu\bar{u}_\varepsilon\,,\, u - \bar{u}_\varepsilon) \geq 0, \quad \forall u \in U_{ad} \tag{15}$$
$$\phi(\varepsilon)\bar{p}_\varepsilon + \psi(\varepsilon)\bar{v}_\varepsilon - \xi(\varepsilon)\bar{\mu}_\varepsilon = 0, \quad a.e.\ in\ \Omega \tag{16}$$
$$(\bar{\mu}_\varepsilon\,,\, y_c - \bar{y}_\varepsilon - \xi(\varepsilon)\bar{v}_\varepsilon) = 0, \quad \bar{\mu}_\varepsilon \geq 0, \quad \bar{y}_\varepsilon \geq y_c - \xi(\varepsilon)\bar{v}_\varepsilon \quad a.e.\ in\ \Omega. \tag{17}$$

The convergence of a sequence of regularized optimal controls $\bar{u}_\varepsilon$ to an optimal solution of the original problem (P) and the uniqueness of dual variables was discussed in [18].

3.3 Equivalence of the concepts

In this section, we will point out the equivalence of the Moreau-Yosida approximation to a special case of the virtual control concept. More precisely, we will demonstrate that the two optimal control problems admit the same optimal controls $\bar{u}_\varepsilon = \bar{u}_\gamma$ and we will then call the regularization concepts and the respective optimal control problems equivalent.

We observe the problems (P^{VC}) for the specific choice $\phi(\varepsilon) \equiv 0$, i.e.:

$$\left. \begin{array}{l} \min \quad J^{VC}(y_\varepsilon, u_\varepsilon, v_\varepsilon) := J(y_\varepsilon, u_\varepsilon) + \dfrac{\psi(\varepsilon)}{2}\|v_\varepsilon\|^2_{L^2(\Omega)} \\[2mm] \qquad A y_\varepsilon + d(x, y) = u_\varepsilon \quad in\ \Omega \\[2mm] \qquad\qquad \partial_{n_A}y_\varepsilon = 0 \quad on\ \Gamma \\[2mm] \qquad u_a \leq u(x) \leq u_b \quad a.e.\ in\ \Omega \\[2mm] \qquad y_\varepsilon(x) \geq y_c(x) - \xi(\varepsilon)v_\varepsilon \quad a.e.\ in\ \Omega, \end{array} \right\} \tag{Q^{VC}}$$

As one can easily see, there is no longer a coupling of both control variables by the state equation of the problem.

First, we consider both types of problems (Q^{VC}) and (P^{MY}) without any notice on the optimality conditions. We start investigating the mixed control-state constraints in (Q^{VC}) pointwise, where we split the domain Ω into two disjoint subsets $\Omega = \Omega_1 \cup \Omega_2$:

$$\Omega_1 := \{x \in \Omega : y_c(x) - y_\varepsilon(x) < 0 \text{ a.e. in } \Omega\}$$
$$\Omega_2 := \{x \in \Omega : y_c(x) - y_\varepsilon(x) \geq 0 \text{ a.e. in } \Omega\}.$$

Initially, we consider Ω_1. The mixed constraints are given by $y_c(x) - y_\varepsilon(x) \leq \xi(\varepsilon)v_\varepsilon(x)$ a.e. in Ω. Due to the minimization of the L^2-norm of the virtual control v_ε in the objective of (Q^{VC}), we derive

$$v_\varepsilon \equiv 0 \quad \text{a.e. in } \Omega_1.$$

Considering Ω_2, the inequality

$$\xi(\varepsilon)v_\varepsilon(x) \geq y_c(x) - y_\varepsilon(x) \geq 0$$

has to be satisfied. Choosing the virtual control as small as possible, we deduce

$$v_\varepsilon = \frac{1}{\xi(\varepsilon)}(y_c - y_\varepsilon) \quad \text{a.e. in } \Omega_2.$$

Concluding, the mixed control-state constraints can be replaced by the equation

$$v_\varepsilon = \frac{1}{\xi(\varepsilon)}(y_c - y_\varepsilon)_+.$$

Thus, the optimal control problem (Q^{VC}) can be rewritten equivalently in the form

$$\min \quad J(y_\varepsilon, u_\varepsilon) + \frac{\psi(\varepsilon)}{2\xi(\varepsilon)^2}\|(y_c - y_\varepsilon)_+\|_{L^2(\Omega)}^2$$
$$Ay_\varepsilon + d(x, y_\varepsilon) = u_\varepsilon \quad \text{in } \Omega$$
$$\partial_{n_A} y_\varepsilon = 0 \quad \text{on } \Gamma$$
$$u_a \leq u_\varepsilon(x) \leq u_b \quad \text{a.e. on } \Omega.$$

Consequently, we formulate the following result.

Corollary 1. *For the specific parameter function $\phi(\varepsilon) \equiv 0$, the problem (P^{VC}) is equivalent to the problem (P^{MY}) arising by the Moreau-Yosida regularization, if the regularization parameter $\gamma > 0$ is defined by $\gamma := \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}$.*

For the sake of completeness, we will additionally elaborate on the equivalence by the different first order necessary optimality conditions. Due to Proposition 2 and $\phi(\varepsilon) \equiv 0$, an optimal control $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ of (Q^{VC}) satisfies

$$A\bar{y}_\varepsilon + d(x, \bar{y}_\varepsilon) = \bar{u}_\varepsilon \qquad A^*\bar{p}_\varepsilon + d_y(x, \bar{y}_\varepsilon)\bar{p}_\varepsilon = \bar{y}_\varepsilon - y_d - \bar{\mu}_\varepsilon$$
$$\partial_{n_A}\bar{y}_\varepsilon = 0 \qquad\qquad \partial_{n_{A^*}}\bar{p}_\varepsilon = 0 \tag{18}$$

$$(\bar{p}_\varepsilon + \nu\bar{u}_\varepsilon , u - \bar{u}_\varepsilon) \geq 0, \quad \forall u \in U_{ad} \tag{19}$$
$$\psi(\varepsilon)\bar{v}_\varepsilon - \xi(\varepsilon)\bar{\mu}_\varepsilon = 0, \quad \text{a.e. in } \Omega \tag{20}$$
$$(\bar{\mu}_\varepsilon , y_c - \bar{y}_\varepsilon - \xi(\varepsilon)\bar{v}_\varepsilon) = 0, \quad \bar{\mu}_\varepsilon \geq 0, \quad \bar{y}_\varepsilon \geq y_c - \xi(\varepsilon)\bar{v}_\varepsilon \quad \text{a.e. in } \Omega \tag{21}$$

Since the multiplier $\bar{\mu}_\varepsilon$ is a regular function, it is well known that the complementary slackness conditions in (21) are equivalent to

$$\bar{\mu}_\varepsilon - \max\{0, \bar{\mu}_\varepsilon + c(y_c - \bar{y}_\varepsilon - \xi(\varepsilon)\bar{v}_\varepsilon)\} = 0$$

for every $c > 0$. Using the specific choice $c = \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}$, we obtain

$$\bar{\mu}_\varepsilon = \max\{0, \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(y_c - \bar{y}_\varepsilon)\} = \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(y_c - \bar{y}_\varepsilon)_+.$$

instead of (20) and (21). Due to (20), the virtual control satisfies

$$\bar{v}_\varepsilon = \frac{\xi(\varepsilon)}{\psi(\varepsilon)}\bar{\mu}_\varepsilon = \frac{1}{\xi(\varepsilon)}(y_c - \bar{y}_\varepsilon)_+. \tag{22}$$

By means of Proposition 1, it is easily seen that the optimality systems of (P^{MY}) and (Q^{VC}) are equivalent and we conclude with the following result.

Corollary 2. *Let $(\bar{y}_\varepsilon, \bar{u}_\varepsilon, \bar{v}_\varepsilon)$ be a stationary point of (P^{VC}). If we set $\phi(\varepsilon) \equiv 0$, then the virtual control can be represented by $\bar{v}_\varepsilon = 1/\xi(\varepsilon)(y_c - \bar{y}_\varepsilon)_+$. Moreover, $(\bar{y}_\varepsilon, \bar{u}_\varepsilon)$ is also a stationary point of (P^{MY}) for the specific choice $\gamma = \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}$. Conversely, a stationary point of (P^{MY}) is also a stationary point of (P^{VC}) if the conditions above are satisfied.*

4   Sufficient optimality conditions for the Moreau-Yosida approximation

Now we will formulate a sufficient optimality condition for the Moreau-Yosida approximation based on a second order sufficient optimality condition for the respective equivalent virtual control concept (Q^{VC}). We first define the Lagrangian of problem (Q^{VC}) by

$$
\begin{aligned}
\mathcal{L}^{VC}(u, v, \mu) = {} & \frac{1}{2}\|G(u) - y_d\|^2 + \frac{\nu}{2}\|u\|^2 + \frac{\psi(\varepsilon)}{2}\|v\|^2 \\
& + \int_\Omega (y_c - G(u) - \xi(\varepsilon)v)\mu \, dx
\end{aligned}
\tag{23}
$$

using the control-to-state operator G, given in (2). Straight forward computations show that the second derivative of the Lagrangian is given by

$$
\begin{aligned}
\frac{\partial^2 \mathcal{L}^{VC}(u, v, \mu)}{\partial(u, v)^2}[h_1, h_2] = {} & (G'(u)h_{u,1}, G'(u)h_{u,2}) + \\
& + (G(u) - y_d, G''(u)[h_{u,1}, h_{u,2}]) + \nu(h_{u,1}, h_{u,2}) + \\
& + \psi(\varepsilon)(h_{v,1}, h_{v,2}) - (G''(u)[h_{u,1}, h_{u,2}], \mu)
\end{aligned}
\tag{24}
$$

for $h_i = (h_{u,i}, h_{v,i}) \in L^2(\Omega)^2$, $i = 1, 2$. In the sequel, let $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ be a local solution of (Q^{VC}) with associated Lagrange multiplier $\bar{\mu}_\varepsilon$, i.e. (18)-(21) are satisfied. We proceed with establishing the second order sufficient optimality condition.

Assumption 2. *There exists a constant $\alpha \geq 0$ such that*

$$
\frac{\partial^2 \mathcal{L}^{VC}(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{\mu}_\varepsilon)}{\partial(u, v)^2}[h_u, h_v]^2 \geq \alpha\|h_u\|^2 + \psi(\varepsilon)\|h_v\|^2
\tag{25}
$$

is valid for all $h_u \in L^2(\Omega)$.

Remark 2. *Condition (25) can be deduced from the strong second order sufficient condition (10) for the unregularized problem* (P), *cf. [18].*

Remark 3. *Note, that the coercivity condition of the second derivative of the Lagrangian with respect to directions $h_v \in L^2(\Omega)$ is satisfied by construction, see (24). Coercivity with respect to directions h_u can again be formulated with the help of strongly active sets, cf. [7]. However, the strong formulation (25) matches the strong formulation (10) for the unregularized problem.*

Based on the previous coercivity condition, one can prove a quadratic growth condition for problem (Q^{VC}) that ensures local optimality of $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$.

Proposition 3. *Let $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ be a control satisfying the first order necessary optimality conditions (18)-(21). Additionally, $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ fulfills Assumption 2. Then, there exist constants $\beta > 0$ and $\delta > 0$ such that*

$$J^{VC}(G(u), u, v) \geq J^{VC}(G(\bar{u}_\varepsilon), \bar{u}_\varepsilon, \bar{v}_\varepsilon) + \beta(\|u - \bar{u}_\varepsilon\|^2 + \|v - \bar{v}_\varepsilon\|^2) \quad (26)$$

for all feasible controls (u, v) of problem (Q^{VC}) with $\|u - \bar{u}_\varepsilon\| \leq \delta$.

Proof. First, let us mention that there is a specific difference to the standard proofs, since no smallness condition for $\|v - \bar{v}_\varepsilon\|$ is required. Let $(u, v) \in U_{ad} \times L^2(\Omega)$ be an admissible control of problem (Q^{VC}), i.e. mainly $y_c - \xi(\varepsilon)v - G(u) \leq 0$. In view of the positivity of the optimal Lagrange multiplier $\bar{\mu}_\varepsilon$, we can estimate the cost functional J^{VC} by the Lagrange functional:

$$J^{VC}(G(u), u, v) \geq J^{VC}(G(u), u, v) + \int_\Omega (y_c - G(u) - \xi(\varepsilon)v)\bar{\mu}_\varepsilon \, dx = \mathcal{L}(u, v, \bar{\mu}_\varepsilon).$$

Under Assumption 1, the Lagrange functional is twice continuously differentiable with respect to the $L^2(\Omega)$-norms, since the solution operator G has this property, see Theorem 2. Then, a Taylor expansion is given by

$$\mathcal{L}^{VC}(u, v, \bar{\mu}_\varepsilon) = \mathcal{L}^{VC}(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{\mu}_\varepsilon) + \frac{\partial \mathcal{L}^{VC}(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{\mu}_\varepsilon)}{\partial(u, v)}(u - \bar{u}_\varepsilon, v - \bar{v}_\varepsilon)$$
$$+ \frac{1}{2}\frac{\partial^2 \mathcal{L}^{VC}(\tilde{u}, \tilde{v}, \bar{\mu}_\varepsilon)}{\partial(u, v)^2}(u - \bar{u}_\varepsilon, v - \bar{v}_\varepsilon)^2$$

with $\tilde{u} = \bar{u}_\varepsilon + \theta(u - \bar{u}_\varepsilon)$, $\tilde{v} = \bar{v}_\varepsilon + \theta(v - \bar{v}_\varepsilon)$ for a $\theta \in (0, 1)$. Since $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ satisfies the first order necessary optimality conditions (18)-(21) and $\bar{\mu}_\varepsilon$ is the associated Lagrange multiplier, we have

$$\frac{\partial \mathcal{L}^{VC}(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{\mu}_\varepsilon)}{\partial(u, v)}(u - \bar{u}_\varepsilon, v - \bar{v}_\varepsilon) \geq 0 \text{ and } \mathcal{L}^{VC}(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{\mu}_\varepsilon) = J^{VC}(G(\bar{u}_\varepsilon), \bar{u}_\varepsilon, \bar{v}_\varepsilon),$$

which implies

$$\mathcal{L}^{VC}(u, v, \bar{\mu}_\varepsilon) \geq J^{VC}(G(\bar{u}_\varepsilon), \bar{u}_\varepsilon, \bar{v}_\varepsilon) + \frac{1}{2}\frac{\partial^2 \mathcal{L}^{VC}(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{\mu}_\varepsilon)}{\partial(u, v)^2}(u - \bar{u}_\varepsilon, v - \bar{v}_\varepsilon)^2$$
$$+ \frac{1}{2}\left(\frac{\partial^2 \mathcal{L}^{VC}(\tilde{u}, \tilde{v}, \bar{\mu}_\varepsilon)}{\partial(u, v)^2} - \frac{\partial^2 \mathcal{L}^{VC}(\bar{u}_\varepsilon, \bar{v}_\varepsilon, \bar{\mu}_\varepsilon)}{\partial(u, v)^2}\right)(u - \bar{u}_\varepsilon, v - \bar{v}_\varepsilon)^2.$$

Using the SSC of Assumption 2, we obtain

$$
\begin{aligned}
\mathcal{L}^{VC}(u,v,\bar{\mu}_\varepsilon) \geq \quad & J^{VC}(G(\bar{u}_\varepsilon),\bar{u}_\varepsilon,\bar{v}_\varepsilon) + \alpha\|u-\bar{u}_\varepsilon\|^2 + \psi(\varepsilon)\|v-\bar{v}_\varepsilon\|^2 \\
& + \tfrac{1}{2}\left(\frac{\partial^2\mathcal{L}^{VC}(\tilde{u},\tilde{v},\bar{\mu}_\varepsilon)}{\partial(u,v)^2} - \frac{\partial^2\mathcal{L}^{VC}(\bar{u}_\varepsilon,\bar{v}_\varepsilon,\bar{\mu}_\varepsilon)}{\partial(u,v)^2}\right)(u-\bar{u}_\varepsilon,v-\bar{v}_\varepsilon)^2.
\end{aligned}
$$

One can easily see that the second derivative (24) is independent of the virtual control v since the control-to-state operator is only applied to the control variable u and linear mixed control-state constraints are considered. Moreover, one can prove under Assumption 1 that the second derivative of the Lagrangian (24) is locally Lipschitz continuous with respect to u, i.e. there exists a positive constant C_L such that the estimate

$$
\left|\left(\frac{\partial^2\mathcal{L}^{VC}(u_1,v,\mu)}{\partial(u,v)^2} - \frac{\partial^2\mathcal{L}^{VC}(u_2,v,\mu)}{\partial(u,v)^2}\right)h^2\right| \leq C_L\|u_1-u_2\|\|h\|^2
$$

holds true for $\|u_1 - u_2\| \leq \delta$ and $\delta > 0$ sufficiently small, see for instance [30, Lemma 4.24]. By means of the Lipschitz property concerning u and the independency of v, see (24), we conclude

$$
\begin{aligned}
\mathcal{L}^{VC}(u,v,\bar{\mu}_\varepsilon) \ \geq \ & J^{VC}(G(\bar{u}_\varepsilon),\bar{u}_\varepsilon,\bar{v}_\varepsilon) + \alpha\|u-\bar{u}_\varepsilon\|^2 + \psi(\varepsilon)\|v-\bar{v}_\varepsilon\|^2 \\
& -c\|u-\bar{u}_\varepsilon\|(\|u-\bar{u}_\varepsilon\|^2 + \|v-\bar{v}_\varepsilon\|^2) \\
\geq \ & J^{VC}(G(\bar{u}_\varepsilon),\bar{u}_\varepsilon,\bar{v}_\varepsilon) + (\alpha - c\delta)\|u-\bar{u}_\varepsilon\|^2 + \\
& + (\psi(\varepsilon) - c\delta)\|v-\bar{v}_\varepsilon\|^2,
\end{aligned}
$$

provided that $\|u-\bar{u}_\varepsilon\| \leq \delta$. For sufficiently small $\delta > 0$, we find a positive constant $\beta > 0$ such that the assertion is fulfilled. $\qquad\square$

Forthcoming, we will rewrite the second order sufficient optimality condition of problem (Q^{VC}) in terms of the equivalent Moreau-Yosida regularization (P^{MY}) using relations between the respective variables derived in the previous section.

Due to Corollary 2, the control $\bar{u}_\varepsilon$ satisfies the first order optimality conditions (11)-(13) of (P^{MY}) with $\gamma = \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}$. Thus, we set

$$
\bar{u}_\lambda = \bar{u}_\varepsilon, \quad \bar{\lambda}_\gamma = \bar{\mu}_\varepsilon = \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}(y_c - \bar{y}_\varepsilon)_+
$$

and the SSC (25) of Assumption 2 yields the following

$$\|G'(\bar{u}_\gamma)h_u\|^2 + \nu\|h_u\|^2 + (G(\bar{u}_\gamma) - y_d, G''(\bar{u}_\gamma)h_u^2)$$
$$-(\bar{\lambda}_\gamma, G''(\bar{u}_\gamma)h_u^2) \geq \alpha\|h_u\|^2 \qquad (27)$$

for all $h_u \in L^2(\Omega)$, written in terms of the Moreau-Yosida regularization. Summarizing, one ends up with

$$J''(G(\bar{u}_\gamma), \bar{u}_\gamma)h_u^2 - (\bar{\lambda}_\gamma, G''(\bar{u}_\gamma)h_u^2) \geq \alpha\|h_u\|^2.$$

Concluding, we can state the following result.

Theorem 5. *Let $\bar{u}_\gamma \in U_{ad}$, with associated state $\bar{y}_\gamma = G(\bar{u}_\gamma)$, be a control satisfying the first order necessary optimality conditions (11)-(13). Additionally, there exists a constant $\alpha > 0$ such that for all $h_u \in L^2(\Omega)$ the following condition is fulfilled:*

$$J''(G(\bar{u}_\gamma), \bar{u}_\gamma)h_u^2 - \gamma((y_c - G(\bar{u}_\gamma))_+, G''(\bar{u}_\gamma)h_u^2) \geq \alpha\|h_u\|^2, \qquad (28)$$

i.e. there exists a constant $\alpha > 0$ such that

$$\int_\Omega (y_{h_u}^2 - \bar{p}_\gamma d_{yy}(x, \bar{y}_\gamma)y_{h_u}^2 + \nu h_u^2)\, dx \geq \alpha\|h_u\|^2 \qquad (29)$$

is satisfied for all $(h_u, y_{h_u}) \in L^2(\Omega) \times H^1(\Omega)$ with $y_{h_u} = G'(\bar{u}_\gamma)h_u$, and $\bar{p}_\gamma$ defined in (11).

Then, there exist constants $\beta > 0$ and $\delta > 0$ so that the quadratic growth condition

$$J^{MY}(G(u_\gamma), u_\gamma) \geq J^{MY}(G(\bar{u}_\gamma), \bar{u}_\gamma) + \beta\|u_\gamma - \bar{u}_\gamma\|^2 \qquad (30)$$

holds for all $u_\gamma \in U_{ad}$ with $\|u_\gamma - \bar{u}_\gamma\| \leq \delta$. In particular, $(G(\bar{u}_\gamma), \bar{u}_\gamma)$ is a locally optimal solution of (P^{MY}).

Proof. Due to Corollary 2 and (22), the pair $(\bar{u}_\gamma, \bar{v}_\gamma := \frac{1}{\xi(\varepsilon)}(y_c - \bar{y}_\gamma)_+)$ satisfies the first order optimality conditions (18)-(21) of problem (Q^{VC}), where the parameter functions $\psi(\varepsilon)$ and $\xi(\varepsilon)$ are chosen in a way such that $\gamma = \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}$. The associated Lagrange multiplier in the optimality conditions is denoted

by $\bar{\mu}_\gamma$. Due to the former argumentation, one can easily see, that (28) implies the coercivity condition (25) in the point $(\bar{u}_\gamma, \bar{v}_\gamma, \bar{\mu}_\gamma)$, i.e.

$$\frac{\partial^2 \mathcal{L}^{VC}(\bar{u}_\gamma, \bar{v}_\gamma, \bar{\mu}_\gamma)}{\partial(u, v)^2}[h_u, h_v] \geq \alpha \|h_u\|^2 + \psi(\varepsilon)\|h_v\|^2$$

for all $h_u \in L^2(\Omega)$. Thus, Assumption 2 is satisfied and we proceed by applying Proposition 26. Hence, there exist constants $\beta > 0$ and $\delta > 0$ such that

$$J^{VC}(G(u), u, v) \geq J^{VC}(G(\bar{u}_\gamma), \bar{u}_\gamma, \bar{v}_\gamma) + \beta(\|u - \bar{u}_\gamma\|^2 + \|v - \bar{v}_\gamma\|^2)$$

for all feasible (u, v) of problem (Q^{VC}) with $\|u - \bar{u}_\gamma\| \leq \delta$. Now, we consider an arbitrary control $u \in U_{ad}$ with $\|u - \bar{u}_\gamma\| \leq \delta$. Furthermore, the pair of controls $(u, v := \frac{1}{\xi(\varepsilon)}(y_c - G(u))_+)$ is feasible for problem (Q^{VC}) since

$$\xi(\varepsilon)v = (y_c - G(u))_+ \geq y_c - G(u).$$

By means of the equivalence of the problems (P^{MY}) and (Q^{VC}) and $\gamma = \frac{\psi(\varepsilon)}{\xi(\varepsilon)^2}$, we deduce

$$J^{MY}(G(\bar{u}_\gamma), \bar{u}_\gamma) = J^{VC}(G(\bar{u}_\gamma), \bar{u}_\gamma \bar{v}_\gamma) \text{ and } J^{MY}(G(u), u) = J^{VC}(G(u), u, v).$$

Concluding, we obtain the assertion

$$J^{MY}(G(u), u) \geq J^{MY}(G(\bar{u}_\gamma), \bar{u}_\gamma) + \beta \|u - \bar{u}_\gamma\|^2$$

for all $u \in U_{ad}$ with $\|u - \bar{u}_\gamma\| \leq \delta$. $\qquad\square$

The quadratic growth condition (30) for Problem (P^{MY}) from the last theorem has essentially been proven under condition (28) for the regularized problem formulation. In [18], we have deduced second-order sufficient conditions for the regularized problem (P^{VC}) on assumptions on the unregularized problem (P) only. By the previously shown equivalence of the two regularization concepts the same is true for (P^{MY}). As an analogue to [18, Theorem 4.5] we obtain that (28) also follows from (10):

Corollary 3. *Let $\bar{u}$ fulfill the first order necessary optimality conditions of Theorem 3 with unique dual variables $\bar{\mu}$ and $\bar{p}$, as well as the second order sufficient condition (10). Then there exists a constant $\alpha > 0$ such that*

$$\int_\Omega (y_{h_u}^2 - \bar{p}_\gamma d_{yy}(x, \bar{y}_\gamma)y_{h_u}^2 + \nu h_u^2)\, dx \geq \alpha \|h_u\|^2$$

is fulfilled for all $h_u \in L^2(\Omega)$ provided that γ is sufficiently large.

5   Generalizations

In this section we want to point out that the theory presented in this paper can be generalized to large classes of semilinear optimal control problems. Let us start with an elliptic boundary control problem. The virtual control formulation with $\phi(\varepsilon) = 0$ is given by

$$
\left.
\begin{aligned}
\min \quad & J(y_\varepsilon, u_\varepsilon, v_\varepsilon) := \frac{\alpha_1}{2}\|y_\varepsilon - y_{d,\Omega}\|_{L^2(\Omega)}^2 + \frac{\alpha_2}{2}\|y_\varepsilon - y_{d,\Gamma}\|_{L^2(\Gamma)}^2 \\
& \quad + \frac{\nu}{2}\|u_\varepsilon\|_{L^2(\Gamma)}^2 + \frac{\psi(\varepsilon)}{2}\|v_\varepsilon\|_{L^2(\Omega)}^2 \\
& \quad A y_\varepsilon + d(x, y_\varepsilon) = 0 \quad \text{in } \Omega \\
& \quad \partial_{n_A} y_\varepsilon + b(x, y_\varepsilon) = u_\varepsilon \quad \text{on } \Gamma \\
& \quad u_a \le u_\varepsilon(x) \le u_b \quad \text{a.e. in } \Gamma \\
& \quad y_\varepsilon(x) \ge y_c(x) - \xi(\varepsilon) v_\varepsilon \quad \text{a.e. in } \Omega,
\end{aligned}
\right\} \quad (\mathrm{Q}_1^{VC})
$$

and the corresponding equivalent Moreau-Yosida regularization is presented by

$$
\left.
\begin{aligned}
\min \quad & J(y_\gamma, u_\gamma, v_\gamma) := \frac{\alpha_1}{2}\|y_\gamma - y_{d,\Omega}\|_{L^2(\Omega)}^2 + \frac{\alpha_2}{2}\|y_\gamma - y_{d,\Gamma}\|_{L^2(\Gamma)}^2 \\
& \quad + \frac{\nu}{2}\|u_\gamma\|_{L^2(\Gamma)}^2 + \frac{\gamma}{2}\|(y_c - y_\gamma)_+\|_{L^2(\Omega)}^2 \\
& \quad A y_\gamma + d(x, y_\gamma) = 0 \quad \text{in } \Omega \\
& \quad \partial_{n_A} y_\gamma + b(x, y_\gamma) = u_\gamma \quad \text{on } \Gamma \\
& \quad u_a \le u_\gamma(x) \le u_b \quad \text{a.e. in } \Gamma.
\end{aligned}
\right\} \quad (\mathrm{P}_1^{MY})
$$

The theory presented in section 3 can be adapted by only changing the corresponding sets. The results of section 4 depend on the dimension of the domain. For dimension $N = 3$ we get a two norm discrepancy in the second order sufficient optimality condition of proposition 3 in the virtual control approach, but only for the original control u. Of course, the corresponding sufficient optimality condition for the Moreau-Yosida regularization in Theorem 5 contains a two norm setting, too. Let us mention that in this case sufficient optimality conditions for the unregularized problems are challenging due to regularity problems. Therefore, Corollary 3 is then not verified by our theory.

It is also possible to generalize the theory to the regularized version of parabolic optimal control problems like

$$
\left.
\begin{aligned}
\min \quad & J(y,u) := \frac{\alpha_1}{2}\|y - y_d\|_{L^2(Q)}^2 + \frac{\alpha_2}{2}\|y(T) - y_T\|_{L^2(\Omega)}^2 \\
& + \frac{\alpha_3}{2}\|y - y_\Sigma\|_{L^2(\Sigma)}^2 + \frac{\nu}{2}\|u\|_{L^2(Q)}^2 \\
& y_t + Ay + d(t,x,y) = u \quad \text{in } Q = (0,T) \times \Omega \\
& \partial_{n_A} y + b(t,x,y) = 0 \quad \text{on } \Sigma = (0,T) \times \Gamma \\
& u_a \le u(t,x) \le u_b \quad \text{a.e. in } Q \\
& y(t,x) \ge y_c(t,x) \quad \text{a.e. in } Q, \\
& y(0) = y_0.
\end{aligned}
\right\}
\tag{P_2}
$$

Due to the weaker differentiability properties of parabolic control-to-state operators, a two norm discrepancy will have to be taken into account in proposition 3 and theorem 5 for spatial dimensions greater than one. Similarly to the elliptic problem, Corollary 3 is then not verified.

Moreover, it is possible to discuss more general objectives and nonlinearities in the partial differential equations with respect to the control u. However, then the discussion of the differentiability of the control-to-state mapping becomes more involved. In addition, one needs several technical assumptions on the nonlinearities to get the desired results. Such assumptions are essentially that ones that were needed for the derivation of sufficient second order conditions, see [26]. These discussions go beyond the scope of the paper.

6 Numerical example

In this section, we present a numerical example and motivate how the theoretical results shown in this article are used in numerical computations. We aim at solving the optimal control problem

$$
\left.
\begin{aligned}
\min \quad & J(y,u) := \frac{1}{2}\|y - y_d\|_{L^2(\Omega)}^2 + \frac{\nu}{2}\|u\|_{L^2(\Omega)}^2 \\
& \Delta y + y + y^3 = u + f \quad \text{in } \Omega \\
& \partial_n y = 0 \quad \text{on } \Gamma \\
& u_a \le u(x) \le u_b \quad \text{a.e. in } \Omega \\
& y(x) \ge y_c(x) \quad \text{a.e. in } \bar{\Omega},
\end{aligned}
\right\}
\tag{PT}
$$

with $\Omega = [0,1]^2$ and Tikhonov regularization parameter $\nu = 1 \cdot 10^{-3}$, where the remaining data is chosen such that

$$\bar{u}(x) = \Pi_{[u_a, u_b]} \left\{ -\frac{p(x)}{\nu} \right\}$$

with $u_a = 150$ and $u_b = 850$ is an optimal control with associated optimal state $\bar{y}$, adjoint state p, and Lagrange multiplier μ, given by

$$\bar{y}(x) = -16x_1^4 + 32x_1^3 - 16x_1^2 + 1,$$
$$p(x) = 2x_1^3 - 3x_1^2,$$
$$\mu(x) = \max\{0, \bar{y}(x_1 = 0.2) - \bar{y}(x)\}.$$
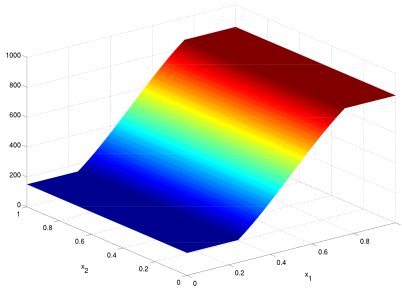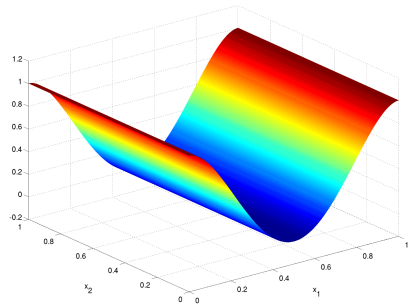
It can be verified that this is obtained with

$$y_c(x) = \min\{\bar{y}(x_1 = 0.2), \bar{y}(x)\},$$
$$f = -\Delta \bar{y} + \bar{y} + \bar{y}^3 - \bar{u},$$
$$y_d = \Delta p - p - 3\bar{y}^2 p + \bar{y} - \mu.$$

The second order sufficient conditions are also satisfied, which is easily proven by computing $-p(x)d_{yy}(x, \bar{y}(x)) \geq 0$ on $[0,1]^2$, that guarantees (29). Notice, that the active sets associated to the pure state constraints and active set corresponding to the control constraints are not disjoint, so that regularization by the virtual control approach is reasonable.

We solve this problem with the help of the Moreau-Yosida regularization approach, i.e. the virtual control approach with $\phi = 0$, and denote the regularized problem by (PT^{MY}). We apply an SQP method, cf. for instance [15] and [29]. We point out that a key argument in the proof of convergence of SQP methods are second order sufficient conditions, which are now guaranteed for the Moreau-Yosida regularized problem, and it is reasonable to investigate the convergence behavior of the solution algorithm.

For completeness, let us mention that a primal-dual active set strategy is used for solving the linear quadratic subproblems, see e.g. [2, 11, 12, 19] and the references therein. Moreover, all functions are discretized by piecewise linear ansatz functions, defined on a uniform finite element mesh. The number of intervals in one dimension, denoted by N, is related to the mesh size by $h = \sqrt{2}N$. In the following all computations are performed with $N = 192$. The Figures 1-4 show the numerical solution of the Moreau-Yosida

Figure 1: Control u_γ

Figure 2: State y_γ

approximation of problem (PT) for the fixed penalization parameter $\gamma = 1 \cdot 10^5$. In Figure 4 one can see irregularities of the multiplier approximation on the boundary and in the parts of the domain, where the active sets of the original problem (PT) associated to the different constraints are not disjoint. We obtain the following error of the numerical solution of problem (PT^{MY}):

$$\|u_\gamma - \bar{u}\| \approx 3.1426e{-}02, \ \ \|y_\gamma - \bar{y}\| \approx 2.7497e{-}05, \ \ \|p_\gamma - p\| \approx 1.5147e - 04. \quad (31)$$
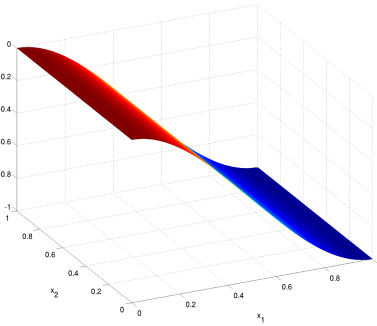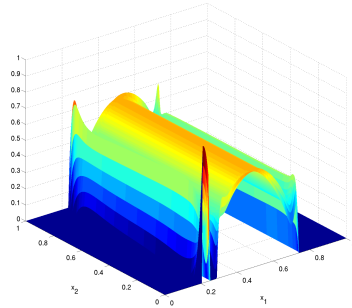
The convergence behavior of the SQP method is presented in Table 1. We display the value of the cost functional J^{MY} for each step of SQP algorithm as well as the relative difference between two iterates, which is defined by

$$\delta_\gamma = \frac{1}{3} \left(\frac{\|u_\gamma^{(n)} - u_\gamma^{(n+1)}\|}{\|u_\gamma^{(n+1)}\|} + \frac{\|y_\gamma^{(n)} - y_\gamma^{(n+1)}\|}{\|y_\gamma^{(n+1)}\|} + \frac{\|p_\gamma^{(n)} - p_\gamma^{(n+1)}\|}{\|p_\gamma^{(n+1)}\|} \right).$$

This quantity is used for a termination condition of the SQP method. In all numerical tests the algorithm stops if $\delta < 1 \cdot 10^{-6}$. In addition the number of iterations of the primal-dual active set strategy is shown.

We also test the regularization algorithm for increasing regularization parameters, noting that convergence of $\bar{u}_\gamma$ towards $\bar{u}$ is discussed in, e.g., [22]. We mention Hintermüller and Kunisch in [14, 13], where path-following methods associated to the Moreau-Yosida regularization parameter are developed. In this numerical test, we use only a simple nested approach: the numerical solution of the problem is taken as the starting point for the SQP-method with respect to the next regularization parameter. The convergence behavior for increasing regularization parameters γ is displayed in Table 2. As expected,

it_{SQP}	J^{MY}	δ_γ	$\#it_{AS}$
1	$1.497214e + 02$	$1.414707e + 00$	13
2	$1.766473e + 02$	$3.585474e - 01$	32
3	$1.767212e + 02$	$4.605969e - 02$	12
4	$1.767218e + 02$	$7.115167e - 04$	6
5	$1.767218e + 02$	$2.087841e - 07$	1

Table 1: Convergence of SQP-method for (PT^{MY})

Figure 3: Adjoint state p_γ

Figure 4: Approximation of Lagrange multiplier λ_γ

the errors $\|\bar{u}_\gamma - \bar{u}\|$ and $\|\bar{y}_\gamma - \bar{y}\|$ are decreasing for increasing parameters γ. Moreover, an influence of the discretization error is visible in the difference of the controls.

7   Conclusions

In this article, we have investigated the well-known Moreau-Yosida regularization concept for state-constrained optimal control problems governed by semilinear elliptic equations with respect to sufficient optimality conditions. These are important in numerous ways such as convergence of numerical algorithms, stability with respect to perturbations, and also discretization error estimates, and hence play an essential role in the analysis of numerical methods for nonlinear optimal control problems. For the Moreau-Yosida

γ	$\|\bar{u}_\gamma - \bar{u}\|$	$\|\bar{y}_\gamma - \bar{y}\|$	$\#it_{SQP}$	$\#it_{AS}$
20	$5.114051e - 01$	$1.495545e - 02$	8	34
40	$3.161757e - 01$	$7.893723e - 03$	3	5
80	$1.853852e - 01$	$4.056569e - 03$	2	3
160	$1.059317e - 01$	$2.053420e - 03$	2	3
320	$6.072430e - 02$	$1.030834e - 03$	2	3
640	$3.559594e - 02$	$5.152590e - 04$	2	2
1280	$2.176439e - 02$	$2.568771e - 04$	2	3
2560	$1.442802e - 02$	$1.277243e - 04$	2	2
5120	$1.093492e - 02$	$6.323253e - 05$	2	2
10240	$9.458933e - 03$	$3.104869e - 05$	2	2
20480	$8.883362e - 03$	$1.500880e - 05$	2	2
40960	$8.656709e - 03$	$7.063147e - 06$	2	3

Table 2: Convergence of (PT^{MY})

regularization, a standard second order analysis is not possible, since the regularized objective function is not twice differentiable. However, by the equivalence of the Moreau-Yosida regularization to a specific setting of the virtual control concept, we were able to bypass these restrictions. As a byproduct of our analysis, we obtained that a sufficient condition for the Moreau-Yosida regularization can be deduced from an SSC for the unregularized problem.

References

[1] J.-J. Alibert and J.-P. Raymond. Boundary control of semilinear elliptic equations with discontinuous leading coefficients and unbounded controls. *Numer. Funct. Anal. and Optimization*, 3 & 4:235–250, 1997.

[2] M. Bergounioux, K. Ito, and K. Kunisch. Primal-dual strategy for constrained optimal control problems. *SIAM J. Control and Optimization*, 37:1176–1194, 1999.

[3] J.F. Bonnans and A. Shapiro. *Perturbation Analysis of Optimization Problems.* Springer, New York, 2000.

[4] E. Casas. Control of an elliptic problem with pointwise state constraints. *SIAM J. Control and Optimization*, 4:1309–1322, 1986.

[5] E. Casas. Optimal control in coefficients of elliptic equations with state constraints. *Appl. Math.*, 26:21–37, 1992.

[6] E. Casas. Boundary control of semilinear elliptic equations with pointwise state constraints. *SIAM J. Control and Optimization*, 31:993–1006, 1993.

[7] E. Casas, J.C. de los Reyes, and F. Tröltzsch. Sufficient second-order optimality conditions for semilinear control problems with pointwise state constraints. *SIAM J. on Optimization*, 19(2):616–643, 2008.

[8] E. Casas and F. Tröltzsch. Second order necessary and sufficient optimality conditions for optimization problems and applications to control theory. *SIAM J. Optimization*, 13:406–431, 2002.

[9] E. Casas, F. Tröltzsch, and A. Unger Second order sufficient optimality conditions for some state-constrained control problems of semilinear elliptic equations. *SIAM J. Control and Optimization*, 38:1369–1391, 2000.

[10] S. Cherednichenko, K. Krumbiegel, and A. Rösch. Error estimates for the Lavrentiev regularization of elliptic optimal control problems. *Inverse Problems*, 24(5):055003, 2008.

[11] W. Hager and G. Ianulescu. Dual approximation in optimal control. *SIAM J. on Control and Optimization*, 22:423–466, 1984.

[12] M. Hintermüller, K. Ito, and K. Kunisch. The primal-dual active set method as a semi-smooth Newton method. *SIAM Journal Optimization*, 13:865–888, 2003.

[13] M. Hintermüller and K. Kunisch. Feasible and non-interior path-following in constrained minimization with low multiplier regularity. *SIAM J. Control Optim.*, 45(4):1198–1221, 2006.

[14] M. Hintermüller and K. Kunisch. Path-following methods for a class of constrained minimization problems in function space. *SIAM J. on Optimization*, 17(1):159–187, 2006.

[15] K. Ito and K. Kunisch. Augmented Lagrangian-SQP-methods for non-linear optimal control problems of tracking type. *SIAM J.Control Opt.*, 34(3):874–891, 1996.

[16] K. Ito and K. Kunisch. Semi-smooth Newton methods for state-constrained optimal control problems. *Systems and Control Letters*, 50:221–228, 2003.

[17] K. Krumbiegel and A. Rösch A virtual control concept for state constrained optimal control problems. *COAP*, 43(2):213–233, 2009.

[18] K. Krumbiegel, I. Neitzel, and A. Rösch. Regularization error estimates for semilinear elliptic optimal control problems with pointwise state and control constraints. 2010. Weierstrass Institute for Applied Analysis and Stochastics, WIAS Preprint 1480.

[19] K. Kunisch and A. Rösch. Primal-dual active set strategy for a general class of constrained optimal control problems. *SIAM Journal Optimization*, 13(2):321–334, 2002.

[20] C. Meyer, A. Rösch, and F. Tröltzsch. Optimal control of PDEs with regularized pointwise state constraints. *Computational Optimization and Applications*, 33(2-3):209–228, March 2006.

[21] C. Meyer and F. Tröltzsch. On an elliptic control problem with pointwise mixed control-state constraints. *In A. Seeger, editor, Recent Advances in Optimization. Proceedings of the 12th French-German-Spanish Conference on Optimization*, pages 187–204, 2005.

[22] C. Meyer and I. Yousept. Regularization of state-constrained elliptic optimal control problems with nonlocal radiation interface conditions. *Computational Optimization and Applications*, 44(2):183–212, 2009.

[23] I. Neitzel and F. Tröltzsch. On convergence of regularization methods for nonlinear parabolic optimal control problems with control and state constraints. *Control and Cybernetics*, 37(4):1013–1043, 2008.

[24] I. Neitzel and F. Tröltzsch. On regularization methods for the numerical solution of parabolic control problems with pointwise state constraints. *ESAIM Control, Optimisation and Calculus of Variations*, 15(2):426–453, 2009.

[25] A. Rösch and F. Tröltzsch. Existence of regular Lagrange multipliers for elliptic optimal control problem with pointwise control-state constraints. *SIAM J. Optimization*, 45(2):548–564, 2006.

[26] A. Rösch and F. Tröltzsch. Sufficient second-order optimality conditions for an elliptic optimal control problem with pointwise control-state constraints. *SIAM J. Control and Optimization*, 17(3):776–794, 2006.

[27] A. Rösch and F. Tröltzsch. On regularity of solutions and Lagrange multipliers of optimal control problems for semilinear equations with mixed pointwise control-state constraints. *SIAM J. Control and Optimization*, 46(3):1098–1115, 2007.

[28] A. Schiela. Barrier methods for optimal control problems with state constraints. *SIAM J. Optim*, 20(2):1002–1031, 2009.

[29] F. Tröltzsch. An SQP method for the optimal control of a nonlinear heat equation. *Control and Cybernetics*, 23(1/2):267–288, 1994.

[30] F. Tröltzsch. *Optimal Control of Partial Differential Equations: Theory, Methods and Applications*. AMS, Providence, 2010.

[31] F. Tröltzsch and I. Yousept. A regularization method for the numerical solution of elliptic boundary control problems with pointwise state constraints. *COAP*, 42(1):43–66, 2009.