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ROBUST STABILITY AND ROBUST STABILIZATION OF DISCRETE-TIME LINEAR STOCHASTIC SYSTEMS*

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Abstract

In this paper the problem of robust stabilization of a general class of discrete-time linear stochastic systems subject to Markovian jumping and independent random perturbations is investigated. A stochastic version of the bounded real lemma is derived and the small gain theorem is proved. Finally, methodology for the designing of a stabilizing feedback gain for discrete-time linear stochastic system with structured parametric uncertainties is proposed.

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1 Introduction

In many applications the mathematical model of the controlled process is not completely known. Even if the multiplicative white noise perturbations are introduced in order to model the stochastic environmental perturbations which are hard to quantify, it is also possible that some parametric uncertainties occur in the coefficients of the stochastic system. Thus a robust

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stabilization problem, ask to construct a control law in a static or dynamic feedback form which stabilizes all discrete-time linear stochastic systems into a neighborhood of a given system often called the nominal system.

To be more specific, let us consider the controlled system:

$$x(t+1) = (A_0(\eta_t) + \varDelta_A(t,\eta_t) + \sum_{k=1}^{r} w_k(t)A_k(\eta_t))x(t) + B(\eta_t)u(t) \qquad (1)$$

where $A_k(i)$, $0 \leq k \leq r$, $B(i)$, $1 \leq i \leq N$, ar known matrices of appropriate dimensions, while $\varDelta_A(t,i)$, $t \geq 0$ are unknown matrices. A robust stabilization problem, via state feedback control law, ask to construct a control $u(t) = F(\eta_t)x(t)$ such that the zero state equilibrium of the nominal system

$$x(t+1) = (A_0(\eta_t) + B(\eta_t)F(\eta_t) + \sum_{k=1}^{r} w_k(t)A_k(\eta_t))x(t) \qquad (2)$$

and the zero state equilibrium of the perturbed system

$$x(t+1) = (A_0(\eta_t) + B(\eta_t)F(\eta_t) + \varDelta_A(t,\eta_t) + \sum_{k=1}^{r} w_k(t)A_k(\eta_t))x(t) \qquad (3)$$

are exponentially stable in mean square (ESMS) for all uncertainties $\varDelta_A(t,i)$ in a neighborhood of the origin in $\mathbf{R}^{n\times n}$.

It is known that if the zero state equilibrium of the nominal system (2) is ESMS then the zero state equilibrium of the perturbed system (3) is still ESMS for some "small perturbations" $\varDelta_A(t,i)$. In a robust stability problem, as well as in a robust stabilization problem, the goal is to preserve the stability of the nominal system for the perturbed systems in the case of the variation of the coefficients of the system which are not necessarily small.

In this paper we shall investigate different aspects of the problem of robust stability and robust stabilization of discrete-time linear stochastic systems (1) with structured parametric uncertainties of the form:

$$\varDelta_A(t,\eta_t) = (G_0(\eta_t) + \sum_{k=1}^{r} w_k(t)G_k(\eta_t))\varDelta(\eta_t)C(\eta_t)$$

where the matrices $G_k(i)$, $0 \leq k \leq r$, $C(i)$, $1 \leq i \leq N$ are assumed to be known, and $\varDelta(i)$, $1 \leq i \leq N$ are unknown matrices of appropriate dimensions. We shall see that in the definition of the set of the uncertainties

$\Delta = (\Delta(1), ..., \Delta(N))$ for which the exponential stability in mean square is preserved, an important role is played by the norm of linear operator adequately chosen, named input-output operator.

For this reason we shall start with the proof of the stochastic version of the Bounded Real Lemma. This result allows us to obtain information about the norm of an input-output operator. Further we shall prove a stochastic version of the Small Gain Theorem which is a powerful tool in the estimation of the stability radius of a perturbed system, with structured parametric uncertainties.

Bounded Real Lemma and other H_∞ control problems for discrete-time linear systems affected by independent random perturbations were considered in [1, 4, 5, 11, 12, 13, 15, 17, 19] while in the Markovian case in [2, 3, 14, 16, 18, 20, 21]. The proof of the Bounded Real Lemma in this paper follows the ideas in [1, 16]. In fact, the result of this paper is the discrete-time counter part of the ones developed in chapter 6 in [6].

2 Input-output operators

Let us consider the system (G) with the state space representation:

$$x(t+1) = (A_0(\eta_t) + \sum_{k=1}^{r} w_k(t) A_k(\eta_t)) x(t) + (B_0(\eta_t) + \sum_{k=1}^{r} w_k(t) B_k(\eta_t)) v(t) \quad (4)$$
$$z(t) = C(\eta_t) x(t) + D(\eta_t) v(t)$$

where $x(t) \in \mathbf{R}^n$ is the state of the system, $v(t) \in \mathbf{R}^{m_v}$ is the external input and $z(t) \in \mathbf{R}^{n_z}$ is the output; $\{w(t)\}_{t\geq 0}$, $\big(w(t) = (w_1(t), w_2(t), ..., w_r(t))^T\big)$ is a sequence of independent random vectors and the triple $(\{\eta_t\}_{t\geq 0}, P, \mathcal{D})$ is an homogeneous Markov chain, on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with the set of the states $\mathcal{D} = \{1, 2, ..., N\}$ and the transition probability matrix $P = (p(i,j))_{i,j=1}^{N}$.

Concerning the processes $\{\eta_t\}_{t\geq 0}, \{w(t)\}_{t\geq 0}$ the following assumptions are made:

$\mathbf{H}_1$) $\{w(t)\}_{t\geq 0}$ is a sequence of independent random vectors with the following properties:

$$E[w(t)] = 0, E[w(t)w^T(t)] = I_r, t \geq 0,$$

I_r being the identity matrix of size r.

$\mathbf{H}_2$) The stochastic processes $\{w(t)\}_{t\geq 0}$ and $\{\eta(t)\}_{t\geq 0}$ are independent.

Throughout the paper we assume that together with the hypotheses $H_1) - H_2)$, the Markov chain verifies the additional assumption:

$\mathbf{H_3}$) (i) The transition probability matrix P is a nondegenerate stochastic matrix, that is

$$\sum_{j=1}^{N} p(j,i) > 0, \quad (\forall) \quad 1 \leq i \leq N.$$

(ii) $\pi_0(i) = \mathcal{P}\{\eta_0 = i\} > 0$, $1 \leq i \leq N$.

It is easy to verify by induction that the assumption $\mathbf{H_3}$) holds iff $\pi_t(i) = \mathcal{P}\{\eta_t = i\} > 0$ for all $t \in \mathbf{Z}_+$ and $i \in \mathcal{D}$.

In (4) $A_k(i), B_k(i)$, $0 \leq k \leq r$, $C(i)$, $D(i)$, $1 \leq i \leq N$ are given matrices of appropriate dimensions.

For each $t \geq 0$ we denote $\mathcal{F}_t = \sigma(w(s); 0 \leq s \leq t)$ and $\mathcal{G}_t = \sigma(\eta_s; 0 \leq s \leq t)$. Let $\mathcal{H}_t = \mathcal{F}_t \vee \mathcal{G}_t, t \in \mathbf{Z}_+$. $\tilde{\mathcal{H}}_t = \mathcal{F}_{t-1} \vee \mathcal{G}_t$ if $t \geq 1$ and $\tilde{\mathcal{H}}_0 = \sigma(\eta_0)$.

In the following $\ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^m\}$ stands for the space of all finite sequences $\{v(t)\}_{0 \leq t \leq \tau}$ of m-dimensional random vectors with the properties that for all $0 \leq t \leq \tau$, $v(t)$ is $\tilde{\mathcal{H}}_t$-measurable and $E[|v(t)|^2] < \infty$. Also $\ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^m\}$ is the space of all sequences $\{v(t)\}_{t \geq 0}$ of m-dimensional random vectors with the properties that for all $t \geq 0$, $v(t)$ is $\tilde{\mathcal{H}}_t$-measurable and $\sum_{t=0}^{\infty} E[|v(t)|^2] < \infty$.

In this paper, the inputs $v = \{v(t)\}_{t \geq 0}$ are stochastic processes either in $\ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_v}\}$ for $\tau > 0$ or in $\ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^{m_v}\}$. Both $\ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_v}\}$ and $\ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^{m_v}\}$ are real Hilbert spaces.

The norms induced by the usual inner product on each of this Hilbert space are:

$$||v||_{\ell^2_{\tilde{\mathcal{H}}}\{0,\tau;\mathbf{R}^{m_v}\}} = (\sum_{t=0}^{\tau} E[|v(t)|^2])^{\frac{1}{2}}$$

for all $v \in \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_v}\}$ and

$$||\overline{v}||_{\ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\}} = (\sum_{t=0}^{\infty} E[|\overline{v}(t)|^2])^{\frac{1}{2}}$$

respectively, for all $\overline{v} \in \ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^{m_v}\}$.

Let $x(t, 0, v)$ be the solution of the system (4) corresponding to the input $v = \{v(t)\}_{t \geq 0}$ with the initial condition $x(0, 0, v) = 0$. Let

$$z(t, 0, v) = C(\eta_t)x(t, 0, v) + D(\eta_t)v(t) \tag{5}$$

the corresponding output. One can see that if $v \in \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_v}\}$ for some $\tau \geq 1$, then $x(t, 0, v)$ is $\mathcal{H}_{t-1}$-measurable and $E[|x(t, 0, v)|^2] < \infty$.

Hence from (5) it follows that $\{z(t, 0, v)\}_{0 \leq t \leq \tau} \in \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{n_z}\}$.

Consider the linear system:

$$x(t+1) = (A_0(\eta_t) + \sum_{k=1}^{r} w_k(t)A_k(\eta_t))x(t) \tag{6}$$

Definition 2.1 We say that the zero state equilibrium of the system (6) is exponentially stable in mean square (ESMS) if there exist $\beta \geq 1$ and $q \in (0, 1)$ such that

$$E[|x(t, 0, x_0)|^2 \leq \beta q^t |x_0|^2$$

for all $t \in \mathbf{Z}_+, x_0 \in \mathbf{R}^n$, where $x(t, 0, x_0)$ is the solution of (6) starting from x_0 at time $t = 0$.

Applying Lemma 4.3 in [9] we deduce that if the zero state equilibrium of (6) is ESMS, then there exists $\gamma > 0$ such that

$$\sum_{t=0}^{\infty} E[|z(t, 0, v)|^2] \leq \gamma^2 \sum_{t=0}^{\infty} E[|v(t)|^2] \tag{7}$$

for all $v \in \ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^{m_v}\}$.

It can be remarked that in the absence of the property of the exponential stability in mean square of the linear system (6) one can prove that for each $\tau \geq 1$ there exists $\gamma(\tau) > 0$ such that

$$\sum_{t=0}^{\tau} E[|z(t, 0, v)|^2] \leq \gamma^2(\tau) \sum_{t=0}^{\tau} E[|v(t)|^2] \tag{8}$$

for all $v \in \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_v}\}$.

Since $v \to z(t, 0, v)$ is a linear dependence, we deduce that if the state equilibrium of (6) is ESMS, we may define a linear operator $\mathcal{T} \colon \ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^{m_v}\} \to \ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^{n_z}\}$ by:

$$(\mathcal{T}v)(t) = z(t, 0, v) \tag{9}$$

for all $v \in \ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^{m_v}\}$.

In the absence of the assumption of exponential stability for each $\tau \geq 1$, the equality (9) defines a linear operator $\mathcal{T}_\tau : \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_v}\} \to \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{n_z}\}$.

From (7) and (8) one obtains that $\mathcal{T}$ and $\mathcal{T}_\tau$ are bounded operators.

The linear operator $\mathcal{T}$ introduced by (9) will be called input-output operator defined by the system (4) while, the system (4) is known as a state space representation of the operator $\mathcal{T}$. From the definition of the input-output operator one sees that a such operator maps only finite-energy disturbance signal v into the corresponding finite energy output signal z of the considered system.

To obtain an estimate of a robustness radius of the stabilization achieved by a control law, an important role is played by the norm of an input-output operator. It is well known, from the deterministic context, that the norm of an input-output operator cannot be explicitly computed as in the case of H_2-norms. That is why, we are looking for necessary and sufficient conditions which guarantee the fact that the norm of an input-output operator is smaller than a prescribed level $\gamma > 0$.

Such conditions are provided by the well known Bounded Real Lemma.

In the last part of this section we present several auxiliary results useful in the developments of the next sections.

Firstly, we remark that it is easy to prove the next inequality:

$$\|\mathcal{T}_\tau\| \leq \|\mathcal{T}\| \tag{10}$$

for all $\tau \geq 1$.

Let $\gamma > 0$, $0 < \tau \in \mathbf{Z} \cup \{\infty\}$ and $x_0 \in \mathbf{R}^n$ be arbitrary but fixed. We consider the following cost functionals

$$J_\gamma(\tau, x_0, i, v) = \sum_{t=0}^{\tau} E[|z(t, x_0, v)|^2 - \gamma^2 |v(t)|^2 | \eta_0 = i] \tag{11}$$

$i \in \mathcal{D}$ and

$$\tilde{J}_\gamma(\tau, x_0, v) = \sum_{t=0}^{\tau} E[|z(t, x_0, v)|^2 - \gamma^2 |v(t)|^2] \tag{12}$$

for all $v = \{v(t)\}_{0 \leq t \leq \tau} \in \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_v}\}$.

It should be noted that if (11) and (12) are written for $\tau = +\infty$ we assume tacitly that the zero state equilibrium of the system (6) is ESMS. It is clear that $\|\mathcal{T}_\tau\| \le \gamma$ if and only if $\tilde{J}_\gamma(\tau, 0, v) \le 0$ for all $v \in \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_v}\}$ and $\|\mathcal{T}\| \le \gamma$ if and only if $\tilde{J}_\gamma(\infty, 0, v) \le 0$ for all $v \in \ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^{m_v}\}$.

Throughout this paper $\mathcal{S}_n^N = \mathcal{S}_n \oplus \mathcal{S}_n \oplus .... \oplus \mathcal{S}_n, \mathcal{S}_n$ being the Hilbert space of $n \times n$ symmetric matrices. If $X(t) = (X(t,1), ..., X(t,N)) \in \mathcal{S}_n^N$ we shall use the notations:

$$[\varPi X(t+1)](i) = \begin{pmatrix} \varPi_{1i}X(t+1) & \varPi_{2i}X(t+1) \\ (\varPi_{2i}X(t+1))^T & \varPi_{3i}X(t+1) \end{pmatrix} = \tag{13}$$
$$\sum_{k=0}^{r} \begin{pmatrix} A_k(i) & B_k(i) \end{pmatrix}^T \mathcal{E}_i(X(t+1)) \begin{pmatrix} A_k(i) & B_k(i) \end{pmatrix}$$

with $\mathcal{E}_i(X(t+1)) = \sum\limits_{j=1}^{N} p(i,j)X(t+1,j)$, $1 \le i \le N$.

Let $F(t) = (F(t,1), ..., F(t,N))$, $F(t,i) \in \mathbf{R}^{m_v \times n}$, $0 \le t \le \tau$, $\tau \ge 1$.

Let $X_F^\gamma(t) = (X_F^\gamma(t,1), ..., X_F^\gamma(t,N))$ be the solution of the following problem with the given final value

$$\begin{aligned} X(t,i) &= \sum_{k=0}^{r}(A_k(i) + B_k(i)F(t,i))^T \mathcal{E}_i(X(t+1))(A_k(i) + \\ &\quad + B_k(i)F(t,i)) + (C(i) + D(i)F(t,i))^T(C(i) + \\ &\quad + D(i)F(t,i)) - \gamma^2 F^T(t,i)F(t,i) \\ X(\tau+1, i) &= 0, 1 \le i \le N. \end{aligned} \tag{14}$$

Let $x_F = \{x_F(t)\}_{0 \le t \le \tau+1}$ be the solution of the following problem with the initial given value:

$$\begin{aligned} x(t+1) &= [A_0(\eta_t) + B_0(\eta_t)F(t,\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t) + \\ &\quad + B_k(\eta_t)F(t,\eta_t))]x(t) + [B_0(\eta_t) + \\ &\quad + \sum_{k=1}^{r} w_k(t)B_k(\eta_t)]v(t) \\ x(0) &= x_0. \end{aligned} \tag{15}$$

Applying Lemma 3.2 in [9] we obtain:

Lemma 2.1. *Let* $F = \{F(t)\}_{0\leq t\leq \tau}$, $F(t) = (F(t,1), ..., F(t,N))$, $F(t,i) \in \mathbf{R}^{m_v\times n}$ *be a sequence of gain matrices. If* $\{X_F^\gamma(t)\}_{0\leq t\leq \tau+1}$ *is the solution of the problem (14), then we have:*

$$J_\gamma(\tau, x_0, i, v + Fx_F) = x_0^T X_F^\gamma(0,i)x_0 +$$

$$\sum_{t=0}^{\tau} E[v^T(t)\mathfrak{H}_\gamma(X_F^\gamma(t+1),\eta_t)v(t) + 2v^T(t)\mathfrak{N}(X_F^\gamma(t+1),\eta_t)x_F(t)|\eta_0 = i]$$

for all $i \in \mathcal{D}$, $x_0 \in \mathbf{R}^n$, $v \in \ell^2_{\tilde{\mathcal{H}}}\{0,\tau;\mathbf{R}^{m_v}\}$, $x_F(t)$ *being the solution of the problem (15) corresponding to the input* v *and*

$$\mathfrak{H}_\gamma(X_F^\gamma(t+1),i) = \Pi_{3i}X_F^\gamma(t+1) + D^T(i)D(i) - \gamma^2 I_{m_v} \tag{16}$$

$$\mathfrak{N}(X_F^\gamma(t+1),i) = (\Pi_{2i}X_F^\gamma(t+1) + C^T(i)D(i))^T + \mathfrak{H}_\gamma(X_F^\gamma(t+1),i)F(t,i). \tag{17}$$

Proof may be done by direct calculations. It is omitted for shortness.

Now we prove:

Proposition 2.2. *If for an integer* $\tau \geq 1$ *and a real number* $\gamma > 0$, $\|\mathcal{T}_\tau\| < \gamma$, *then*

$$\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X_F^\gamma(t+1))B_k(i) + D^T(i)D(i) - \gamma^2 I_{m_v} \leq -\varepsilon_0 I_{m_v} \tag{18}$$

for all $0 \leq t \leq \tau$, *with* $\varepsilon_0 \in (0, \gamma^2 - \|\mathcal{T}_\tau\|^2)$.

Proof. Let us remark that (18) can be rewritten

$$\mathfrak{H}_\gamma(X_F^\gamma(t+1),i) \leq -\varepsilon_0 I_{m_v}, \;\; 0 \leq t \leq \tau. \tag{19}$$

We prove (19) in two steps. First we show that

$$\mathfrak{H}_\gamma(X_F^\gamma(t+1),i) \leq 0 \tag{20}$$

for all $0 \leq t \leq \tau$, $i \in \mathcal{D}$.

In the second step, using (20), we shall show the validity of (19). Let us assume by contrary that (20) is not true. This implies that there exist $0 \le t_0 \le \tau$, $i_0 \in \mathcal{D}$ and $v \in \mathbf{R}^{m_v}$ with $|v| = 1$, such that

$$v^T \mathfrak{H}_\gamma(X_F^\gamma(t_0+1), i_0)v = \nu_0 > 0 \tag{21}$$

for a $\nu_0 > 0$.

Let $\hat{v} = \{\hat{v}(t)\}_{0 \le t \le \tau}$ defined as follows:

$$\hat{v}(t) = \begin{cases} \chi_{\{\eta_{t_0}=i_0\}} v, & if \quad t = t_0 \\ 0, \quad if \quad t \ne t_0 \end{cases}$$

It is clear that $\hat{v} \in \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_v}\}$.

Let also $\hat{x} = \{\hat{x}(t)\}_{0 \le t \le \tau+1}$ be the solution of (15) with zero initial value and corresponding to the input $\hat{v}$.

Let $\check{v}(t) = \hat{v}(t) + F(t, \eta_t)\hat{x}(t)$. It is clear that $\check{v} = \{\check{v}(t)\}_{0 \le t \le \tau}$ lies in $\ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_v}\}$.

Hence

$$\tilde{J}_\gamma(\tau, 0, \check{v}) = \|\mathcal{T}_\tau \check{v}\|^2_{\ell^2_{\tilde{\mathcal{H}}}\{0,\tau;\mathbf{R}^{n_z}\}} - \gamma^2 \|\check{v}\|^2_{\ell^2_{\tilde{\mathcal{H}}}\{0,\tau;\mathbf{R}^{m_v}\}} \le 0. \tag{22}$$

Since $\hat{x}(t) = 0$ for $t \le t_0$ and $\tilde{J}_\gamma(\tau, x_0, v) = \sum_{i=1}^{N} \pi_0(i) J_\gamma(\tau, x_0, i, v)$ we obtain via Lemma 2.1 and the inequality (22) that

$$0 \ge \sum_{i=1}^{N} \pi_0(i) J_\gamma(\tau, 0, i, \check{v}) = \sum_{i=1}^{N} \pi_0(i) E[v^T \mathfrak{H}_\gamma(X_F^\gamma(t_0+1), i_0) v \chi_{\{\eta_{t_0}=i_0\}} | \eta_0 = i] = \tag{23}$$

$$\nu_0 \sum_{i=1}^{N} \pi_0(i) E[\chi_{\{\eta_{t_0}=i_0\}} | \eta_0 = i] = \nu_0 \pi_{t_0}(i_0) > 0.$$

This is a contradiction, hence (20) is correct. Note that $\pi_{t_0}(i_0) > 0$ is a consequence of the assumption $\mathbf{H_3}$).

Let $0 < \varepsilon_0 < \gamma^2 - \|\mathcal{T}_\tau\|^2$. Set $\tilde{\gamma} = (\gamma^2 - \varepsilon_0)^{\frac{1}{2}}$. We have $\|\mathcal{T}_\tau\| < \tilde{\gamma}$. Hence (20) is fulfilled for γ replaced by $\tilde{\gamma}$. This means that

$$\mathfrak{H}_{\tilde{\gamma}}(X_F^{\tilde{\gamma}}(t+1), i) \le 0, \quad i \in \mathcal{D}, \quad 0 \le t \le \tau. \tag{24}$$

We deduce recursively that

$$X_F^{\tilde{\gamma}}(t,i) \geq X_F^{\gamma}(t,i), \;\; 0 \leq t \leq \tau, \;\; i \in \mathcal{D}. \tag{25}$$

Therefore

$$\mathfrak{H}_{\tilde{\gamma}}(X_F^{\gamma}(t+1),i) \leq \mathfrak{H}_{\tilde{\gamma}}(X_F^{\tilde{\gamma}}(t+1),i) \leq 0.$$

Having in mind the definition of $\tilde{\gamma}$ we obtain that

$$\mathfrak{H}_{\gamma}(X_F^{\gamma}(t+1),i) \leq -\varepsilon_0 I_{m_v}, \;\; 0 \leq t \leq \tau, \;\; i \in \mathcal{D}$$

which completes the proof.

Let $X^{\gamma}(t) = (X^{\gamma}(t,1),...,X^{\gamma}(t,N))$ be the solution of the problem (14) in the special case $F(t) = 0$. One obtains recursively for $t \in \{\tau+1, \tau, ..., 0\}$, $i \in \mathcal{D}$ that $X^{\gamma}(t,i) \geq 0$. Applying Proposition 2.2 for $X^{\gamma}(t)$ instead of $X_F^{\gamma}(t)$ one obtains:

Corollary 2.3 *If there exists an integer $\tau \geq 1$ such that $\|\mathcal{T}_\tau\| < \gamma$, then*

$$\gamma^2 I_{m_v} - D^T(i)D(i) > 0, \quad i \in \mathcal{D}.$$

3 Stochastic version of Bounded Real Lemma

In the developments of this section an important role is played by the following backward discrete-time stochastic generalized Riccati equations (DTSGRE):

$$\begin{aligned} X(t,i) \;\; = \;\; & \sum_{k=0}^{r} A_k^T(i)\mathcal{E}_i(X(t+1))A_k(i) + C^T(i)C(i) - \\ & -(\sum_{k=0}^{r} A_k^T(i)\mathcal{E}_i(X(t+1))B_k(i) + \\ & +C^T(i)D(i))(\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X(t+1))B_k(i) + \\ & +D^T(i)D(i) - \gamma^2 I_{m_v})^{-1} \times (\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X(t+1))A_k(i) \\ & +D^T(i)C(i)), \;\; 1 \leq i \leq N. \end{aligned} \tag{26}$$

Using the notation introduced in (13) we may rewrite (26) in the following compact form:

$$X(t)=\Pi_1 X(t+1)+M-(\Pi_2 X(t+1)+L)(\Pi_3 X(t+1)+R)^{-1}(\Pi_2 X(t+1)+L)^T \quad (27)$$

where

$$M=(M(1),M(2),...,M(N))\in\mathcal{S}_n^N,\quad M(i)=C^T(i)C(i),$$

$$L=(L(1),L(2),...,L(N))\in\mathcal{M}_{n,m_v}^N,\quad L(i)=C^T(i)D(i),$$

$$R=(R(1),R(2),...,R(N))\in\mathcal{S}_{m_v}^N,\quad R(i)=D^T(i)D(i)-\gamma^2 I_{m_v}.$$

For each integer $\tau\geq 1$, let $X_\tau(t)=(X_\tau(t,1),...,X_\tau(t,N))$ be the solution of DTSGRE (26) with the final value

$$X_\tau(\tau+1,i)=0,\ \ i\in\mathcal{D}. \tag{28}$$

By using Proposition 2.2 we can prove by induction the next result:

Lemma 3.1 *If for an integer $\tau\geq 1$ and a real number $\gamma>0$ we have $\|\mathcal{T}_\tau\|<\gamma$, then the solution $X_\tau(t)$ of the problem (26)-(28) is well defined for all $0\leq t\leq\tau$ and it has the properties:*

$X_\tau(t,i)\geq 0$ and

$$\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X_\tau(t+1))B_k(i)+D^T(i)D(i)-\gamma^2 I_{m_v}\leq -\varepsilon_0 I_{m_v} \tag{29}$$

for all $0\leq t\leq\tau$, $i\in\mathcal{D}$, where $\varepsilon_0\in(0,\gamma^2-\|\mathcal{T}_\tau\|^2)$.

Lemma 3.2 *Assume: a) the zero state equilibrium of the system (6) is ESMS,*

b) the input-output operator $\mathcal{T}$ associated to the system (4) satisfies $\|\mathcal{T}\|<\gamma$.

Then there exists $\rho>0$ such that $\tilde{J}_\gamma(\infty,x_0,v)\leq\rho|x_0|^2$ for all $x_0\in\mathbf{R}^n$ and $v\in\ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\}$.

Proof. Under the assumption a) and Theorem 3.5 in [7] it follows that the linear equation

$$Z(i)=\sum_{k=0}^{r} A_k^T(i)\mathcal{E}_i(Z)A_k(i)+C^T(i)C(i),\ \ 1\leq i\leq N. \tag{30}$$

has a unique solution $Z = (Z(1), Z(2), ..., Z(N)) \in \mathcal{S}_n^{N+}$. We recall that under the assumption a), if $v \in \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\}$ then, from Lemma 4.3 in [9], we have $\lim_{t\to\infty} E[|x(t,x_0,v)|^2] = 0$.

Applying Lemma 2.1 in the special case $F(t,i) = 0, X(t,i) = Z(i)$ and taking the limit for $\tau \to \infty$, one gets:

$$\begin{aligned}\tilde{J}_\gamma(\infty,x_0,v) \;&= \sum_{i=1}^{N} \pi_0(i) x_0^T Z(i) x_0 + \sum_{t=0}^{\infty} E[v^T(t)\mathfrak{H}_\gamma(Z,\eta_t)v(t) + \\ &\quad +2x^T(t,x_0,v)\mathfrak{N}^T(Z,\eta_t)v(t)]\end{aligned} \tag{31}$$

for all $v \in \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\}$ and all $x_0 \in \mathbf{R}^n$, where $\mathfrak{N}(Z,i)$ and $\mathfrak{H}_\gamma(Z,i)$ are as in (16) and (17) with $Z(i)$ instead of $X_F^\gamma(t,i)$.

Let ε be such that $\|\mathcal{T}\|^2 < \gamma^2 - \varepsilon^2$. Thus we may write:

$$\tilde{J}_\gamma(\infty,0,v) = \|\mathcal{T}v\|^2_{\ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{n_z}\}} - \gamma^2\|v\|^2_{\ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\}} \le -\varepsilon^2\|v\|^2_{\ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\}}.$$

Therefore

$$\tilde{J}_\gamma(\infty;x_0,v) \le \sum_{i=1}^{N} \pi_0(i)x_0^T Z(i)x_0 + \sum_{t=0}^{\infty} E[2x^T(t,x_0,0)\mathfrak{N}^T(Z,\eta_t)v(t) - \varepsilon^2|v(t)|^2]$$

or

$$\tilde{J}_\gamma(\infty;x_0,v) \le \sum_{i=1}^{N} \lambda_{max}(Z(i))|x_0|^2 + \frac{1}{\varepsilon^2}\sum_{t=0}^{\infty} E[|\mathfrak{N}(Z,\eta_t)x(t,x_0,0)|^2]$$

$$-\sum_{t=0}^{\infty} E[|\varepsilon v(t) - \frac{1}{\varepsilon}\mathfrak{N}(Z,\eta_t)x(t,x_0,0)|^2].$$

Let $\nu > 0$ such that $max|\mathfrak{N}(Z,i)| \le \nu$. Thus we have

$$\tilde{J}_\gamma(\infty;x_0,v) \le \sum_{i=1}^{N} \lambda_{max}(Z(i))|x_0|^2 + \frac{\nu^2}{\varepsilon^2}\sum_{t=0}^{\infty} E[|x(t,x_0,0)|^2]. \tag{32}$$

From the assumption a) we deduce that there exists $\rho_1 > 0$ not depending upon x_0 such that $\sum_{t=0}^{\infty} E[|x(t,x_0,0)|^2] \le \rho_1|x_0|^2$. Introducing the last inequality in (32) one obtains the inequality from the statement with $\rho = \sum_{i=1}^{N} \lambda_{\max} Z(i) + \rho_1 \frac{\nu^2}{\varepsilon^2}$. Thus the proof is complete.

If $X_\tau(t)$, $0 \le t \le \tau+1$ is the solution of the problem with given final value (26)-(28) we define $K(t) = (K(t,1), ..., K(t,N))$ by

$$K(t,i) = X_\tau(\tau+1-t, i). \tag{33}$$

We see that $K(0,i) = X_\tau(\tau+1,i) = 0$, $1 \le i \le N$. Also, by direct calculation one obtains that $K = \{K(t)\}_{t\ge 0}$ solves the following forward nonlinear equation on $\mathcal{S}_n^N$:

$$\begin{aligned} K(t+1,i) = \Pi_{1i}K(t) + C^T(i)C(i) - (\Pi_{2i}K(t) + C^T(i)D(i))(\Pi_{3i}K(t) \\ +D^T(i)D(i) - \gamma^2 I_{m_v})^{-1}(\Pi_{2i}K(t) + C^T(i)D(i))^T. \end{aligned} \tag{34}$$

Let us denote $K_0(t) = (K_0(t,1), ..., K_0(t,N))$ the solution of (34) with given initial value $K_0(0,i) = 0$, $1 \le i \le N$.

Several properties of the solution $K_0(t)$ are summarized in the next result:

Proposition 3.3 *Assume: a) the zero state equilibrium of (6) is ESMS. b)* $\|\mathcal{T}\| < \gamma$.

Then the solution $K_0(t)$ *of the forward equation (34) with the given initial value* $K_0(0,i) = 0$ *is defined for all* $t \ge 0$. *It has the properties:*

(i)

$$\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(K_0(t))B_k(i) + D^T(i)D(i) - \gamma^2 I_{m_v} \le -\varepsilon_0 I_{m_v} \tag{35}$$

where $\varepsilon_0 \in (0, \gamma^2 - \|\mathcal{T}\|^2)$.

(ii) $0 \le K_0(\tau,i) \le K_0(\tau+1,i) \le cI_n, \quad (\forall)\ t,i \in \mathbf{Z}_+ \times \mathcal{D}$, *where* $c > 0$ *is a constant not depending upon* t, i.

Proof. Based on (10) we obtain that $\|\mathcal{T}_\tau\| \le \|\mathcal{T}\| < \gamma$ for all $\tau \ge 1$. Therefore, we deduce, via Lemma 3.1, that for any integer $\tau \ge 1$ the solution $X_\tau(t)$ of the problem with given final value (26), (28) is well defined for $0 \le t \le \tau+1$ and it verifies (29). Thus we deduce via (33) that $K_0(t)$ is well defined for all $t \ge 0$. If $0 < \varepsilon_0 < \gamma^2 - \|\mathcal{T}\|^2$ it follows that $\varepsilon_0 < \gamma^2 - \|\mathcal{T}_\tau\|^2$ for all $\tau \ge 1$.

Hence in (29) we may choose ε_0 independent of τ. Writing (29) for $t = 0$ and taking into account that $K_0(\tau,i) = X_\tau(1,i)$ we obtain that (i) is fulfilled. Further, from (35) and (34) we deduce that $K_0(t,i) \ge 0$ for all $(t,i) \in \mathbf{Z}_+ \times \mathcal{D}$.

Let $X_\tau(t)$ and $X_{\tau+1}(t)$ be the solutions of the DTSGRE (26) with the final value $X_\tau(\tau+1)=0$ and $X_{\tau+1}(\tau+2)=0$ in $\mathcal{S}_n^N$. Under the considered assumptions we know that these two solutions are well defined for $0\le t\le \tau+1$ and $0\le t\le\tau+2$, respectively.

Let $Z_\tau(t,i)=X_{\tau+1}(t,i)-X_\tau(t,i)$, $0\le t\le\tau+1$, $1\le i\le N$.

We can deduce recursively that $Z_\tau(t,i)\ge 0$ for $0\le t\le\tau+1$.

This means that $X_\tau(t,i)\le X_{\tau+1}(t,i)$, $0\le t\le\tau+1$, $i\in\mathcal{D}$. Particularly $X_\tau(1,i)\le X_{\tau+1}(1,i)$, $i\in\mathcal{D}$.

Using (33) we see that the above inequality is equivalent to $K_0(\tau,i)\le K_0(\tau+1,i)$, $i\in\mathcal{D}$, $\tau\ge 1$. Further we consider $v_\tau=\{v_\tau(t)\}_{0\le t\le\tau}$ defined by $v_\tau(t)=F_\tau(t,\eta_t)x_\tau(t)$ where $x_\tau(t)$ is the solution of (4) corresponding to $v_\tau(t)$ and F_τ is defined by

$$\begin{aligned}F_\tau(t,i)=&-(\sum_{k=0}^{r}B_k^T(i)\mathcal{E}_i(X_\tau(t+1))B_k(i)+D^T(i)D(i)-\gamma^2 I_{m_v})^{-1}\\&\times(\sum_{k=0}^{r}B_k^T(i)\mathcal{E}_i(X_\tau(t+1))A_k(i)+D^T(i)C(i)),\quad i\in\mathcal{D}.\end{aligned}\tag{36}$$

Let $\overline{v}_\tau=\{\overline{v}_\tau(t)\}_{t\ge 0}\in\ell^2_{\mathcal{H}}\{0,\infty;\mathbf{R}^{m_v}\}$ be the natural extension of v_τ taking $\overline{v}_\tau(t)=0$ for $t\ge\tau+1$.

Applying Lemma 3.2 from above and Lemma 3.2 in [9] we may write successively

$$\pi_0(i)x_0^TX_\tau(0,i)x_0\le E[x_0^TX_\tau(0,\eta_0)x_0]=\tilde{J}_\gamma(\tau,x_0,v_\tau)\le\tilde{J}_\gamma(\infty;x_0,\overline{v}_\tau)\le\rho|x_0|^2$$

for all $x_0\in\mathbf{R}^n$, $i\in\mathcal{D}$. Hence

$$\pi_0(i)x_0^TX_\tau(0,i)x_0\le\rho|x_0|^2\tag{37}$$

for all $x_0\in\mathbf{R}^n$, $i\in\mathcal{D}$ and for all initial distribution $\pi_0=(\pi_0(1),...,\pi_0(N))$ with $\pi_0(i)>0$. Particulary, (37) is valid for the special case $\pi_0(i)=\frac{1}{N}$.

This leads to $x_0^TX_\tau(0,i)x_0\le N\rho|x_0|^2$ for all $i\in\mathcal{D}$. Thus $x_0^TK_0(\tau+1,i)x_0\le c|x_0|^2$ $(\forall)$ $\tau\ge 1$, $i\in\mathcal{D}$, $x_0\in\mathbf{R}^n$ where $c=N\rho$. Thus the proof is complete.

Let us consider the following system of discrete-time coupled algebraic Riccati equations (DTSARE):

$$\begin{aligned} X(i) \;=\; & \sum_{k=0}^{r} A_k^T(i)\mathcal{E}_i(X)A_k(i) + C^T(i)C(i) - \\ & -(\sum_{k=0}^{r} A_k^T(i)\mathcal{E}_i(X)B_k(i) + C^T(i)D(i)) \\ & \times(\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X)B_k(i) + D^T(i)D(i) - \gamma^2 I_{m_v})^{-1} \\ & (\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X)A_k(i) + D^T(i)C(i)). \end{aligned} \tag{38}$$

We have:

Corollary 3.4 *Under the assumptions of the Proposition 3.3 the DTSARE (38) has a solution* $\tilde{X} = (\tilde{X}(1), ..., \tilde{X}(N)) \in \mathcal{S}_n^{N+}$ *with the additional property:*

$$\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X)B_k(i) + D^T(i)D(i) - \gamma^2 I_{m_v} < 0, 1 \le i \le N. \tag{39}$$

Proof. From Proposition 3.3 one obtains that the sequences $\{K_0(\tau, i)\}_{\tau\ge 1}$, $1 \le i \le N$ are convergent. Let $\tilde{X}(i) = \lim_{\tau\to\infty} K_0(\tau, i)$. Taking the limit for $t \to \infty$ in (34) one obtains that $\tilde{X} = (\tilde{X}(1), ..., \tilde{X}(N))$ is a solution of DTSARE (38). Finally, taking the limit for $t \to \infty$ in (35) we deduce that (39) is fulfilled. The proof ends.

We say that a solution $X_s = (X_s(1), ..., X_s(N))$ of the DTSARE (38) is a stabilizing solution if the zero state equilibrium of the closed-loop system $x_s(t+1) = [A_0(\eta_t) + B_0(\eta_t)F_s(\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t) + B_k(\eta_t)F_s(\eta_t))]x_s(t)$ is ESMS, where

$$\begin{aligned} F_s(i) = & -(\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X_s)B_k(i) + D^T(i)D(i) - \gamma^2 I_{m_v})^{-1} \\ & \times(\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X_s)A_k(i) + D^T(i)C(i)) \end{aligned} \tag{40}$$

$1 \le i \le N$.

Before to prove the main result of this section we recall several definitions and results from [10].

Consider the discrete-time general Riccati equation

$$X = \Pi_1 X + M - (L + \Pi_2 X)(R + \Pi_3 X)^{-1}(L + \Pi_2 X)^T \tag{41}$$

where

$$X \to \Pi X = \begin{pmatrix} \Pi_1 X & \Pi_2 X \\ (\Pi_2 X)^T & \Pi_3 X \end{pmatrix} \tag{42}$$

is a linear and positive operator defined on $\mathcal{S}_n^N$ taking values in $\mathcal{S}_{n+m_v}^N$ and $\mathcal{Q} = \begin{pmatrix} M & L \\ L^T & R \end{pmatrix} \in \mathcal{S}_{n+m_v}^N$. To the pair $\Sigma = (\Pi, \mathcal{Q})$ (which defines the equation (41)) we associate the so called dissipation operator $\mathbf{D}^\Sigma : \mathcal{S}_n^N \to \mathcal{S}_{n+m_v}^N$ by: $\mathbf{D}^\Sigma X = (\mathbf{D}_1^\Sigma X, ..., \mathbf{D}_N^\Sigma X)$ where

$$\mathbf{D}_i^\Sigma X = \begin{pmatrix} \Pi_{1i} X - X(i) + M(i) & L(i) + \Pi_{2i} X \\ (L(i) + \Pi_{2i} X)^T & R(i) + \Pi_{3i} X \end{pmatrix} \tag{43}$$

for all $X = (X(1), ..., X(N)) \in \mathcal{S}_n^N$.

If $\Pi : \mathcal{S}_n^N \to \mathcal{S}_{n+m_v}^N$ is a linear operator and $F = (F(1), F(2), ..., F(N))$, $F(i) \in \mathbf{R}^{m_v \times n}$ then we denote $\Pi_F X = ((\Pi_F X)(1), (\Pi_F X)(2), ..., (\Pi_F X)(N))$ with

$$(\Pi_F X)(i) = \begin{pmatrix} I_n & F^T(i) \end{pmatrix} \begin{pmatrix} \Pi_{1i} X & \Pi_{2i} X \\ (\Pi_{2i} X)^T & \Pi_{3i} X \end{pmatrix} \begin{pmatrix} I_n \\ F(i) \end{pmatrix}. \tag{44}$$

Definition 3.1 We say that a linear and positive operator $\Pi : \mathcal{S}_n^N \to \mathcal{S}_{n+m_v}^N$ is stabilizable if there exists $F = (F(1), F(2), ..., F(N))$, $F(i) \in \mathbf{R}^{m_v \times n}$ with the property that the eigenvalues of the operator Π_F are located in the inside of the disk $|\lambda| < 1$.

It should be remarked that in the special case of Π introduced by (13) the concept of stabilizability introduced in Definition 3.1 is equivalent to the concept of stochastic stabilizability introduced in [8].

Definition 3.2 A solution $X_s = (X_s(1), ..., X_s(N))$ of (41) is a stabilizing solution if the eigenvalues of the operator Π_{F_s} are in the inside of the disk $|\lambda| < 1$, where Π_{F_s} is defined as in (44) with F replaced by $F_s = (F_s(1), ..., F_s(N))$,

$$F_s(i) = -(R(i) + \Pi_{3i} X_s)^{-1}(L(i) + \Pi_{2i} X_s)^T. \tag{45}$$

The next result provides a set of necessary and sufficient conditions for the existence of a stabilizing solution of (41).

Theorem 3.5 ([10]) *With the considered notations, the following are equivalent:*

(i) the linear and positive operator Π is stabilizable and there exists $\hat{X} \in \mathcal{S}_n^N$,
$\hat{X} = (\hat{X}(1), \hat{X}(2), ..., \hat{X}(N))$ such that

$$\mathbf{D}_i^{\Sigma}\hat{X} > 0, (\forall) \quad i \in \{1, 2, ..., N\}; \tag{46}$$

(ii) the algebraic Riccati equation (41) has a stabilizing solution $X_s = (X_s(1), X_s(2), ..., X_s(N))$ which satisfies

$$R(i) + \Pi_{3i}X_s > 0, 1 \leq i \leq N. \tag{47}$$

The main result of this section is:

Theorem 3.6 (Bounded Real Lemma) *Under the considered assumptions, for a given scalar $\gamma > 0$, the following are equivalent:*

(i) the zero state equilibrium of (6) is ESMS and the input-output operator $\mathcal{T}$ defined by the system (4) satisfies $\|\mathcal{T}\| < \gamma$.

(ii) there exists $X = (X(1), ..., X(N)) \in \mathcal{S}_n^N$, $X(i) > 0$, $1 \leq i \leq N$, which solves the following system of LMI's:

$$\begin{pmatrix} \Pi_{1i}X - X(i) + C^T(i)C(i) & \Pi_{2i}X + C^T(i)D(i) \\ (\Pi_{2i}X + C^T(i)D(i))^T & \Pi_{3i}X + D^T(i)D(i) - \gamma^2 I_{m_v} \end{pmatrix} < 0,$$
$$1 \leq i \leq N, \tag{48}$$

where the operators Π_{li} are introduced by (13);

(iii) the DTSARE (38) has a stabilizing solution $\tilde{X} = (\tilde{X}(1), ..., \tilde{X}(N)) \in \mathcal{S}_n^N$ with $\tilde{X}(i) \geq 0, 1 \leq i \leq N$ which satisfies (39);

(iv) there exists $Y = (Y(1), Y(2), ..., Y(N)) \in \mathcal{S}_n^N$, $Y(i) > 0$, $1 \leq i \leq N$, which solves the following system of LMIs

$$\begin{pmatrix} -Y(i) & \Psi_{0i}(Y) & \Psi_{1i}(Y) & ... & \Psi_{ri}(Y) & Y(i)C^T(i) \\ \Psi_{0i}^T(Y) & \mathfrak{G}_{00}(i) - \mathcal{Y} & \mathfrak{G}_{01}(i) & ... & \mathfrak{G}_{0r}(i) & \mathfrak{G}_{0r+1}(i) \\ \Psi_{1i}^T(Y) & \mathfrak{G}_{01}^T(i) & \mathfrak{G}_{11}(i) - \mathcal{Y} & ... & \mathfrak{G}_{1r}(i) & \mathfrak{G}_{1r+1}(i) \\ ... & ... & ... & ... & ... & ... \\ \Psi_{ri}^T(Y) & \mathfrak{G}_{0r}^T(i) & \mathfrak{G}_{1r}^T(i) & ... & \mathfrak{G}_{rr}(i) - \mathcal{Y} & \mathfrak{G}_{rr+1}(i) \\ C(i)Y(i) & \mathfrak{G}_{0r+1}^T(i) & \mathfrak{G}_{1r+1}^T(i) & ... & \mathfrak{G}_{rr+1}^T(i) & D(i)D^T(i) - \gamma^2 I_{n_z} \end{pmatrix} < 0 \tag{49}$$

where

$$\Psi_{ki}(Y) = \left(\sqrt{p(i,1)}Y(i)A_k^T(i) \quad \sqrt{p(i,2)}Y(i)A_k^T(i) \; ... \; \sqrt{p(i,N)}Y(i)A_k^T(i) \right),$$

$\mathcal{Y} = diag(Y(1), ..., Y(N)) \in \mathcal{S}_{nN}$

$$\mathfrak{G}_{lk}(i) = \mathfrak{I}^T(i)B_l(i)B_k^T(i)\mathfrak{I}(i), 0 \leq l \leq k \leq r,$$

$$\mathfrak{G}_{lr+1}(i) = \mathfrak{I}^T(i)B_l(i)D^T(i)$$

and

$$\mathfrak{I}(i) = \left(\sqrt{p(i,1)}I_n \quad \sqrt{p(i,2)}I_n \quad ... \quad \sqrt{p(i,N)}I_n \right).$$

Proof. Let us assume that (i) holds. If $\delta > 0$ denote $\mathcal{T}_\delta : \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\} \to \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{n+n_z}\}$ the linear operator defined by $v \to (\mathcal{T}_\delta v)(t) = C_\delta(\eta_t)x(t,0,v) + D_\delta(\eta_t)v(t)$ where $x(t,0,v)$ is the zero initial value solution of (4) corresponding to the input v and $C_\delta(i) = \begin{pmatrix} C(i) \\ \delta I_n \end{pmatrix}$, $D_\delta(i) = \begin{pmatrix} D(i) \\ 0 \end{pmatrix}$. Based on (7) we deduce that for $\delta > 0$ sufficiently small we have $\|\mathcal{T}_\delta\| < \gamma$. Applying Corollary 3.4 we deduce that there exists $X_\delta = (X_\delta(1), ..., X_\delta(N)), X_\delta(i) \geq 0$ solving the DTSARE:

$$\begin{aligned} X_\delta(i) = \Pi_{1i}X_\delta - (\Pi_{2i}X_\delta + C^T(i)D(i))(\Pi_{3i}X_\delta + D^T(i)D(i) - \gamma^2 I_{m_v})^{-1} \\ (\Pi_{2i}X_\delta + C^T(i)D(i))^T + C^T(i)C(i) + \delta^2 I_n\,,\; 1 \leq 1 \leq N, \end{aligned} \tag{50}$$

with additional property

$$\Pi_{3i}X_\delta + D^T(i)D(i) - \gamma^2 I_{m_v} < 0, \quad 1 \leq 1 \leq N. \tag{51}$$

Since the right hand side of (50) is positive definite it follows that $X_\delta(i) > 0$, $1 \leq i \leq N$.

Also (50) implies

$$\begin{gathered} \Pi_{1i}X_\delta - X_\delta(i) + C^T(i)C(i) - (\Pi_{2i}X_\delta + \\ +C^T(i)D(i))(\Pi_{3i}X_\delta + D^T(i)D(i) - \gamma^2 I_{m_v})^{-1} \\ (\Pi_{2i}X_\delta + C^T(i)D(i))^T < 0, \;\; 1 \leq 1 \leq N. \end{gathered} \tag{52}$$

By a Schur complement technique one obtains that (51) and (52) are equivalent to (48) and thus the proof of the implication $(i) \to (ii)$ is complete.

To prove the converse implication, $(ii) \to (i)$ we remark that if (ii) is fulfilled then the (1;1) block of (48) is negative definite. Thus we obtained that there exists $X = (X(1), ..., X(N)) \in \mathcal{S}_n^N$ with $X(i) > 0$, such that $X(i) > \sum_{k=0}^{r} A_k^T(i)\mathcal{E}_i(X)A_k(i)$, $1 \le i \le N$. Applying Corollary 4.8 in [8] we deduce that the zero state equilibrium of the system (6) is ESMS. Further, applying Corollary 3.3 in[9] for $X(t,i) = X(i)$, $0 \le t \le \tau$, $\tau \ge 1$, $1 \le i \le N$, and taking the limit for $\tau \to \infty$ we have:

$$\tilde{J}_\gamma(\infty; 0, v) = \sum_{t=0}^{\infty} E[\begin{pmatrix} x(t,0,v) \\ v(t) \end{pmatrix}^T \mathbf{Q}(X, \eta_t) \begin{pmatrix} x(t,0,v) \\ v(t) \end{pmatrix}] \tag{53}$$

where $\mathbf{Q}(X,i)$ is the left hand side of (48). If $X = (X(1), ..., X(N))$ verifies (48) then for $\varepsilon > 0$ small enough we have

$$\mathbf{Q}(X,i) \le -\varepsilon^2 I_{n+m_v}, \quad 1 \le 1 \le N. \tag{54}$$

Combining (53) and (54) we deduce

$$\tilde{J}_\gamma(\infty; 0, v) \le -\varepsilon^2 \sum_{t=0}^{\infty} E[|x(t,0,v)|^2] + E[|v(t)|^2]$$

or equivalently

$$\tilde{J}_{\tilde{\gamma}}(\infty; 0, v) \le -\varepsilon^2 \sum_{t=0}^{\infty} E[|x(t,0,v)|^2] < 0$$

for all $v \in \ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^{m_v}\}$ where $\tilde{\gamma} = (\gamma^2 - \varepsilon^2)^{\frac{1}{2}}$.

The last inequality may be written:

$$\|\mathcal{T}v\|^2_{\ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{n_z}\}} \le \tilde{\gamma}^2 \|v\|^2_{\ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\}}$$

for all $v \in \ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^{m_v}\}$. This leads to $\|\mathcal{T}\|^2 \le \gamma^2 - \varepsilon^2$ and thus the implication $(ii) \to (i)$ is proved.

To prove the equivalence $(ii) \leftrightarrow (iii)$ let us consider the DTSGRE:

$$X = \Pi_1 X + \hat{M} - (\Pi_2 X + \hat{L})(\Pi_3 X + \hat{R})^{-1}(\Pi_2 X + \hat{L})^T \tag{55}$$

where

$$\hat{M}(i) = -C^T(i)C(i), \quad \hat{L}(i) = -C^T(i)D(i), \quad \hat{R}(i) = \gamma^2 I_{m_v} - D^T(i)D(i),$$

$1 \leq i \leq N$. One can sees that (55) is a nonlinear equation of type (41) defined by the pair $\Sigma = (\Pi, \hat{\mathcal{Q}})$ with $\hat{\mathcal{Q}} = \begin{pmatrix} \hat{M} & \hat{L} \\ \hat{L}^T & \hat{R} \end{pmatrix} \in \mathcal{S}^N_{n+m_v}$. One can check that if $X = (X(1), X(2), ..., X(N))$ solves (48) then $\hat{X} = (\hat{X}(1), ..., \hat{X}(N))$ with $\hat{X}(i) = -X(i)$, $i \in \mathcal{D}$ solves the corresponding LMIs (46).

Also if (ii) is fulfilled then from (1,1) block of (48) one obtains that $\Pi_{1i}X - X(i) < 0$, $1 \leq i \leq N$. Using the implication $(vii) \to (i)$ of Theorem 3.4 in [7] in the special case of the positive operator Π_1 we deduce that the eigenvalues of this operator are located in the inside of the disc $|\lambda| < 1$. This means that the operator Π defined by $\Pi X = \begin{pmatrix} \Pi_1 X & \Pi_2 X \\ (\Pi_2 X)^T & \Pi_3 X \end{pmatrix}$ is stabilizable (in the sense of Definition 3.1 from above). Thus we obtain that if (ii) is fulfilled then in the case of DTSGRE (55) the assertion (i) in Theorem 3.5 is fulfilled. Hence, (55) has a stabilizing solution $X_s = (X_s(1), X_s(2), ..., X_s(N))$ which satisfies

$$\Pi_{3i}X_s - R(i) > 0, 1 \leq i \leq N.x \tag{56}$$

A simple computation shows that $\tilde{X} = (\tilde{X}(1), ..., \tilde{X}(N))$ defined by $\tilde{X}(i) = -X_s(i)$ is the stabilizing solution of DTSARE (38) which satisfies (39). Since the eigenvalues of the positive operator Π_1 are located in the inside of the disk $|\lambda| < 1$ from Theorem 3.5 in [7] it follows that $\tilde{X}(i) \geq 0, i \in \mathcal{D}$ and then $(ii) \to (iii)$ is true.

Conversely, let $\tilde{X} = (\tilde{X}(1), ..., \tilde{X}(N))$ be the stabilizing solution of the DTSARE (38) which satisfies (39). If $X_s(i) = -\tilde{X}(i), 1 \leq i \leq N$, then $X_s = (X_s(1), ..., X_s(N))$ is the stabilizing solution of (55) which satisfies the condition (56). Applying Theorem 3.5 in the case of (55) one deduces that there exists $\hat{X} = (\hat{X}(1), ..., \hat{X}(N))$ which solves

$$\begin{pmatrix} \Pi_{1i}\hat{X} - \hat{X}(i) + \hat{M}(i) & \Pi_{2i}\hat{X} + \hat{L}(i) \\ (\Pi_{2i}\hat{X} + \hat{L}(i))^T & \Pi_{3i}\hat{X} + \hat{R}(i) \end{pmatrix} > 0, 1 \leq i \leq N. \tag{57}$$

On the other hand from Proposition 5.1 in [10] we deduce that X_s coincides with the maximal solution of (55). Therefore, $\hat{X}(i) \leq X_s(i) = -\tilde{X}(i) \leq 0, 1 \leq i \leq N$.

Let $\Delta_i(\hat{X})$ be defined by $\Delta_i(\hat{X}) = \Pi_{1i}\hat{X} - \hat{X}(i) + \hat{M}(i)$. Since $\Delta_i(\hat{X})$ is the (1,1) block of the matrix from the left hand side of (57) we have $\Delta_i(\hat{X}) > 0, 1 \leq i \leq N$. Writing $\hat{X}(i) = \Pi_{1i}\hat{X} + \hat{M}(i) - \Delta_i(\hat{X})$ and taking into account that $\hat{M}(i) \leq 0$, we conclude that $\hat{X}(i) < 0, 1 \leq i \leq N$. Taking $X(i) = -\hat{X}(i)$ one sees that $X = (X(1), X(2), ..., X(N))$ solves (48) and $X(i) > 0, 1 \leq i \leq N$.

This completes the proof of the implication $(iii) \to (ii)$.

The equivalence $(ii) \leftrightarrow (iv)$ follows immediately by a Schur complement technique. This completes the proof of the theorem.

If the system (4) is either in the case $N = 1$ or $N \geq 2$, with $A_k(i) = 0$, $B_k(i) = 0$, $1 \leq k \leq r$, $i \in \mathcal{D}$, the result proved in Theorem 3.6 recover as special cases the stochastic version of the Bounded Real Lemma for discrete-time linear stochastic systems perturbed by independent random perturbations and the discrete-time linear stochastic systems with Markovian switching, respectively.

Let us remark that if the zero state equilibrium of (6) is ESMS from Theorem 3.6, it follows that

$$\|\mathcal{T}\| = inf\{\gamma > 0, \text{ for which it exists } X \in \mathcal{S}_n^N, X > 0,$$

$$\text{such that (48) holds}\} = inf\{\gamma > 0,$$

$$\text{DTSARE (38) has a positive semidefinite solution verifying (39)}\}$$

4 The small gain theorem and robust stability

One of the important consequence of the Bounded Real Lemma is the so called Small Gain Theorem. It is known that this result is a powerful tool in the derivation of some estimates of the stability radius with respect to several classes of parametric uncertainties. We start with an auxiliary result which is interesting in itself:

Theorem 4.1 *Regarding the system (4) we assume that the following assumptions are fulfilled:*

a) the number of inputs equals the number of outputs (i.e. $m_v = n_z = m$);

b) the zero state equilibrium of the corresponding linear system (6) is ESMS;

c) the input-output operator $\mathcal{T}$ associated to the system (4) satisfies $\|\mathcal{T}\| < 1$.

Under these assumptions we have:

(i) the matrices $I_m \pm D(i)$, $i \in \{1, 2, ..., N\}$ are invertible.

(ii) the zero state equilibrium of the system

$$x(t+1) = (\overline{A}(\eta_t) + \sum_{k=1}^{r} w_k(t)\overline{A}_k(\eta_t))x(t) \tag{58}$$

is ESMS, where either $\overline{A}_k(i) = A_k(i) - B_k(i)(I_m + D(i))^{-1}C(i)$ *or* $\overline{A}_k(i) = A_k(i) + B_k(i)(I_m - D(i))^{-1}C(i)$.

Proof. Based on (10) and assumption c) we deduce that $\|\mathcal{T}_\tau\| < 1$ for any integer $\tau \geq 1$. Thus applying Corollary 2.3 one obtains that $I_m - D^T(i)D(i) > 0$, $i \in \{1, 2..., N\}$. Therefore for each i the eigenvalues of the matrix $D(i)$ are located in the inside of the disk $|\lambda| < 1$. Hence $det(I_m \pm D(i)) \neq 0$, $1 \leq i \leq N$. Thus we obtain that (i) is true. To prove (ii) we use the implication $(i) \to (ii)$ of Theorem 3.6. Thus if the assumptions b) and c) are fulfilled, then there exist $X = (X(1), ..., X(N)) \in \mathcal{S}_n^N$, $X(i) > 0$ such that (48) hold with $\gamma = 1$.

Taking

$$F(i) = \pm(I_m \mp D(i))^{-1}C(i) \tag{59}$$

by direct calculation one obtains via (13) and (48) that

$$\sum_{k=0}^{r}[A_k(i) + B_k(i)F(i)]^T\mathcal{E}_i(X)[A_k(i) + B_k(i)F(i)] - \tag{60}$$
$$X(i) + (C(i) + D(i)F(i))^T(C(i) + D(i)F(i)) - F^T(i)F(i) < 0$$

$1 \leq i \leq N$.

If we take into account (59) we obtain $C(i)+D(i)F(i) = (I_m \mp D(i))^{-1}C(i)$. Thus we have $(C(i) + D(i)F(i))^T(C(i) + D(i)F(i)) - F^T(i)F(i) = 0$. Hence (60) becomes:

$$\sum_{k=0}^{r}\overline{A}_k^T(i)\mathcal{E}_i(X)\overline{A}_k(i) - X(i) < 0, \quad X(i) > 0, \ 1 \leq i \leq N. \tag{61}$$

Applying Corollary 4.8 in [8] one deduces that the zero state equilibrium of the system (58) is ESMS. This completes the proof.

Consider the system

$$
\begin{aligned}
x(t+1) &= [A_0(\eta_t) + \sum_{k=1}^{r} w_k(t) A_k(\eta_t)]x(t) + [B_0(\eta_t) + \\
&\quad + \sum_{k=1}^{r} w_k(t) B_k(\eta_t)]u(t) \qquad (62)\\
z(t) &= C(\eta_t)x(t)
\end{aligned}
$$

with the input $u(t) \in \mathbf{R}^m$ and the output $z(t) \in \mathbf{R}^p$.

Let $\hat{D} = (\hat{D}(1), ..., \hat{D}(N)), \hat{D}(i) \in \mathbf{R}^{m \times p}$. By definition $|\hat{D}| = max\{|\hat{D}(i)|, 1 \leq i \leq N\}$.

Theorem 4.2 (The small gain theorem). Assume:

a) The zero state equilibrium of the system (6) is ESMS.

b) $\|\tilde{\mathcal{T}}\| < \gamma$ where $\tilde{\mathcal{T}} : \ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^m\} \to \ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^p\}$ is the input-output operator defined by the system (62).

c) $|\hat{D}| < \gamma^{-1}$.

Under these conditions the zero state equilibrium of the system

$$
\begin{aligned}
x(t+1) &= [A_0(\eta_t) + B_0(\eta_t)\hat{D}(\eta_t)C(\eta_t) + \\
&\quad \sum_{k=1}^{r} w_k(t)(A_k(\eta_t) + B_k(\eta_t)\hat{D}(\eta_t)C(\eta_t))]x(t) \qquad (63)
\end{aligned}
$$

is ESMS.

Proof. Define the linear bounded operator $\hat{\mathcal{T}} : \ell^2_{\tilde{\mathcal{H}}}\{0, \infty, \mathbf{R}^p\} \to \ell^2_{\tilde{\mathcal{H}}}\{0, \infty, \mathbf{R}^m\}$ by

$$(\hat{\mathcal{T}}v)(t) = D(\eta_t)v(t)$$

with $v(t) \in \ell^2_{\tilde{\mathcal{H}}}\{0, \infty, \mathbf{R}^p\}$. Reasoning as in the proof of Proposition 13 in [6] page 234 one can prove that $\|\hat{\mathcal{T}}\| = |\hat{D}|$.

Let us consider the system:

$$
\begin{aligned}
\hat{x}(t+1) &= [A_0(\eta_t) + \sum_{k=1}^{r} w_k(t) A_k(\eta_t)]\hat{x}(t) + [B_0(\eta_t) + \\
&\quad + \sum_{k=1}^{r} w_k(t) B_k(\eta_t)]\hat{D}(\eta_t)v(t) \qquad (64)\\
z(t) &= C(\eta_t)\hat{x}(t).
\end{aligned}
$$

We observe that $\tilde{\mathcal{T}}\hat{\mathcal{T}}$ is the input output operator associated with the system (64). Since $\|\tilde{\mathcal{T}}\hat{\mathcal{T}}\| < 1$, the conclusion follows via Theorem 4.1. Thus the proof is complete.

In this section the problem of the robust stability is investigated for a class of discrete-time linear stochastic systems subject to linear parametric uncertainties.

Let us consider the discrete-time linear stochastic system described by:

$$x(t+1) = [A_0(\eta_t) + B_0(\eta_t)\Delta(\eta_t)C(\eta_t) + \\ + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t) + B_k(\eta_t)\Delta(\eta_t)C(\eta_t))]x(t) \tag{65}$$

where $A_k(i) \in \mathbf{R}^{n\times n}$, $B_k(i) \in \mathbf{R}^{n\times m}$, $0 \le k \le r$, $C(i) \in \mathbf{R}^{p\times n}$ are assumed to be known matrices, $\Delta(i) \in \mathbf{R}^{m\times p}$ are unknown matrices. The system (65) is a perturbed model of the nominal system (6).

The matrices $B_k(i)$, $C(i)$ occurring in (65) determine the structure of the parametric uncertainties presented in the perturbed model.

If the zero state equilibrium of the nominal system (6) is ESMS we will analyze if the zero state equilibrium of the perturbed model (65) remains ESMS for some $\Delta(i) \neq 0$. This would be, in few words the formulation of the problem of the robust stability. For a more precise formulation of the robust stability problem we introduced a norm in the set of the uncertainties.

If $\Delta = (\Delta(1), \Delta(2), ..., \Delta(N)) \in \mathcal{M}_{m,p}^N$ i.e. $\Delta(i)$ are $m \times p$ real matrices, we set

$$|\Delta| = \max_{i\in\mathcal{D}} |\Delta(i)| = \max_{i\in\mathcal{D}} (\lambda_{\max}(\Delta^T(i)\Delta(i)))^{\frac{1}{2}}. \tag{66}$$

As a measure of the robustness of the stability we introduce the concept of stability radius.

Definition 4.1 The stability radius of the nominal system (6), or equivalently, the stability radius of the pair $(\mathbf{A}, P)$ with respect to the structured parametric uncertainties with the structure determined by the pair $(\mathbf{B}, C)$ is the number $\rho_L[\mathbf{A}, P|\mathbf{B}, C] = \inf\{\rho > 0|(\exists)\Delta = (\Delta(1), ..., \Delta(N)) \in \mathcal{M}_{m,p}^N$ with $|\Delta| \le \rho$ that the zero state equilibrium of the corresponding system (4.8) is not ESMS$\}$.

The next result provides a lower bound of the stability radius introduced in the above definition. To this end, let us consider the fictitious system constructed based on the known matrices occurring in the perturbed model (65):

$$x(t+1) = (A_0(\eta_t) + \sum_{k=1}^{r} w_k(t)A_k(\eta_t))x(t) + (B_0(\eta_t) + \\ + \sum_{k=1}^{r} w_k(t)B_k(\eta_t))v(t); \quad z(t) = C(\eta_t)x(t) \tag{67}$$

Theorem 4.3 *Assume that the zero state equilibrium of the nominal system (6) is ESMS. Let* $\mathcal{T} : \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^m\} \to \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^p\}$ *be the input output operator defined by the fictitious system (67). Then we have:*

$$\rho_L[\mathbf{A}, P|\mathbf{B}, C] \geq \|\mathcal{T}\|^{-1} \tag{68}$$

Proof. Let $\rho < \|\mathcal{T}\|^{-1}$ be arbitrary but fixed. We show that for any perturbation $\varDelta = (\varDelta(1), \varDelta(2), ..., \varDelta(N)) \in \mathcal{M}^N_{m,p}$ with $|\varDelta| < \rho$, the zero state equilibrium of the perturbed system (65) is ESMS. Let $\varDelta \in \mathcal{M}^N_{m,p}$ be a perturbation with $|\varDelta| < \rho$. Setting $\gamma = \rho^{-1}$, we have $\|\mathcal{T}\| < \gamma$ and $|\varDelta| < \gamma^{-1}$. Hence the fictitious system (67) and the perturbation $\varDelta$ are in the conditions of Theorem 4.2. Thus the prooof is complete.

5 The disturbance attenuation problem and the robust stabilization

Consider the control system:

$$\begin{aligned} x(t+1) = A_0(\eta_t)x(t) + G_0(\eta_t)v(t) + B_0(\eta_t)u(t) + \\ \sum_{k=1}^{r} w_k(t)[A_k(\eta_t)x(t) + G_k(\eta_t)v(t) + B_k(\eta_t)u(t)] \\ y(t) = x(t) \\ z(t) = C_z(\eta_t)x(t) + D_{zv}(\eta_t)v(t) + D_{zu}(\eta_t)u(t). \end{aligned} \tag{69}$$

If we take

$$u(t) = F(\eta_t)x(t). \tag{70}$$

the closed-loop system obtained when coupling (70) and (69) is:

$$\begin{aligned} x(t+1) &= [A_0(\eta_t) + B_0(\eta_t)F(\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t) + B_k(\eta_t)F(\eta_t))]x(t) + \\ &\quad (G_0(\eta_t) + \sum_{k=1}^{r} w_k(t)G_k(\eta_t))v(t) \qquad (71) \\ z(t) &= (C_z(\eta_t) + D_{zu}(\eta_t)F(\eta_t))x(t) + D_{zv}(\eta_t)v(t). \end{aligned}$$

If $F = (F(1), F(2), ..., F(N))$ is a stabilizing feedback gain, that is (71) with $v(t) = 0$ is ESMS, then the system (71) defines an input output operator, $\mathcal{T}_F : \ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^{m_v}\} \to \ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^{n_z}\}$ by $(\mathcal{T}_F v)(t) = (C_z(\eta_t) + D_{zu}(\eta_t)F(\eta_t))x(t, 0, v) + D_{zv}(\eta_t)v(t)$, $t \in \mathbf{Z}_+$.

The disturbance attenuation problem with level of attenuation $\gamma > 0$ asks for constructing a stabilizing feedback gain F, such that $\|\mathcal{T}_F\| < \gamma$.

Remark 5.1 The disturbance attenuation problem (DAP) stated before extends to this general framework the H_∞ control problem from the deterministic context. Therefore, this problem will be often named stochastic H_∞-problem.

The solution of the above problem is given in the next result:

Theorem 5.1 *For the system (69) and a given scalar $\gamma > 0$, the following are equivalent:*

(i) there exists a control law $u(t) = F(\eta_t)x(t)$ such that the zero state equilibrium of the linear system $x(t+1) = [A_0(\eta_t) + B_0(\eta_t)F(\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t) + B_k(\eta_t)F(\eta_t))]x(t)$ is ESMS and $\|\mathcal{T}_F\| < \gamma$.

(ii) there exist $Y = (Y(1), Y(2), ..., Y(N)) \in \mathcal{S}_n^N$ and $\Gamma = (\Gamma(1), \Gamma(2), ..., \Gamma(N)) \in \mathcal{M}_{mn}^N$, $Y(i) > 0$, $1 \leq i \leq N$, which solve the following system of LMIs:

$$\begin{pmatrix} -Y(i) & \mathcal{W}_{0i}(Y,\Gamma) & \mathcal{W}_{1i}(Y,\Gamma) & ... & \mathcal{W}_{ri}(Y,\Gamma) & Y(i)C_z^T(i)+\Gamma^T(i)D_{zu}^T(i) \\ \mathcal{W}_{0i}^T(Y,\Gamma) & \mathfrak{G}_{00}-\mathcal{Y} & \mathfrak{G}_{01}(i) & ... & \mathfrak{G}_{0r}(i) & \mathfrak{G}_{0r+1}(i) \\ \mathcal{W}_{1i}^T(Y,\Gamma) & \mathfrak{G}_{01}^T(i) & \mathfrak{G}_{11}(i)-\mathcal{Y} & ... & \mathfrak{G}_{1r}(i) & \mathfrak{G}_{1r+1}(i) \\ ... & ... & ... & ... & ... & ... \\ \mathcal{W}_{ri}^T(Y,\Gamma) & \mathfrak{G}_{0r}^T(i) & \mathfrak{G}_{1r}^T(i) & ... & \mathfrak{G}_{rr}(i)-\mathcal{Y} & \mathfrak{G}_{rr+1}(i) \\ C_z(i)Y(i)+D_{zu}(i)\Gamma(i) & \mathfrak{G}_{0r+1}^T(i) & \mathfrak{G}_{1r+1}^T(i) & ... & \mathfrak{G}_{rr+1}^T(i) & D_{zv}^T(i)D_{zv}^T(i)-\gamma^2 I_{n_z} \end{pmatrix} < 0 \quad (72)$$

where $\mathcal{W}_{ki}(Y,\Gamma) = (Y(i)A_k^T(i) + \Gamma^T(i)B_k^T(i))\Im(i)$, $0 \le k \le r$,

$$\begin{aligned}\Im(i) = \big(\ \sqrt{p(i,1)}I_n \quad \sqrt{p(i,2)}I_n \quad ... \quad \sqrt{p(i,N)}I_n\ \big)\\ \mathfrak{G}_{lk}(i) = \Im^T(i)G_l(i)G_k^T(i)\Im(i),\quad 0 \le l \le k \le r,\\ \mathfrak{G}_{lr+1}(i) = \Im^T(i)G_l(i)D_{zv}^T(i),\quad 0 \le l \le r\\ \mathcal{Y} = diag(Y(1), Y(2), ..., Y(N)).\end{aligned} \tag{73}$$

Moreover, if (Y,Γ) *is a solution of the above LMI (72), then a solution of the disturbance attenuation problem under consideration is given by* $F = (F(1), F(2), ..., F(N))$, $F(i) = \Gamma(i)Y^{-1}(i)$, $1 \le i \le N$.

Proof. It follows immediately via the equivalence $(i) \leftrightarrow (iv)$ in Theorem 3.6 specialized in the case of the system (71) and taking $\Gamma(i) = F(i)Y(i)$.

We shall apply Theorem 5.1 in order to solve a robust stabilization problem.

Consider the system described by:

$$\begin{aligned}x(t+1) = [A_0(\eta_t) + \hat{G}_0(\eta_t)\Delta_1(\eta_t)\hat{C}(\eta_t)]x(t) + \\ +[B_0(\eta_t) + \hat{B}_0(\eta_t)\Delta_2(\eta_t)\hat{D}(\eta_t)]u(t)\\ +\sum_{k=1}^{r} w_k(t)\{[A_k(\eta_t) + \hat{G}_k(\eta_t)\Delta_1(\eta_t)\hat{C}(\eta_t)]x(t) + \\ +[B_k(\eta_t)\Delta_2(\eta_t)\hat{D}(\eta_t)]u(t)\}\end{aligned} \tag{74}$$

where $A_k(\eta_t), \hat{G}_k(i), B_k(i), \hat{C}(i), \hat{D}(i), 0 \le k \le r, i \in \mathcal{D}$ are known matrices of appropriate dimensions and $\Delta_1 = (\Delta_1(1), ..., \Delta_1(N))$ and $\Delta_2 = (\Delta_2(1), ..., \Delta_2(N))$ are unknown matrices and they describe the magnitude of the uncertainties of the system (74). It is assumed that the whole state vector is accessible for measurements.

The robust stabilization problem considered here can be stated as follows:

For a given $\rho > 0$ find a control $u(t) = F(\eta_t)x(t)$ stabilizing (74) for any Δ_1 and Δ_2 such that $max(|\Delta_1|, |\Delta_2|) < \rho$.

The closed-loop system obtained with $u(t) = F(\eta_t)x(t)$ is given by

$$\begin{aligned}x(t+1) = \{A_0(\eta_t)+B_0(\eta_t)F(\eta_t)+G_0(\eta_t)\Delta(\eta_t)[C(\eta_t)+D(\eta_t)F(\eta_t)]\}x(t)\\ +\sum_{k=1}^{r} w_k(t)\{A_k(\eta_t)+B_k(\eta_t)F(\eta_t)+G_k(\eta_t)\Delta(\eta_t)[C(\eta_t)+D(\eta_t)F(\eta_t)]\}x(t)\end{aligned} \tag{75}$$

where $G_k(i) = \begin{pmatrix} \hat{G}_k(i) & \hat{B}_k(i) \end{pmatrix}, C(i) = \begin{pmatrix} \hat{C}(i) \\ 0 \end{pmatrix}, D(i) = \begin{pmatrix} 0 \\ \hat{D}(i) \end{pmatrix}, \mathit{\Delta}(i) = \begin{pmatrix} \mathit{\Delta}_1(i) & 0 \\ 0 & \mathit{\Delta}_2(i) \end{pmatrix}$.

If the zero state equilibrium of the linear system obtained from (75)taking $\mathit{\Delta} = 0$ is ESMS, then from Theorem 4.3 it follows that the zero state equilibrium of (75) is ESMS for all $\mathit{\Delta}$ with $|\mathit{\Delta}| < \rho$, $|\mathit{\Delta}| = max(|\mathit{\Delta}_1|, |\mathit{\Delta}_2|)$, if the input-output operator $\mathcal{T}_F$ associated to the system (71) with $z(t) = [C(\eta_t) + D(\eta_t)F(\eta_t)]x(t)$ satisfies the condition $\|\mathcal{T}_F\| < \frac{1}{\rho}$.

Therefore, F is a robust stabilizing feedback with the robustness radius ρ if it is a solution of the DAP with level of attenuation $\gamma = \frac{1}{\rho}$ for the system (69) with $z(t) = C(\eta_t)x(t) + D(\eta_t)u(t)$ where the matrices $C(i)$ and $D(i)$ were defined above.

The next result follows directly from Theorem 5.1:

Theorem 5.2 *Suppose that there exist $Y \in \mathcal{S}_n^N$ and $\mathit{\Gamma} \in \mathcal{M}_{mn}^N$, $Y > 0$ verifying the system of LMIs (72) where $C_z(i) = C(i), D_{zu}(i) = D(i)$, $D_{zv}(i) = 0, \gamma = \frac{1}{\rho}$. Then the state feedback gain $F(i) = \mathit{\Gamma}(i)Y^{-1}(i)$ is a solution of the robust stabilization problem.*

References

[1] A. El Bouhtouri, D. Hinrichsen, A.J. Pritchard, H_∞ type control for discrete-time stochastic systems, *Int. J. Robust Nonlinear Control*, 9, (1999), 923-948.

[2] O.L.V. Costa, R.P. Marques, Mixed H_2/H_∞ control of discrete-time Markovian jump linear systems, *IEEE Trans. Autmatic Control*, AC-43, 1, (1998), 95-100.

[3] O.L.V. Costa, M.D.Fragoso, R.P.Marques, *Discrete-time Markov jump linear systems,* Ed. Springer, 2005.

[4] J.C.Doyle, K. Glover, P.P.Khargonekaar, B.A.Francis, State space solutions to standard H_2 and H_∞ control pproblems, *IEEE Trans. Automatic Control*, 34, 8, (1989), 831-847.

[5] V.Dragan, A.Stoica, A robust stabilization problem for discrete time time- varying stochastic systems with multiplicative noise, *Math. Reports,* 2(52), 3, (2000), 275-293.

[6] V. Dragan, T. Morozan, A.M. Stoica, Mathematical methods in robust control of linear stochastic systems, *Mathematical Concepts and Methods in Science and Engineering*, series Editor Angelo Miele, Volume 50, Springer, (2006).

[7] V. Dragan, T. Morozan, Exponential stability for discrete time linear equations defined by positive operators. *Integral Equ. Oper. Theory*, 54, (2006), 465-493.

[8] V. Dragan, T.Morozan, Mean square exponential stability for some stochastic linear discrete time systems, *European Journal of Control*, 12, (2006), 373-395.

[9] V. Dragan, T. Morozan, The linear quadratic optimization problem for a class of discrete-time stochastic linear systems, *Int. J. of Innovative Computing, Inform. and Control*, 4, 9, (2008), 2127-2137.

[10] V. Dragan, T. Morozan, A class of discrete-time generalized Riccati equations, *Journal of Difference equations and Applications*, 16, 4 (2010), 291-320

[11] E.Ghershon, U.Shaked, I.Yaesh, H_∞ control and filtering of discrete-time stochastic systems with multiplicative noise, *Automatica*, 37, (2001), 409-417.

[12] E. Ghershon, U. Shaked, I. Yaesh, *H_∞ Control and Estimation of State-Multiplicative Linear Systems*, Lecture Notes in Control and Information Sciences, 318,(2005), Springer-Verlag London Limited.

[13] M. Green, D.J.N.Limebeer, *Linear robust control,* Prentice Hall, New-York, 1995.

[14] L. Hu, P. Shi, P. Frank, Robust sampled data control for Markovian jump linear systems, *Automatica*, 42, (2006), 2025-2030.

[15] T. Morozan, Parametrized Riccati equations and input output operators for discrete-time systems with state dependent noise, *Stochastic Anal. and Appl.*, 16,5, (1998), 915-931.

[16] T. Morozan, Parametrized Riccati equations for linear discrete-time systems with Markov perturbations, *Rev. Roum. Math. Pures et Appl.*,43, 7-8, (1998), 761-777.

[17] C.E. de Souza, L.Xie, On the discrete-time bounded real lemma with application in the characterization of static state feedback H_∞ controllers, *System and Control Letters*, 18, (1992), 61-71.

[18] E. Yaz, X. Niu, New results on the robustness of discrete-time systems with stochastic perturbations, *Proceedings of American Control conference*, (1991), Boston MA, 2698-2699.

[19] X. Yu, C.S. Hsu, Reduced order H_∞ filter design for discrete-time invariant systems, *International Journal of Robust and Nonlinear Control*, 7, (1997), 797-809.

[20] L. Zhang, P. Shi, E. K. Boukas, C. Wang, H_∞ Model Reduction for Uncertain Switched Linear Discrete-Time Systems, *Automatica,* 44, 11, (2008), 2944-2949.

[21] L. Zhang, P. Shi, E. K. Boukas, H_∞ model reduction for discrete-time Markov jump linear systems with partially known transition, *Int. J. of Control*, 82,2, (2009), 243-351.