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OPTIMAL CONTROL OF A NONLINEAR COUPLED ELECTROMAGNETIC INDUCTION HEATING SYSTEM WITH POINTWISE STATE CONSTRAINTS*

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Abstract

An optimal control problem arising in the context of 3D electromagnetic induction heating is investigated. The state equation is given by a quasilinear stationary heat equation coupled with a semilinear time-harmonic eddy current equation. The temperature-dependent electrical conductivity and the presence of pointwise inequality state-constraints represent the main challenge of the paper. In the first part of the paper, the existence and regularity of the state are addressed. The second part of the paper deals with the analysis of the corresponding linearized equation. Some sufficient conditions are presented which guarantee the solvability of the linearized system. The final part of the paper is concerned with the optimal control. The aim of the optimization is to find the optimal voltage such that a desired temperature can be achieved optimally. The corresponding first-order necessary optimality condition is presented.

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1 Introduction

In the recent past, there has been growing interest in the analysis and numerical modeling of electromagnetic induction heating. Generally speaking, its mathematical model is given by nonlinear heat equations coupled with Maxwell equations. From among many contributions to this topic, we only mention Bossavit and Rodrigues [6], Bodart et al. [4], Clain and Touzani [9], Hömberg [15], Parietti and Rappaz [21], Rappaz and Swierkosz [22]. An important issue arising in the context of electromagnetic induction heating in modern industry is mainly how to control the process in a way that a desired temperature of the targeted object can be achieved optimally. In addition, in order to avoid undesired damage or melting, the temperature (state of the system) has to be uniformly bounded during the heating process. Thus, it is necessary to include pointwise inequality state constraints in the optimal control problem. From the theoretical and numerical point of view, the treatment of such a problem is challenging. There are two main reasons for this: On the one hand, higher regularity of the state is required for the existence of Lagrange multipliers. On the other hand, Lagrange multipliers associated with pointwise state constraints are in general only Borel measures (cf. [1, 7, 8, 23]).

Eddy current equations

Neglecting the electrical displacement and free charges in the full Maxwell equations leads to the eddy current equations (cf. [5]). For a fixed angular frequency $\omega > 0$, the time-harmonic eddy current equations read as follows

$$\nabla \times H = J \quad \text{in } \mathcal{D} \qquad \text{(Ampère's law)} \tag{1}$$
$$\nabla \times E = -i\omega B \quad \text{in } \mathcal{D} \qquad \text{(Faraday's law)} \tag{2}$$
$$\nabla \cdot B = 0 \quad \text{in } \mathcal{D} \qquad \text{(Gauss's law for magnetism)} \tag{3}$$
$$J = \sigma_{\mathcal{D}} E \quad \text{in } \mathcal{D} \qquad \text{(Ohm's law)} \tag{4}$$
$$B = \mu H \quad \text{in } \mathcal{D} \qquad \text{(Constitutive relation).} \tag{5}$$

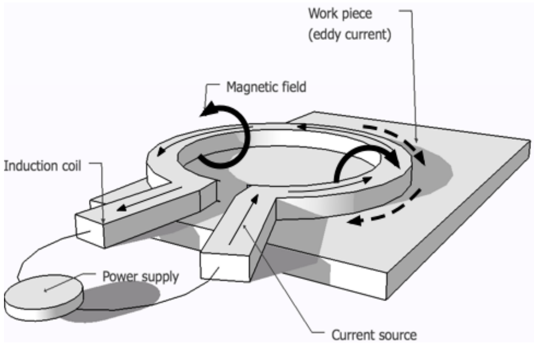

Figure 1: Illustration of electromagnetic induction heating.

In the above setting, E and H denote the electric field intensity and the magnetic field intensity occupying some bounded domain $\mathcal{D} \subset \mathbb{R}^3$. The vector field B describes the magnetic induction, J represents the total current density, and i denotes the imaginary unit. Further, μ is the magnetic permeability and $\sigma_{\mathcal{D}}$ is the electrical conductivity of $\mathcal{D}$. Let us remark that Gauss's law for magnetism (3) implies the existence of a magnetic vector potential A satisfying

$$\nabla \times A = B \text{ in } \mathcal{D} \qquad\qquad \nabla \cdot A = 0 \text{ in } \mathcal{D}. \tag{6}$$

Then, applying (6) to the system (1)–(5), a second-order equation for A can be derived (see [12, 15]). The corresponding formulation for our model will be presented shortly.

Induction heating process

In principle, an electromagnetic induction heating system consists of two essential components: an induction coil connected to an alternating current (AC) power supply and an electrically conductive workpiece (heated material). See Figure 1 for an illustration of induction heating. The AC power supply injects alternating current into the induction coil which produces in turn an alternating magnetic field. Since the workpiece is electrically conductive, the magnetic field generates an eddy current within it. Then, the resistance to the eddy current induces heat in the workpiece (cf. the monograph [16]). A 3D electromagnetic induction heating model involving a thermomechanical effect for induction hardening has been recently investigated by Hömberg in [15]. We follow his model with a further simplification which does not involve the thermomechanical effect. Let $\overline{\Omega}, \overline{R} \subset \mathcal{D}$ denote the workpiece and the induction coil, respectively, and we suppose that $\overline{\Omega} \cap \overline{R} = \emptyset$. The region $\mathcal{D} \setminus \overline{(\Omega \cup R)}$ is assumed to be the surrounding air and hence, as air is non-conducting, $\sigma_{\mathcal{D}}$ can be decomposed into:

$$\sigma_{\mathcal{D}} = \begin{cases} \sigma & \text{in } \Omega \\ \sigma_R & \text{in } R \\ 0 & \text{in } \mathcal{D} \setminus \overline{(\Omega \cup R)}, \end{cases}$$

where σ and σ_R represent the electrical conductivities of Ω and R, respectively. In our model, we suppose that the induction coil R is connected to some external source and there is no impressed current source in the workpiece Ω so that we arrive at the following magnetic vector potential formulation:

$$\begin{cases} \nabla \times \left(\mu^{-1} \nabla \times A\right) + i\omega\sigma \ A &= 0 & \text{in } \Omega \\ \nabla \times \left(\mu^{-1} \nabla \times A\right) + i\omega\sigma_R A &= J_{source} & \text{in } R \\ \nabla \times \left(\mu^{-1} \nabla \times A\right) &= 0 & \text{in } \mathcal{D} \setminus \overline{(\Omega \cup R)} \\ \nabla \cdot A &= 0 & \text{in } \mathcal{D} \\ A \times \vec{n} &= 0 & \text{on } \partial\mathcal{D}. \end{cases} \tag{7}$$

Here and in what follows, $\vec{n}$ denotes the outward unit normal to the corresponding surface and J_{source} is the impressed current source. Note that the boundary condition $A \times \vec{n} = 0$ on $\partial\mathcal{D}$ physically means that $\partial\mathcal{D}$ is a perfect conductor. In addition to this boundary condition, we also include the following interface conditions:

$$[\left(\mu^{-1} \nabla \times A\right) \times \vec{n}]_{\partial R} = 0 \quad \text{on } \partial R \quad \text{and} \quad [\left(\mu^{-1} \nabla \times A\right) \times \vec{n}]_{\partial \Omega} = 0 \quad \text{on } \partial\Omega, \tag{8}$$

where $[\cdot]_{\partial R}$ and $[\cdot]_{\partial\Omega}$ denote the jumps of a quantity across the interfaces ∂R and $\partial\Omega$, respectively. By (5) and (6), the above interface conditions are equivalent to

$$[H \times \vec{n}]_{\partial R} = 0 \quad \text{on } \partial R \qquad\qquad [H \times \vec{n}]_{\partial\Omega} = 0 \quad \text{on } \partial\Omega.$$

In other words, the tangential trace of the magnetic field intensity H is assumed to be continuous across the interfaces $\partial\Omega$ and ∂R.

Let us now explain, how the impressed current source J_{source} in (7) looks like: Throughout the paper, we assume that:

- The induction coil R is given by the union $R = \bigcup_{i=1}^{n} R_i$ $(n \geq 1)$ where $\overline{R}_1, \ldots, \overline{R}_n$ are assumed to be pairwise disjoint rings.

- For every $j = 1, \ldots, n$, the voltage $u_j \in \mathbb{R}^+$ in every coil R_j can be maintained constant and the current source J_{source} in every coil R_j is assumed to be influenced only by applying the voltage u_j.

Based on the above assumption, the impressed current source J_{source} can be written as follows

$$J_{source}(x) = \sum_{j=1}^{n} u_j J_j(x). \tag{9}$$

The control parameter for our system is given by $u_j \in \mathbb{R}^+$, $j = 1, \ldots, n$. On the other hand, every vector field $J_j : R_j \longrightarrow \mathbb{R}^3$ is *fixed given data* and, as J_{source} represents current, it has to satisfy the physical consistency assumption:

$$\nabla \cdot J_j = 0 \text{ in } R_j \qquad J_j \cdot \vec{n} = 0 \text{ on } \partial R_j. \tag{10}$$

An example for J_j is given as follows:

$$J_j(x) = (-x_2/\sqrt{x_1^2 + x_2^2}\,,\, x_1/\sqrt{x_1^2 + x_2^2}\,,\, 0)^T \quad \forall x = (x_1, x_2, x_3)^T \in R_j. \tag{11}$$

As every R_j is a ring (torus), it is straightforward to show that J_j as given above satisfies (10). Further examples for J_j can be found in Druet et al. [11].

Stationary induction heating

Assuming that the oscillation period $2\,\pi/\omega$ of the electromagnetic fields is much smaller than the heat diffusion time, the Joule heat source can be approximated by its averaged value over one oscillation period (see [9]). This approximation leads to the following stationary induction heating system:

$$\begin{cases} -\nabla \cdot (\kappa(x,y)\,\nabla y) + d(x,y) &= \frac{1}{2}\omega^2\sigma(x,y)|A|^2 & \text{in } \Omega \\ y &= 0 & \text{on } \partial\Omega \\ \nabla \times \left(\mu^{-1}\nabla \times A\right) + i\omega\sigma(x,y)A &= 0 & \text{in } \Omega \\ \nabla \times \left(\mu^{-1}\nabla \times A\right) + \quad i\omega\sigma_R A &= \sum_{j=1}^{n} u_j J_j & \text{in } R \\ \nabla \times \left(\mu^{-1}\nabla \times A\right) &= 0 & \text{in } \mathcal{D}\backslash\overline{(\Omega\cup R)} \\ \nabla \cdot A &= 0 & \text{in } \mathcal{D} \\ A \times \vec{n} &= 0 & \text{on } \partial\mathcal{D} \\ [\left(\mu^{-1}\nabla \times A\right) \times \vec{n}]_{\partial R} = 0 \text{ on } \partial R & [\left(\mu^{-1}\nabla \times A\right) \times \vec{n}]_{\partial\Omega} = 0 & \text{on } \partial\Omega. \end{cases} \tag{12}$$

In this setting, y denotes the temperature and κ is the thermal conductivity of Ω. The two-way nonlinear coupling between the quasilinear stationary heat equation and the time-harmonic eddy current equation arises from the dependence of σ on the temperature y. In fact, the temperature dependence effect of thermal and electrical conductivities cannot be ignored as it has been confirmed by many experimental studies (see e.g. [10, 16]). Notice that, instead of the homogeneous Dirichlet-type boundary condition, the subsequent analysis applies also to the nonlinear Neumann- or Robin-type boundary conditions such as $\frac{\partial y}{\partial \vec{n}} + b(x,y) = y_0$ on $\partial\Omega$ with a sufficiently regular right hand side y_0 and nonlinearity b satisfying some local boundedness and monotonicity assumptions. The author is moreover convinced that the subsequent considerations can be extended to the associated system wit h nonlocal boundary radiation conditions arising from heat transfer problems in crystal growth (cf. [11, 17, 18]).

Optimal control

Let $y_d \in L^2(\Omega)$ be a desired temperature and $z_d \in L^2(\Omega)^3$ be a desired temperature gradient. In addition, let $\alpha \geq 0$ and $\beta > 0$. Our focus is set on the following optimal control problem:

$$\text{minimize } \frac{1}{2}\int_\Omega |y - y_d|^2\,dx + \frac{\alpha}{2}\int_\Omega |\nabla y - z_d|^2\,dx + \frac{\beta}{2}|u|^2 \tag{P}$$

subject to (12) and the following inequality control- and state-constraints:

$$\begin{cases} u_j^a & \leq & u_j & \leq & u_j^b & \text{for all } j = 1, \ldots, n \\ y_a(x) & \leq & y(x) & \leq & y_b(x) & \text{for a.a. } x \in \Omega. \end{cases} \tag{13}$$

The lower and upper control-bounds $u^a, u^b \in \mathbb{R}^n$ satisfy $0 \leq u_j^a < u_j^b$ for all $j = 1, \ldots, n$. Further, the lower and upper state-bounds $y_a, y_b \in \mathcal{C}(\overline{\Omega})$ satisfy $y_a(x) < y_b(x)$ for all $x \in \overline{\Omega}$.

It should be underlined that optimal control of 3D stationary induction heating problems in the decoupled case has been recently investigated by Druet et al. [11]. In this work, we considered a *temperature-independent* electrical conductivity such that the stationary heat equation and the eddy current equation could be investigated separately. However, the results in [11] cannot be directly transferred to (P) due to the two-way nonlinear coupling in (12). Also, the linearized system associated with (12) is nonstandard (see (40) on p. 63). Therefore, the analysis of (P) represents the genuine contribution of the present paper and requires us to extend the analysis of the aforementioned reference. Note that the very first results on optimality conditions for optimal control of quasilinear elliptic equations have been recently obtained by Casas and Tröltzsch (see [8]). We shall follow their technique to prove the existence result of the coupled forward problem (12).

The main results of the paper are summarized as follows: First, the existence of solutions to (12) is established in Section 3 (Theorem 1). Then, by means of the maximum elliptic regularity result by Elschner et al. [13], we derive the state regularity in $W_0^{1,q}(\Omega)$ (Proposition 2) which plays a significant role in our analysis. Section 4 is devoted to the analysis of the linearized system associated with (12). Some sufficient conditions shall be established which guarantee the solvability of the linearized system (Theorem 3). A consequence of this result is the *uniqueness* of the solution to (12) (Corollary 1). Finally, the first-order necessary optimality condition of (P) is derived in Section 5.

2 General assumptions and notation

Let us introduce the mathematical setting including the notation used throughout this paper. We denote by c a generic positive constant which can take different values on different occasions. If X is a linear normed function space, then we use the notation $\|\cdot\|_X$ for a standard norm used in X. Furthermore, we set $X^3 := X \times X \times X$. The dual space of X is denoted by X^* and, for the associated duality pairing, we write $\langle\cdot,\cdot\rangle_{X^*,X}$. If it is obvious in which spaces the respective duality pairing is considered, then the subscript is occasionally neglected. Given another linear normed space Y, the space of all bounded linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$ and if X is continuously embedded in Y, then the corresponding injection is denoted by $X \hookrightarrow Y$. For the Fréchet derivative of a differentiable operator $B : X \to Y$ at $x \in X$ in the direction $h \in X$, we write $B'(x)h$. Moreover, the kernel and the image of $B : X \to Y$ are denoted by $\ker B$ and $\operatorname{ran} B$, respectively.

Throughout the paper, for every $1 \leq q \leq \infty$ we denote its conjugate exponent by q'. The Sobolev space on a bounded Lipschitz domain $\mathcal{O} \subset \mathbb{R}^3$ is as usual denoted by $W^{m,q}(\mathcal{O})$ and the corresponding space of complex-valued functions is denoted by $W^{m,q}(\mathcal{O};\mathbb{C})$. Further,

$$
\begin{aligned}
H(curl;\mathcal{O}) &:= \{K \in L^2(\mathcal{O};\mathbb{C})^3 \mid \nabla \times K \in L^2(\mathcal{O};\mathbb{C})^3\} \\
H(div;\mathcal{O}) &:= \{K \in L^2(\mathcal{O};\mathbb{C})^3 \mid \nabla \cdot K \in L^2(\mathcal{O};\mathbb{C})\},
\end{aligned}
$$

where the curl- and div-operators are understood in the distribution sense. Notice that every vector field $K \in H(curl;\mathcal{O})$ has the tangential trace $K \times \vec{n}$ in $H^{-1/2}(\partial\mathcal{O};\mathbb{C})^3$ satisfying

$$
\begin{aligned}
\langle K \times \vec{n}, \psi\rangle_{H^{-1/2}(\partial\mathcal{O};\mathbb{C})^3, H^{1/2}(\partial\mathcal{O};\mathbb{C})^3} = \int_{\mathcal{O}} K \cdot \big(\nabla \times \psi\big)\, dx - \\
\int_{\mathcal{O}} \big(\nabla \times K\big) \cdot \psi \, dx \quad \forall \psi \in H^1(\mathcal{O};\mathbb{C})^3.
\end{aligned}
\tag{14}
$$

We further point out that the real- and imaginary-parts of an element $z \in \mathbb{C}$ are denoted by $\mathcal{R}e\, z$ and $\mathfrak{I}m\, z$, respectively. Further, its complex conjugate is written as $\overline{z}$. Let us now state the general assumption for the data involved in (12).

Assumption 1.

(i) *The domain* $\mathcal{D} \subset \mathbb{R}^3$ *is bounded and simply connected with a connected boundary* $\partial\mathcal{D}$*. The domain* $\mathcal{D}$ *is either of class* $\mathcal{C}^{1,1}$ *or convex. The subdomain* Ω *is assumed to be Lipschitz in the appropriate sense of Grisvard [14].*

(ii) *The functions* $d : \Omega \times \mathbb{R} \to \mathbb{R}$ *and* $\kappa : \Omega \times \mathbb{R} \to \mathbb{R}$ *are Carathéodory functions: For almost all fixed* $x \in \Omega$ *the functions* $d(x,\cdot)$ *and* $\kappa(x,\cdot)$ *are continuous and, for each fixed* $y \in \mathbb{R}$*, the functions* $d(\cdot,y)$ *and* $\kappa(\cdot,y)$ *are Lebesgue measurable. Also, assume that the function* $d(x,\cdot)$ *for almost all fixed* $x \in \Omega$ *is monotone non-decreasing and there exists a constant* $\kappa_l > 0$ *such that*

$$\kappa_l \le \kappa(x,y) \quad \text{for a.a. } x \in \Omega \text{ and all } y \in \mathbb{R}. \tag{15}$$

For every $M > 0$*, there exists* $C_M > 0$ *such that*

$$|d(x,y)| + |\kappa(x,y)| \le C_M \quad \text{for a.a. } x \in \Omega \text{ and all } y \in [-M, M]. \tag{16}$$

(iii) *The function* $\sigma : \Omega \times \mathbb{R} \to \mathbb{R}$ *is also a Carathéodory function. There exist an exponent* $q > 3$*, a positive function* $\sigma^* \in L^q(\Omega)$ *and a constant* $\sigma_l > 0$ *such that*

$$\sigma_l \le \sigma(x,y) \le \sigma^*(x) \quad \text{for a.a. } x \in \Omega \text{ and all } y \in \mathbb{R}. \tag{17}$$

Finally, we assume that $\mu \in L^\infty(\mathcal{D}), \sigma_R \in L^\infty(R)$ *and there exists a constant* $C_0 > 0$ *such that* $C_0 \le \sigma_R(x)$ *for all* $x \in R$ *and* $C_0 \le \mu(x)$ *for all* $x \in \mathcal{D}$*.*

3 Existence and regularity of solutions to (12)

This section addresses the existence and regularity of the solution to the nonlinear coupled system (12).

Definition 1. *The space* $X_{N,0}(\mathcal{D})$ *is defined by*

$$X_{N,0}(\mathcal{D}) := \{K \in H(curl;\mathcal{D}) \cap H(div;\mathcal{D}) \mid \nabla\cdot K = 0 \text{ in } \mathcal{D},\ K \times \vec{n} = 0 \text{ on } \partial\mathcal{D}\}.$$

The upcoming lemma shows that the L^2-norm of a function in $X_{N,0}(\mathcal{D})$ can be estimated by the L^2-norm of its curl (cf. [19] and the references cited there).

Lemma 1 ([19, Corollary 3.51]). *Let $\mathcal{D} \subset \mathbb{R}^3$ be a bounded Lipschitz domain. If $\mathcal{D}$ is simply connected and has a connected boundary $\partial\mathcal{D}$, then there exists a constant $c_{\mathcal{D}} > 0$ such that*

$$\|K\|_{L^2(\mathcal{D};\mathbb{C})^3} \leq c_{\mathcal{D}} \|\nabla \times K\|_{L^2(\mathcal{D};\mathbb{C})^3} \quad \forall K \in X_{N,0}(\mathcal{D}).$$

Another well-known important result ensuring that the space $X_{N,0}(\mathcal{D})$ is continuously embedded in $H^1(\mathcal{D};\mathbb{C})^3$ is summarized in the following lemma:

Lemma 2 ([3, Theorem 2.12 and Theorem 2.17]). *Let $\mathcal{D} \subset \mathbb{R}^3$ be a bounded domain. If $\mathcal{D}$ is of class $\mathcal{C}^{1,1}$ or convex, then the injection $X_{N,0}(\mathcal{D}) \hookrightarrow H^1(\mathcal{D};\mathbb{C})^3$ holds.*

In the upcoming definition, we introduce the notion of (weak) solution to (12), which is derived formally using (14).

Definition 2. *A pair $(y, A) \in H_0^1(\Omega) \times X_{N,0}(\mathcal{D})$ is said to be a solution to* (12) *if and only if it satisfies*

$$\begin{aligned} \int_\Omega \kappa(x,y)\nabla y \cdot \nabla\phi \, dx + \int_\Omega d(x,y)\phi \, dx &= \int_\Omega \frac{\omega^2}{2}\sigma(x,y)|A|^2 \; \phi \, dx \\ \int_{\mathcal{D}} \big(\frac{1}{\mu}\nabla \times A\big) \cdot \big(\nabla \times \overline{\psi}\big) \, dx + i\omega\Big(\int_\Omega \sigma(x,y) A \cdot \overline{\psi} \, dx + \int_R \sigma_R A \cdot \overline{\psi} \, dx\Big) &= \\ \sum_{j=1}^n u_j \int_R J_j \cdot \overline{\psi} \, dx \quad \forall (\phi, \psi) \in H_0^1(\Omega) \times X_{N,0}(\mathcal{D}). \end{aligned}$$

Proposition 1. *Let Assumption 1 be satisfied and let $u \in \mathbb{R}^n$. Then, for every $y \in L^2(\Omega)$, the variational problem*

$$\begin{aligned} \alpha_y(A, \psi) := \int_{\mathcal{D}} \big(\frac{1}{\mu}\nabla \times A\big) \cdot \big(\nabla \times \overline{\psi}\big) \, dx + i\omega\Big(\int_\Omega \sigma(x,y) A \cdot \overline{\psi} \, dx + \\ \int_R \sigma_R A \cdot \overline{\psi} \, dx\Big) = \sum_{j=1}^n u_j \int_R J_j \cdot \overline{\psi} \, dx \quad \forall \psi \in X_{N,0}(\mathcal{D}) \end{aligned} \tag{18}$$

admits a unique solution $A = A(y) \in X_{N,0}(\mathcal{D})$. Furthermore, the solution satisfies the following a priori estimate:

$$\|A(y)\|_{X_{N,0}(\mathcal{D})} \leq c|u|, \tag{19}$$

with a constant $c > 0$ independent of A, y and u. If $y_k \to y$ strongly in $L^2(\Omega)$, then $A(y_k) \to A(y)$ strongly in $X_{N,0}(\mathcal{D})$.

Proof. By virtue of Lemma 1, we may use the following norm

$$\|\psi\|_{X_{N,0}(\mathcal{D})} := \|\nabla \times \psi\|_{L^2(\mathcal{D};\mathbb{C})^3} \quad \forall \psi \in X_{N,0}(\mathcal{D}).$$

Consequently, the sesquilinear form α_y is coercive and bounded in $X_{N,0}(\mathcal{D})$ such that the Lax-Milgram lemma implies that (18) admits a unique solution $A = A(y) \in X_{N,0}(\mathcal{D})$.

Suppose that $\{y_k\}_{k=1}^{\infty} \subset L^2(\Omega)$ such that $y_k \to y$ strongly in $L^2(\Omega)$. We set $A = A(y)$ and $A_k = A(y_k)$ for all $k \in \mathbb{N}$. Then, the difference $A_k - A$ satisfies

$$\int_{\mathcal{D}} \frac{1}{\mu} \nabla \times (A_k - A) \cdot \nabla \times \overline{\psi}\, dx + i\omega\Big(\int_{\Omega} \sigma(x, y_k) A_k \cdot \overline{\psi}\, dx \\ - \int_{\Omega} \sigma(x, y) A \cdot \overline{\psi}\, dx\Big) + i\omega \int_{R} \sigma_R (A_k - A) \cdot \overline{\psi}\, dx = 0 \quad \forall \psi \in X_{N,0}(\mathcal{D})$$

which is equivalent to

$$\int_{\mathcal{D}} \frac{1}{\mu} \nabla \times (A_k - A) \cdot \nabla \times \overline{\psi}\, dx + i\omega\Big(\int_{\Omega} \sigma(x, y_k)(A_k - A) \cdot \overline{\psi}\, dx + \\ \int_{R} \sigma_R (A_k - A) \cdot \overline{\psi}\, dx\Big) = i\omega \int_{\Omega} (\sigma(x, y) - \sigma(x, y_k)) A \cdot \overline{\psi}\, dx \quad \forall \psi \in X_{N,0}(\mathcal{D}).$$

Setting $\psi = A_k - A$ in the above equality results in

$$\Big| \int_{\mathcal{D}} \frac{1}{\mu} |\nabla \times (A_k - A)|^2\, dx + i\omega\Big(\int_{\Omega} \sigma(x, y_k) |A_k - A|^2\, dx + \int_{R} \sigma_R |A_k - A|^2\, dx\Big)\Big| \\ = \Big| i\omega \int_{\Omega} \Big(\sigma(x, y) - \sigma(x, y_k) \Big) A \cdot \overline{(A_k - A)}\, dx \Big|.$$

Consequently, Hölder's inequality along with the injection $X_{N,0}(\mathcal{D}) \hookrightarrow H^1(\mathcal{D};\mathbb{C})^3 \hookrightarrow L^6(\mathcal{D};\mathbb{C})^3$ implies that

$$\|\mu\|^{-1}_{L^\infty(\mathcal{D})} \|A_k - A\|^2_{X_{N,0}(\mathcal{D})} \le \Big| i\omega \int_{\Omega} \Big(\sigma(x, y) - \sigma(x, y_k) \Big) A \cdot \overline{(A_k - A)}\, dx \Big| \\ \le \omega \|\sigma(\cdot, y) - \sigma(\cdot, y_k)\|_{L^2(\Omega)} \|A\|_{L^4(\mathcal{D};\mathbb{C})^3} \|A_k \\ - A\|_{L^4(\mathcal{D};\mathbb{C})^3} \le c \|\sigma(\cdot, y) - \sigma(\cdot, y_k)\|_{L^2(\Omega)} \|A_k - A\|_{X_{N,0}(\mathcal{D})}.$$

Thus, there exists a constant $c > 0$ independent of k such that

$$\|A_k - A\|_{X_{N,0}(\mathcal{D})} \leq c\|\sigma(\cdot, y) - \sigma(\cdot, y_k)\|_{L^2(\Omega)}. \tag{20}$$

On the other hand, in view of Lebesgue's dominated convergence theorem (see e.g. [23, Section 4.2.3]), (17) along with the convergence $y_k \to y$ in $L^2(\Omega)$ yields the convergence $\sigma(\cdot, y_k) \to \sigma(\cdot, y)$ in $L^2(\Omega)$ as $k \to \infty$. This convergence together with (20) completes the proof. □

For the remainder of the presentation, the norm $\|\psi\|_{X_{N,0}(\mathcal{D})} = \|\nabla \times \psi\|_{L^2(\mathcal{D};\mathbb{C})^3}$ is used. With Lemma 2 and Proposition 1 at hand, we establish the existence of solutions to (12) in the following theorem:

Theorem 1. *Let Assumption 1 be satisfied and let $u \in \mathbb{R}^n$. Then, the state equation* (12) *admits a solution $(y, A) \in H_0^1(\Omega) \cap \mathcal{C}(\overline{\Omega}) \times X_{N,0}(\mathcal{D})$ satisfying the following a priori estimate:*

$$\|y\|_{H_0^1(\Omega)\cap\mathcal{C}(\overline{\Omega})} \leq c(|u|^2 + 1) \quad \text{and} \quad \|A\|_{X_{N,0}(\mathcal{D})} \leq c|u| \tag{21}$$

with a constant $c > 0$ independent of y, A and u.

Proof. To prove the assertion, we follow the lines of [8]. First of all, for every $y \in L^2(\Omega)$, let $A(y) \in X_{N,0}(\mathcal{D})$ be the unique solution of (18). Note that Proposition 1 and the embedding

$$X_{N,0}(\mathcal{D}) \hookrightarrow H^1(\mathcal{D};\mathbb{C})^3 \hookrightarrow L^6(\mathcal{D};\mathbb{C})^3 \tag{22}$$

imply that the mapping $y \mapsto A(y)$ is continuous from $L^2(\Omega)$ to $L^6(\mathcal{D};\mathbb{C})^3$. Now the state equation (12) can equivalently be expressed as

$$\begin{cases} -\nabla \cdot (\kappa(\cdot, y)\,\nabla y) + d(\cdot, y) = \dfrac{1}{2}\omega^2\sigma(\cdot, y)|A(y)|^2 & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega. \end{cases} \tag{23}$$

For the time being let $M > 0$ and we introduce the following truncated functions κ_M and d_M:

$$\kappa_M(x, y) := \begin{cases} \kappa(x, y) & |y| \leq M \\ \kappa(x, M) & y > M \\ \kappa(x, -M) & y < -M \end{cases} \quad d_M(x, y) := \begin{cases} d(x, y) & |y| \leq M \\ d(x, M) & y > M \\ d(x, -M) & y < -M. \end{cases}$$

Then, in view of (16), there exists a constant $C_M > 0$ such that

$$|d_M(x,y)| + |\kappa_M(x,y)| \le C_M \qquad \text{for all } y \in \mathbb{R} \text{ and almost all } x \in \Omega. \quad (24)$$

Let us introduce an operator $\mathcal{F} : L^2(\Omega) \to H_0^1(\Omega)$ where $\mathcal{F}(v) = y$ is defined by the unique solution of

$$\begin{cases} -\mathrm{div}(\kappa_M(\cdot,v)\,\nabla y) + d_M(\cdot,v) = \dfrac{1}{2}\omega^2\sigma(\cdot,v)|A(v)|^2 & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega. \end{cases} \quad (25)$$

According to (17) and (24)

$$-d_M(\cdot,v) + \frac{1}{2}\omega^2 \underbrace{\sigma(\cdot,v)}_{\in L^2(\Omega)}\underbrace{|A(v)|^2}_{\in L^3(\Omega)} \in L^{\frac{6}{5}}(\Omega).$$

Hence, by (15) and the embedding $L^{\frac{6}{5}}(\Omega) \hookrightarrow H^{-1}(\Omega)$, the Lax-Milgram lemma immediately implies that (25) admits a unique solution $y = y(v) \in H_0^1(\Omega)$. In addition, by virtue of (17), (19), (22) and (24), the solution satisfies

$$\begin{aligned} \|y(v)\|_{H_0^1(\Omega)} &\le c\|\frac{1}{2}\omega^2\sigma(\cdot,v)|A(v)|^2 - d_M(\cdot,v)\|_{L^{\frac{6}{5}}(\Omega)} \\ &\le c(\|\sigma^*\|_{L^2(\Omega)}\||A(v)|^2\|_{L^3(\Omega)} + \|d_M(\cdot,v)\|_{L^{\frac{6}{5}}(\Omega)}) \\ &\le c(\|A(v)\|^2_{L^6(\mathcal{D};\mathbb{C})^3} + 1) \le c(|u|^2+1) \qquad \forall v \in L^2(\Omega) \end{aligned} \quad (26)$$

with a constant $c > 0$ independent of v, y, u, A.

Let us now consider the operator $\mathcal{F}$ as an operator in $L^2(\Omega)$. In the following, we verify that $\mathcal{F} : L^2(\Omega) \to L^2(\Omega)$ is continuous. Suppose that a sequence $\{v_k\}_{k=1}^\infty \subset L^2(\Omega)$ converges strongly to a $v \in L^2(\Omega)$. The solution of (25) associated with v_k is denoted by $y(v_k) = y_k \in H_0^1(\Omega)$ for all $k \in \mathbb{N}$ and $y(v) = y \in H_0^1(\Omega)$. By (26), $\{y_k\}_{k=1}^\infty$ is uniformly bounded in the $H_0^1(\Omega)$–topology and hence there exists a subsequence $\{y_{k_j}\}_{j=1}^\infty \subset \{y_k\}_{k=1}^\infty$ converging strongly in $L^2(\Omega)$ to a $\tilde{y} \in L^2(\Omega)$. Let us show that $\tilde{y} = y$. First, the difference $y_{k_j} - y$ satisfies

$$\begin{aligned} \int_\Omega \kappa_M(x,v_{k_j})\nabla(y_{k_j} - y)\cdot\nabla\phi\,dx &= \int_\Omega \big(\kappa_M(x,v) - \kappa_M(x,v_{k_j})\big)\nabla y\cdot\nabla\phi\,dx \\ + \int_\Omega \big(d_M(x,v) - d_M(x,v_{k_j})\big)\phi\,dx &+ \frac{\omega^2}{2}\int_\Omega \big(\sigma(x,v_{k_j}) - \sigma(x,v)\big)|A(v)|^2\phi\,dx \\ &+ \frac{\omega^2}{2}\int_\Omega \sigma(x,v_{k_j})\big(|A(v_{k_j})|^2 - |A(v)|^2\big)\phi\,dx \quad \forall\phi \in H_0^1(\Omega). \end{aligned}$$

Setting $\phi = y_{k_j} - y \in H_0^1(\Omega)$ in the latter variational equality, taking (15) and (17) into account and using Hölder's inequality in the resulting inequality, we infer that

$$\begin{aligned}\kappa_l \|y_{k_j} - y\|^2_{H_0^1(\Omega)} \leq{}& \|(\kappa_M(\cdot,v) - \kappa_M(\cdot,v_{k_j}))\nabla y\|_{L^2(\Omega)} \|y_{k_j} - y\|_{H_0^1(\Omega)} \\ &+ \|d_M(\cdot,v) - d_M(\cdot,v_{k_j})\|_{L^2(\Omega)} \|y_{k_j} - y\|_{L^2(\Omega)} + \\ &\bigg(\frac{\omega^2}{2}\|\sigma(\cdot,v_{k_j}) - \sigma(\cdot,v)\|_{L^2(\Omega)} \||A(v)|^2\|_{L^3(\Omega)} + \\ &\frac{\omega^2}{2}\|\sigma^*\|_{L^3(\Omega)} \|\,|A(v_{k_j})|^2 - |A(v)|^2\|_{L^2(\Omega)}\bigg) \|y_{k_j} - y\|_{L^6(\Omega)}.\end{aligned}$$

Hence, it follows that

$$\begin{aligned}\kappa_l \|y_{k_j} - y\|_{H_0^1(\Omega)} \leq c\Big(& \|(\kappa_M(\cdot,v) - \kappa_M(\cdot,v_{k_j}))\nabla y\|_{L^2(\Omega)} \\ &+ \|d_M(\cdot,v) - d_M(\cdot,v_{k_j})\|_{L^2(\Omega)} + \|\sigma(\cdot,v_{k_j}) \\ &- \sigma(\cdot,v)\|_{L^2(\Omega)} + \||A(v_{k_j})|^2 - |A(v)|^2\|_{L^2(\Omega)}\Big)\end{aligned} \tag{27}$$

holds with a constant $c > 0$ independent of k. Analogously to an argument in the proof of Proposition 1, (17) and (24) ensure that

$$\begin{aligned} d_M(\cdot,v_{k_j}) &\to d_M(\cdot,v) \quad \text{in } L^2(\Omega) \quad \text{as } j \to \infty \\ \sigma(\cdot,v_{k_j}) &\to \sigma(\cdot,v) \quad \text{in } L^2(\Omega) \quad \text{as } j \to \infty. \end{aligned} \tag{28}$$

In addition, as mentioned previously, Proposition 1 and the embedding (22) imply that

$$|A(v_{k_j})|^2 \to |A(v)|^2 \quad \text{in } L^2(\Omega) \quad \text{as } j \to \infty. \tag{29}$$

By standard arguments, there exists a subsequence of $\{v_{k_j}\}_{j=1}^{\infty}$ denoted w.l.o.g. again by $\{v_{k_j}\}_{j=1}^{\infty}$ such that $v_{k_j}(x) \to v(x)$ for a.a. $x \in \Omega$ as $j \to \infty$. Consequently, since κ_M is continuous with respect to the second variable, we immediately obtain the following pointwise convergence:

$$\kappa_M(x,v_{k_j}(x))^2|\nabla y(x)|^2 \to \kappa_M(x,v(x))^2|\nabla y(x)|^2 \text{ for a.a. } x \in \Omega \quad \text{as } j \to \infty.$$

Hence, thanks to the uniform boundedness (24), Lebesgue's dominated convergence theorem implies that

$$\kappa_M(\cdot,v_{k_j})\nabla y \to \kappa_M(\cdot,v)\nabla y \quad \text{in } L^2(\Omega) \quad \text{as } j \to \infty. \tag{30}$$

Applying (28)–(30) to (27) implies that $y_{k_j} \to \tilde{y} = y$ strongly in $L^2(\Omega)$. In conclusion, every L^2–converging subsequence of $\{y_k\}_{n=1}^{\infty}$ converges strongly to y in $L^2(\Omega)$ so that, by a standard result, we gain the desired continuity of $\mathcal{F} : L^2(\Omega) \to L^2(\Omega)$. Moreover, the compactness of $\mathcal{F}$ is an immediate consequence of (26) and the fact that the injection $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Hence, along with (26), the Schauder fixpoint theorem implies that $\mathcal{F}$ admits a fixed point y_M. In other words, $y_M \in H_0^1(\Omega)$ is a solution to

$$\begin{cases} -\operatorname{div}(\kappa_M(\cdot, y_M)\,\nabla y_M) + d_M(\cdot, y_M) = \dfrac{1}{2}\omega^2\sigma(\cdot, y_M)|A(y_M)|^2 & \text{in } \Omega \\ y_M = 0 & \text{on } \partial\Omega. \end{cases} \tag{31}$$

We show now that y_M solves the original problem (23). On account of (17),

$$\frac{1}{2}\omega^2 \underbrace{\sigma(\cdot, y_M)}_{\in L^q(\Omega)} \underbrace{|A(y_M)|^2}_{\in L^3(\Omega)} \in L^{\frac{3q}{3+q}}(\Omega).$$

Since $q > 3$, we have $\frac{3q}{3+q} > \frac{3}{2}$. Consequently, taking (15) into account, the application of Stampacchia technique (see Tröltzsch [23, Theorem 7.3]) implies that y_M is bounded and there exists a constant $c > 0$ independent of y_M, $A(y_M)$, $\kappa_M(\cdot, y_M)$ and d_M such that

$$\|y_M\|_{L^\infty(\Omega)} \le c\|\frac{\omega^2}{2}\sigma(\cdot, y_M)|A(y_M)|^2 - d(\cdot, 0)\|_{L^{\frac{3q}{3+q}}(\Omega)}.$$

Thus, (17) and (19) yield

$$\|y_M\|_{L^\infty(\Omega)} \le c_\infty(|u|^2 + 1) \tag{32}$$

with a constant $c_\infty > 0$ independent of y_M, $A(y_M)$, $\kappa_M(\cdot, y_M)$, d_M and u. To show that $y_M \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a solution to the original problem (23), we choose $M > c_\infty(|u|^2 + 1)$, then (32) implies

$$\kappa_M(x, y_M(x)) = \kappa(x, y_M(x)), \quad d_M(x, y_M(x)) = d(x, y_M(x)) \quad \text{for a.a. } x \in \Omega.$$

In conclusion, $y_M \in H_0^1(\Omega) \cap L^\infty(\Omega)$ solves the original problem (23) for sufficiently large M. Finally, the continuity $y_M \in \mathcal{C}(\overline{\Omega})$ follows from a well-known regularity result for elliptic linear problems (see e.g. [1]). □

Let us address the $W_0^{1,q}(\Omega)$–regularity result for the y-solution of (12). For this purpose, we need a further regularity assumption on the domain Ω and κ:

Assumption 2. *The boundary $\partial\Omega$ is assumed to be of class $\mathcal{C}^1$. Further, there exist disjoint subdomains $\Omega_j \subset \Omega$, $j = 1, \ldots, s$. Each boundary $\partial\Omega_j$ does not touch $\partial\Omega$ and is of class $\mathcal{C}^1$. The heat conductivity κ is assumed to be continuous on $\Omega \setminus \{\bigcup_{j=1}^{s} \Omega_j\} \times \mathbb{R}$ and $\Omega_j \times \mathbb{R}$ for all $j = 1, \ldots, s$.*

Proposition 2. *Let Assumption 1 and Assumption 2 be satisfied and let $u \in \mathbb{R}^n$. Then, every solution $(y, A) \in H_0^1(\Omega) \cap \mathcal{C}(\overline{\Omega}) \times X_{N,0}(\mathcal{D})$ of (12) has extra regularity $y \in W_0^{1,q}(\Omega)$ with $q > 3$ as in Assumption 1. Further, the following a priori estimate*

$$\|y\|_{W_0^{1,q}(\Omega)} \leq c(|u|^2 + 1) \tag{33}$$

holds with a constant $c > 0$ independent of A, y and u.

Proof. Let $(y, A) \in H_0^1(\Omega) \cap \mathcal{C}(\overline{\Omega}) \times X_{N,0}(\mathcal{D})$ be a solution to the state equation (12). Then, y satisfies

$$\int_\Omega \kappa(x, y)\nabla y \cdot \nabla\phi \, dx = \int_\Omega \Big(- d(x, y) + \frac{\omega^2}{2}\sigma(x, y)|A|^2 \Big) \phi \, dx \quad \forall \phi \in H_0^1(\Omega). \tag{34}$$

We introduce the elliptic operator $B(y) : W_0^{1,q}(\Omega) \to W^{-1,q}(\Omega)$ defined by

$$\langle B(y)\zeta, \phi\rangle_{W^{-1,q}(\Omega), W_0^{1,q'}(\Omega)} = \int_\Omega \kappa_y \nabla\zeta \cdot \nabla\phi \, dx \quad \forall \phi \in W_0^{1,q'}(\Omega)$$

where the function κ_y defined by

$$\kappa_y(x) = \kappa(x, y(x)) \quad \text{for a.a. } x \in \Omega.$$

Thanks to the regularity $y \in \mathcal{C}(\overline{\Omega})$ and Assumption 2 , κ_y is continuous on $\Omega \setminus \{\bigcup_{j=1}^{s} \Omega_j\}$ and Ω_j for all $j = 1, \ldots, s$. Hence, by the regularity assumption on the interfaces stated in Assumption 2 , the elliptic regularity result [13, Theorem 1.1] immediately implies that $B(y) : W_0^{1,q}(\Omega) \to W^{-1,q}(\Omega)$ is an isomorphism. In the proof of Theorem 1, we have mentioned that $-d(\cdot, y) + \frac{\omega^2}{2}\sigma(\cdot, y)|A|^2$ belongs to $L^{\frac{3q}{3+q}}(\Omega)$. For this reason, on account of the embedding $L^{\frac{3q}{3+q}}(\Omega) \hookrightarrow W^{-1,q}(\Omega)$ (see e.g. Nečas [20, Theorem 3.4]), we can define the element

$$\zeta := B(y)^{-1}\Big(- d(\cdot, y) + \frac{\omega^2}{2}\sigma(\cdot, y)|A|^2 \Big) \in W_0^{1,q}(\Omega). \tag{35}$$

Then, according to the definition of $B(y)$, it follows that ζ is the unique solution of

$$\int_\Omega \kappa(x,y)\nabla\zeta\cdot\nabla\phi\,dx = \int_\Omega \Big(-d(x,y)+\frac{\omega^2}{2}\sigma(x,y)|A|^2\Big)\phi\,dx \quad \forall\phi\in W_0^{1,q'}(\Omega). \tag{36}$$

By classical bootstrapping arguments, (34) and (36) together with (15) yield $y=\zeta$ in $W_0^{1,q}(\Omega)$. Finally, the a priori estimate (33) follows from (35) along with the continuity of $B(y)^{-1}$, (17) and (21). ∎

We point out that the variational form associated with (12) can be concisely written as an operator equation in an appropriate dual space. Later on, this formulation will be interpreted as an equality PDE-constraint in the control problem (P). The corresponding operator is introduced in the upcoming definition. For the remainder of the paper, let $q>3$ be as in Assumption 1.

Definition 3.

(i) The operator $\mathscr{A}:\mathbb{R}^n\times W_0^{1,q}(\Omega)\to X_{N,0}(\mathcal{D})$ *assigns to every element* $(u,y)\in\mathbb{R}^n\times W_0^{1,q}(\Omega)$ *the unique solution* $A\in X_{N,0}(\mathcal{D})$ *of*

$$\int_\mathcal{D} (\frac{1}{\mu}\nabla\times A)\cdot(\nabla\times\overline{\psi})\,dx + i\omega(\int_\Omega \sigma(x,y)A\cdot\overline{\psi}\,dx + \int_R \sigma_R A\cdot\overline{\psi}\,dx) = \sum_{j=1}^n u_j\int_R J_j\cdot\overline{\psi}\,dx \qquad \text{for all } \psi\in X_{N,0}(\mathcal{D}).$$

(ii) The operator $C:\mathbb{R}^n\times W_0^{1,q}(\Omega)\to W^{-1,q}(\Omega)$ *is defined by*

$$\langle C(u,y),\phi\rangle := \int_\Omega \kappa(x,y)\nabla y\cdot\nabla\phi\,dx + \int_\Omega d(x,y)\phi\,dx - \frac{\omega^2}{2}\int_\Omega \sigma(x,y)|\mathscr{A}(u,y)|^2\phi\,dx$$

for all $(u,y)\in\mathbb{R}^n\times W_0^{1,q}(\Omega)$ *and all* $\phi\in W_0^{1,q'}(\Omega)$.

In what follows, we only concentrate on the temperature-reduced system in the sense that the magnetic vector potential A is written in terms of $\mathscr{A}(u,y)$. Thus, taking the operator C into account, the weak formulation of

(12) can be equivalently expressed as the following operator equation with respect to the $W^{-1,q}(\Omega)$–topology:

$$C(u, y) = 0 \quad \text{in } W^{-1,q}(\Omega). \tag{37}$$

According to Theorem 1 and Proposition 2, for every given control $u \in \mathbb{R}^n$, there exists at least one state $y \in W_0^{1,q}(\Omega)$ satisfying (37).

4 Linearized equation

This section deals with the linearized system associated with (12). Our goal is to establish the surjectivity of the derivative of the operator C at any given reference point (u^*, y^*) which is specified later as an optimal solution to (P). This issue is complicated by the non-monotonic structure of the corresponding linearized system, in which case the theorem on monotone operators or the Lax-Milgram lemma are not applicable. Notice that the surjectivity property is mainly important in order to derive the existence of Lagrange multipliers associated with the control problem (P). Once the surjectivity is established, the existence of multipliers can be directly derived by means of the classical result of Kurcyusz and Zowe [24]. In the following, additional assumptions on the functions κ, σ and d are made:

Assumption 3. *The functions κ, σ and d are continuously differentiable with respect to the second variable. There exists a constant $c_0 > 0$ and, for every $M > 0$, there exists a constant $L(M)$ such that*

$$|\frac{\partial \kappa}{\partial y}(x, 0)| + |\frac{\partial d}{\partial y}(x, 0)| + |\frac{\partial \sigma}{\partial y}(x, 0)| \leq c_0$$

$$|\frac{\partial \kappa}{\partial y}(x, y_1) - \frac{\partial \kappa}{\partial y}(x, y_2)| + |\frac{\partial \sigma}{\partial y}(x, y_1) - \frac{\partial \sigma}{\partial y}(x, y_2)| + \\ |\frac{\partial d}{\partial y}(x, y_1) - \frac{\partial d}{\partial y}(x, y_2)| \leq L(M)|y_1 - y_2|$$

hold for a.a. $x \in \Omega$ and all $y_1, y_2 \in [-M, M]$.

Thanks to Assumption 3 and the embedding $W_0^{1,q}(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$, the operators $\mathscr{A} : \mathbb{R}^n \times W_0^{1,q}(\Omega) \to X_{N,0}(\mathcal{D})$ and $C : \mathbb{R}^n \times W_0^{1,q}(\Omega) \to W^{-1,q}(\Omega)$ are continuously differentiable.

In what follows, let $(u^*, y^*) \in \mathbb{R}^n \times W_0^{1,q}(\Omega)$ be a reference point and $A^* = \mathscr{A}(u^*, y^*) \in X_{N,0}(\mathcal{D})$. The derivative of C at (u^*, y^*) in an arbitrary direction $(u, y) \in \mathbb{R}^n \times W_0^{1,q}(\Omega)$ is given by

$$\begin{aligned}\langle C'(u^*, y^*)(u, y), \phi\rangle_{W^{-1,q}(\Omega), W_0^{1,q'}(\Omega)} &= \\ \int_\Omega \Big(\kappa(x, y^*)\,\nabla y + \frac{\partial \kappa}{\partial y}(x, y^*) y \nabla y^*\Big) \cdot \nabla \phi \, dx & \\ + \int_\Omega \frac{\partial d}{\partial y}(x, y^*) y \phi \, dx - \frac{\omega^2}{2} \int_\Omega \frac{\partial \sigma}{\partial y}(x, y^*) y |A^*|^2 \phi \, dx & \\ -\omega^2 \int_\Omega \sigma(x, y^*) \Big(\mathcal{R}e A^* \cdot \mathcal{R}e\big(\mathscr{A}'(u^*, y^*)(u, y)\big) & \\ + \mathfrak{I}m A^* \cdot \mathfrak{I}m\big(\mathscr{A}'(u^*, y^*)(u, y)\big)\Big) \phi \, dx \quad \forall \phi \in W_0^{1,q'}(\Omega), &\end{aligned} \tag{38}$$

where $\mathscr{A}'(u^*, y^*)(u, y) = A \in X_{N,0}(\mathcal{D})$ is given by the unique solution of

$$\begin{aligned}&\int_{\mathcal{D}} \big(\frac{1}{\mu} \nabla \times A\big) \cdot \big(\nabla \times \overline{\psi}\big) \, dx + i\omega\Big(\int_\Omega \sigma(x, y^*) A \cdot \overline{\psi} \, dx + \int_R \sigma_{R} A \cdot \overline{\psi} \, dx\Big) \\ &+ i\omega \int_\Omega \frac{\partial \sigma}{\partial y}(x, y^*) y A^* \cdot \overline{\psi} \, dx = \sum_{j=1}^n u_j \int_R J_j \cdot \overline{\psi} \, dx \qquad \forall \psi \in X_{N,0}(\mathcal{D}).\end{aligned} \tag{39}$$

Note that, for any given $G \in W^{-1,q}(\Omega)$, $C'(u^*, y^*)(u, y) = G$ corresponds to the following (strong) PDE-formulation:

$$\begin{cases} -\nabla \cdot \big(\kappa(\cdot, y^*)\,\nabla y + \frac{\partial \kappa}{\partial y}(\cdot, y^*) \quad y \nabla y^*\big) + \frac{\partial d}{\partial y}(\cdot, y^*) y - \frac{\omega^2}{2} \frac{\partial \sigma}{\partial y}(\cdot, y^*) y |A^*|^2 \\ \qquad = G + \omega \sigma(\cdot \;, y^*)\big(\mathcal{R}e\, A^* \cdot \mathcal{R}e\, A + \mathfrak{I}m\, A^* \cdot \mathfrak{I}m\, A\big) \text{in } \Omega \\ y = 0 \qquad \text{on } \partial\Omega \\ \nabla \times \big(\mu^{-1} \nabla \times A\big) + i\omega\sigma(\cdot, y^*) A = -i\omega \frac{\partial \sigma}{\partial y}(\cdot, y^*) y A^* \qquad \text{in } \Omega \\ \nabla \times \big(\mu^{-1} \nabla \times A\big) + \quad i\omega\sigma_{R} A = \sum_{j=1}^n u_j J_j \qquad \text{in } R \\ \nabla \times \big(\mu^{-1} \nabla \times A\big) = 0 \qquad \text{in } \mathcal{D} \setminus \overline{(\Omega \cup R)} \\ \nabla \cdot A = 0 \qquad \text{in } \mathcal{D} \\ A \times \vec{n} = 0 \qquad \text{on } \partial\mathcal{D} \\ [\big(\mu^{-1} \nabla \times A\big) \times \vec{n}]_{\partial R} = 0 \text{ on } \partial R \qquad [\big(\mu^{-1} \nabla \times A\big) \times \vec{n}]_{\partial\Omega} = 0 \text{ on } \partial\Omega. \end{cases} \tag{40}$$

Our first goal is to establish a condition such that, for every given $G \in W^{-1,q}(\Omega)$, the operator equation

$$\frac{\partial C}{\partial y}(u^*, y^*)y = G \quad \text{in } W^{-1,q}(\Omega) \tag{41}$$

admits a solution $y \in W_0^{1,q}(\Omega)$. The variational form associated with (41) is given by the following linear coupled system:

$$\begin{aligned}\int_\Omega \big(\kappa(x,y^*)\,\nabla y + \frac{\partial \kappa}{\partial y}(x,y^*)y\nabla y^*\big)\cdot\nabla\phi\,dx + \int_\Omega \frac{\partial d}{\partial y}(x,y^*)y\phi\,dx \\ -\frac{\omega^2}{2}\int_\Omega \frac{\partial \sigma}{\partial y}(x,y^*)y|A^*|^2\phi\,dx = \langle G,\phi\rangle_{W^{-1,q}(\Omega),W_0^{1,q'}(\Omega)} \\ +\omega^2\int_\Omega \sigma(x,y^*)\big(\mathcal{R}e\,A^*\cdot\mathcal{R}e\,A + \mathfrak{I}m\,A^*\cdot\mathfrak{I}m\,A\big)\phi\,dx \\ \forall\phi\in W_0^{1,q'}(\Omega)\end{aligned} \tag{42}$$

$$\begin{aligned}\int_{\mathcal{D}} (\frac{1}{\mu}\nabla\times A)\cdot\big(\nabla\times\overline{\psi}\big)\,dx + i\omega(\int_\Omega \sigma(x,y^*)A\cdot\overline{\psi}\,dx + \int_R \sigma_R A\cdot\overline{\psi}\,dx) \\ = -i\omega\int_\Omega \frac{\partial\sigma}{\partial y}(x,y^*)yA^*\cdot\overline{\psi}\,dx \quad \forall\psi\in X_{N,0}(\mathcal{D}).\end{aligned} \tag{43}$$

To devise the existence, we exploit first the regularity structure involved in (42)–(43). Since $y^* \in W_0^{1,q}(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$ holds for $q > 3$, Assumption 3 yields

$$\frac{\partial\kappa}{\partial y}(\cdot,y^*)\in L^\infty(\Omega), \quad \frac{\partial d}{\partial y}(\cdot,y^*)\in L^\infty(\Omega), \quad \frac{\partial\sigma}{\partial y}(\cdot,y^*)\in L^\infty(\Omega). \tag{44}$$

We now introduce the following operators:

$$\begin{aligned}
B(y^*) &: W_0^{1,q}(\Omega)\to W^{-1,q}(\Omega) & \langle B(y^*)v,\xi\rangle &= \int_\Omega \kappa(x,y^*)\nabla v\cdot\nabla\xi\,dx \\
Q(y^*) &: L^\infty(\Omega)\to W^{-1,q}(\Omega) & \langle Q(y^*)v,\xi\rangle &= \int_\Omega \tfrac{\partial\kappa}{\partial y}(x,y^*)v\nabla y^*\cdot\nabla\xi\,dx \\
D(y^*) &: L^\infty(\Omega)\to W^{-1,q}(\Omega) & \langle D(y^*)v,\xi\rangle &= \int_\Omega \tfrac{\partial d}{\partial y}(x,y^*)v\xi\,dx \\
\mathcal{K}(y^*,A^*) &: L^\infty(\Omega)\to W^{-1,q}(\Omega) & \langle \mathcal{K}(y^*,A^*)v,\xi\rangle &= \tfrac{\omega^2}{2}\int_\Omega \tfrac{\partial\sigma}{\partial y}(x,y^*)v|A^*|^2\xi\,dx.
\end{aligned} \tag{45}$$

Note that these operators appear in the left hand side of the variational form (42). On account of (44) as well as the regularity $y^* \in W_0^{1,q}(\Omega)$ and $|A^*|^2 \in L^3(\Omega)$, they are well-defined, continuous and linear in their respective spaces.

Next, let us define the operator associated with the right hand side of (42). For this purpose, we introduce the operator $\mathcal{R}(y^*, A^*) : L^2(\Omega) \to X_{N,0}(\mathcal{D})$ associated with the variational form (43). In other words, for every $v \in L^2(\Omega)$, $\mathcal{R}(y^*, A^*)v = A \in X_{N,0}(\mathcal{D})$ is given by the unique solution of

$$\begin{aligned}\int_{\mathcal{D}} (\frac{1}{\mu}\nabla \times A) \cdot (\nabla \times \overline{\psi})\, dx + i\omega(\int_\Omega \sigma(x, y^*) A \cdot \overline{\psi}\, dx + \int_R \sigma_R A \cdot \overline{\psi}\, dx) \\ = -i\omega \int_\Omega \frac{\partial \sigma}{\partial y}(x, y^*) v A^* \cdot \overline{\psi}\, dx \quad \forall \psi \in X_{N,0}(\mathcal{D}).\end{aligned} \tag{46}$$

Thanks to the regularity $\frac{\partial\sigma}{\partial y}(\cdot, y^*) \in L^\infty(\mathcal{D})$ and the embedding $X_{N,0}(\mathcal{D}) \hookrightarrow L^6(\mathcal{D};\mathbb{C})^3$, the right hand side of (46) given by

$$F_v(\psi) := -i\omega \int_\Omega \frac{\partial \sigma}{\partial y}(x, y^*) v A^* \cdot \overline{\psi}\, dx \quad \forall \psi \in X_{N,0}(\mathcal{D})$$

is well-defined as an element of $X_{N,0}(\mathcal{D})^*$. As a consequence, the Lax-Milgram lemma implies that the operator $\mathcal{R}(y^*, A^*) : L^2(\Omega) \to X_{N,0}(\mathcal{D})$ is well-defined, continuous and linear. Having established the operator $\mathcal{R}(y^*, A^*)$, we define the operator $\mathcal{T}(y^*, A^*) : L^2(\Omega) \to W^{-1,q}(\Omega)$ by

$$\begin{aligned}\langle \mathcal{T}(y^*, A^*)v, \phi\rangle := \omega^2 \int_\Omega \sigma(x, y^*)\big(\mathcal{R}e\, A^* \cdot \mathcal{R}e\,\big(\mathcal{R}(y^*, A^*)v\big) + \\ \mathfrak{Im}\, A^* \cdot \mathfrak{Im}\,\big(\mathcal{R}(y^*, A^*)v\big)\big)\phi\, dx \quad \forall \phi \in W_0^{1,q'}(\Omega)\end{aligned} \tag{47}$$

Let us remark that, due to $\sigma(\cdot, y^*) \in L^q(\Omega)$, $A^* \in X_{N,0}(\mathcal{D})$, $\mathcal{R}(y^*, A^*) \in \mathcal{L}(L^2(\Omega), X_{N,0}(\mathcal{D}))$ and the embedding $X_{N,0}(\mathcal{D}) \hookrightarrow L^6(\mathcal{D};\mathbb{C})^3$, we have

$$\sigma(x, y^*)\big(\mathcal{R}e\, A^* \cdot \mathcal{R}e\,\big(\mathcal{R}(y^*, A^*)v\big) + \mathfrak{Im}\, A^* \cdot \mathfrak{Im}\,\big(\mathcal{R}(y^*, A^*)v\big)\big) \in L^{\frac{3q}{3+q}}(\Omega)$$

$$\forall v \in L^2(\Omega)$$

and hence, by virtue of the embedding $L^{\frac{3q}{3+q}}(\Omega) \hookrightarrow W^{-1,q}(\Omega)$, the operator $\mathcal{T}(u^*, y^*) : L^2(\Omega) \to W^{-1,q}(\Omega)$ is well-defined, linear and continuous.

Employing all the operators defined previously, (41) can be equivalently written as:

$$
\begin{aligned}
\frac{\partial C}{\partial y}(u^*,y^*)y =& \Big(B(y^*) + \big(Q(y^*) + D(y^*) - \\
&\mathcal{K}(y^*,A^*)\big)\mathcal{I}_{\infty,q} - \mathcal{T}(y^*,A^*)\mathcal{I}_{2,q}\Big) y = G.
\end{aligned} \tag{48}
$$

In the above setting, the operators $\mathcal{I}_{\infty,q}$ and $\mathcal{I}_{2,q}$ denote the injections $W_0^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega)$ and $W_0^{1,q}(\Omega) \hookrightarrow L^2(\Omega)$, respectively.

In the proof of Theorem 1, we already mentioned that the elliptic operator $B(y^*) : W_0^{1,q}(\Omega) \to W^{-1,q}(\Omega)$ is a topological isomorphism. Consequently, applying $B(y^*)^{-1}$ to (48) results in

$$
B(y^*)^{-1}\frac{\partial C}{\partial y}(u^*,y^*)y = \big(I - \Psi(y^*,A^*)\big)y = B(y^*)^{-1}G \text{ in } W_0^{1,q}(\Omega) \tag{49}
$$

where $\Psi(y^*,A^*) : W_0^{1,q}(\Omega) \to W_0^{1,q}(\Omega)$ is given by

$$
\begin{aligned}
\Psi(y^*,A^*) := -B(y^*)^{-1}\big((Q(y^*)+D(y^*)-\mathcal{K}(y^*,A^*))\mathcal{I}_{\infty,q} \\
-\mathcal{T}(y^*,A^*)\mathcal{I}_{2,q}\big).
\end{aligned} \tag{50}
$$

This motivates the following assumption:

Assumption 4. *Suppose that $\lambda = 1$ is not an eigenvalue of $\Psi(y^*,A^*) : W_0^{1,q}(\Omega) \to W_0^{1,q}(\Omega)$.*

Theorem 2. *Let Assumptions 1, 2 and 3 be satisfied. Further, let $(u^*,y^*) \in \mathbb{R}^n \times W_0^{1,q}(\Omega)$ and $A^* = \mathscr{A}(u^*,y^*)$. If Assumption 4 is satisfied, then $\frac{\partial C}{\partial y}(u^*,y^*) : W_0^{1,q}(\Omega) \to W^{-1,q}(\Omega)$ is an isomorphism and consequently, for every $G \in W^{-1,q}(\Omega)$, the equation (41) has a solution $(y,A) \in W_0^{1,q}(\Omega) \times X_{N,0}(\mathcal{D})$.*

Proof. Since $q > 3$, the injections $\mathcal{I}_{\infty,q} : W_0^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega)$ and $\mathcal{I}_{2,q} : W_0^{1,q}(\Omega) \hookrightarrow L^2(\Omega)$ are compact such that $\Psi(y^*,A^*) : W_0^{1,q}(\Omega) \to W_0^{1,q}(\Omega)$ is in turn compact. Consequently, Fredholm's theorem along with Assumption 4 implies that the operator $\big(I - \Psi(y^*,A^*)\big) : W_0^{1,q}(\Omega) \to W_0^{1,q}(\Omega)$ is continuously invertible and hence the assertion immediately follows. ∎

Remark 1. *The regularity $y^* \in W_0^{1,q}(\Omega)$ with $q > 3$ is the key point of the whole argumentation. Without such regularity, we would not have the compactness of the operator $\Psi(y^*, A^*)$ and the Fredholm alternative would not be applicable. Notice that, as every compact operator possesses only countably many eigenvalues, Assumption 4 seems to be reasonable.*

An immediate consequence of Theorem 2 is the following uniqueness result for solutions to the state equation (12):

Corollary 1 (Uniqueness result for (12))**.** *Let Assumptions 1, 2 and 3 be satisfied and let $u^* \in \mathbb{R}^n$. Further, let $y^* \in W_0^{1,q}(\Omega)$ satisfy $C(u^*, y^*) = 0$ and $A^* = \mathscr{A}(u^*, y^*)$. If Assumption 4 is fulfilled, then there exists an open neighborhood $\mathcal{B}_{u^*}$ of u^* in $\mathbb{R}^n$ such that for every $u \in \mathcal{B}_{u^*}$ there exists a unique $y \in W_0^{1,q}(\Omega)$ satisfying C(u,y)=0. In conclusion, for every $u \in \mathcal{B}_{u^*}$, the state equation* (12) *admits a unique solution $(y, A) \in W_0^{1,q}(\Omega) \times X_{N,0}(\mathcal{D})$.*

Proof. Thanks to Theorem 2, $\frac{\partial C}{\partial y}(u^*, y^*) : W_0^{1.q}(\Omega) \to W^{-1,q}(\Omega)$ is continuously invertible. Then, the assertion follows immediately from the implicit function theorem. □

In the following, we establish a fairly simple example which meets the condition that $\lambda = 1$ is not an eigenvalue of the compact operator $\Psi(y^*, A^*) : W_0^{1,q}(\Omega) \to W_0^{1,q}(\Omega)$.

Example 1. *Let Assumptions 1, 2 and 3 be satisfied and let $(u^*, y^*) \in \mathbb{R}^n \times W_0^{1,q}(\Omega)$. If $\frac{\partial \sigma}{\partial y}(\cdot, y^*) = 0$, then $\lambda = 1$ is not an eigenvalue of the operator $\Psi(y^*, A^*) : W_0^{1,q}(\Omega) \to W_0^{1,q}(\Omega)$.*

Proof. We justify that

$$\Big(I - \Psi(y^*, A^*)\Big)y = 0 \quad \text{in } W_0^{1,q}(\Omega) \tag{51}$$

admits only the trivial solution $y = 0$. Let $y \in W_0^{1,q}(\Omega)$ be a solution to (51) and hence, by the definition of $\Psi(y^*, A^*)$ in (50), y satisfies

$$\Big(B(y^*) + \big(Q(y^*) + D(y^*) - \mathcal{K}(y^*, A^*)\big)\mathcal{I}_{\infty,q} - \mathcal{T}(y^*, A^*)\mathcal{I}_{2,q}\Big)y = 0.$$

Since $\frac{\partial \sigma}{\partial y}(\cdot, y^*) = 0$, if follows that $\Big(B(y^*) + Q(y^*) + D(y^*)\Big)y = 0$. Hence, according to (45), y satisfies

$$\begin{aligned}\int_\Omega \kappa(x,y^*)\,\nabla y\cdot\nabla\phi\,dx + \frac{\partial\kappa}{\partial y}(x,y^*)y\nabla y^*\cdot\nabla\phi\,dx +\\ \int_\Omega \frac{\partial d}{\partial y}(x,y^*)y\phi\,dx = 0 \quad \forall\phi\in W_0^{1,q'}(\Omega).\end{aligned} \tag{52}$$

From the above equation, the comparison principle of Casas and Tröltzsch [8] implies that $y = 0$. □

Let us now turn to the case where $\lambda = 1$ is an eigenvalue of $\Psi(y^*, A^*) : W_0^{1,q}(\Omega) \to W_0^{1,q}(\Omega)$ which implies that $\big(I - \Psi(y^*, A^*)\big)$ is not an isomorphism. As the continuous invertibility of $\big(I - \Psi(y^*, A^*)\big)$ is not necessary for the surjectivity of $C'(u^*, y^*)$, we shall derive another condition ensuring that $C'(u^*, y^*)$ is surjective. If $\lambda = 1$ is an eigenvalue of $\Psi(y^*, A^*) : W_0^{1,q}(\Omega) \to W_0^{1,q}(\Omega)$, then, by virtue of the Riesz-Schauder theorem (see e.g. [2]), the compactness of $\Psi(y^*, A^*)$ implies that

$$W_0^{1,q}(\Omega) = \operatorname{ran}\big(I - \Psi(y^*, A^*)\big)^l \oplus \ker\big(I - \Psi(y^*, A^*)\big)^l \tag{53}$$

with some $l \in \mathbb{N}$ (Riesz-index), and the kernel $\ker\big(I - \Psi(y^*, A^*)\big)^l \subset W_0^{1,q}(\Omega)$ is finite-dimensional. Next, straightforward computations yield

$$\begin{aligned}\langle\frac{\partial C}{\partial u}(u^*,y^*)u,\phi\rangle_{W^{-1,q}(\Omega),W_0^{1,q'}(\Omega)} = -\omega^2\int_\Omega \sigma(x,y^*)\Big(\mathcal{R}eA^*\cdot\mathcal{R}e\big(\frac{\partial\mathscr{A}}{\partial u}(u^*,y^*)u\big)\\ +\mathfrak{Im}A^*\cdot\mathfrak{Im}\big(\frac{\partial\mathscr{A}}{\partial u}(u^*,y^*)u\big)\Big)\phi\,dx \quad \forall\phi\in W_0^{1,q'}(\Omega),\end{aligned}$$

where $A^* = \mathscr{A}(u^*, y^*)$ and $\frac{\partial\mathscr{A}}{\partial u}(u^*, y^*)u = A \in X_{N,0}(\mathcal{D})$ is given by the solution of

$$\begin{aligned}\int_{\mathcal{D}} \big(\frac{1}{\mu}\nabla\times A\big)\cdot\big(\nabla\times\overline{\psi}\big)\,dx + i\omega\big(\int_\Omega \sigma(x,y^*)A\cdot\overline{\psi}\,dx +\\ \int_R \sigma_R A\cdot\overline{\psi}\,dx\big) = \sum_{j=1}^n u_j\int_R J_j\cdot\overline{\psi}\,dx \quad \forall\psi\in X_{N,0}(\mathcal{D}).\end{aligned} \tag{54}$$

In view of the superposition principle, the operator $\frac{\partial\mathscr{A}}{\partial u}(u^*, y^*)$ can be simplified by making use of the following vector fields:

Definition 4. *For every $j = 1, \dots, n$, let $A_j^* \in X_{N,0}(\mathcal{D})$ be the unique solution of*

$$\int_{\mathcal{D}} \big(\frac{1}{\mu}\nabla \times A_j^*\big) \cdot \big(\nabla \times \overline{\psi}\big)\, dx + i\omega\big(\int_{\Omega} \sigma(x, y^*) A_j^* \cdot \overline{\psi}\, dx + \int_R \sigma_R A_j^* \cdot \overline{\psi}\, dx\big) = \int_R J_j \cdot \overline{\psi}\, dx \quad \forall \psi \in X_{N,0}(\mathcal{D}).$$

Further, for every $j = 1, \dots, n$, let $\mathcal{N}_j^* \in W^{-1,q}(\Omega)$ be defined by

$$\langle \mathcal{N}_j^*, \phi \rangle_{W^{1,-q}(\Omega), W_0^{1,q'}(\Omega)} = -\omega^2 \int_{\Omega} \sigma(x, y^*) \big(\mathcal{R}eA^* \cdot \mathcal{R}eA_j^* + \mathfrak{I}mA^* \cdot \mathfrak{I}mA_j^*\big) \phi\, dx \quad \forall \phi \in W_0^{1,q'}(\Omega).$$

Invoking these vector fields in (54), the superposition principle implies that

$$\frac{\partial \mathscr{A}}{\partial u}(u^*, y^*) u = \sum_{j=1}^n u_j A_j^* \qquad \forall u \in \mathbb{R}^n. \tag{55}$$

By this formula, we can in turn express $\frac{\partial C}{\partial u}(u^*, y^*)$ as

$$\frac{\partial C}{\partial u}(u^*, y^*) u = \sum_{j=1}^n u_j \mathcal{N}_j^* \qquad \forall u \in \mathbb{R}^n, \tag{56}$$

where $\mathcal{N}_j^* \in W^{-1,q}(\Omega)$ is defined as in Definition 4.

Assumption 5. *In the case where $\lambda = 1$ is an eigenvalue of $\Psi(y^*, A^*) : W_0^{1,q}(\Omega) \to W_0^{1,q}(\Omega)$, let $l \geq 1$ be the Riesz-index associated with the corresponding Riesz-decomposition* (53)*. We assume that for every $g \in \ker\big(I - \Psi(y^*, A^*)\big)^l$ there exists a vector $u^{(g)} \in \mathbb{R}^n$ such that*

$$g = \sum_{j=1}^n u_j^{(g)} \underbrace{B(y^*)^{-1} \mathcal{N}_j^*}_{\in W_0^{1,q}(\Omega)}.$$

Theorem 3. *Let Assumptions 1, 2 and 3 be satisfied and let $(u^*, y^*) \in \mathbb{R}^n \times W_0^{1,q}(\Omega)$. If either Assumption 4 or Assumption 5 is satisfied, then $C'(u^*, y^*) : \mathbb{R}^n \times W_0^{1,q}(\Omega) \to W^{-1,q}(\Omega)$ is surjective.*

Proof. We only need to show that Assumption 5 leads to the surjectivity of $C'(u^*, y^*)$. Let $G \in W^{-1,q}(\Omega)$ be arbitrarily fixed. We prove that the following operator equation

$$C'(u^*, y^*)(u, y) = \frac{\partial C}{\partial y}(u^*, y^*)y + \frac{\partial C}{\partial u}(u^*, y^*)u = G \quad \text{in } W^{-1,q}(\Omega) \tag{57}$$

admits a solution $(u, y) \in \mathbb{R}^n \times W_0^{1,q}(\Omega)$. Applying $B(y^*)^{-1} : W^{-1,q}(\Omega) \to W_0^{1,q}(\Omega)$ to the above equation results in

$$B(y^*)^{-1}\frac{\partial C}{\partial y}(u^*, y^*)y + B(y^*)^{-1}\frac{\partial C}{\partial u}(u^*, y^*)u = B(y^*)^{-1}G \quad \text{in } W_0^{1,q}(\Omega)$$

which is, by (49) and (56), equivalent to

$$(I - \Psi(y^*, A^*))y + \sum_{j=1}^{n} u_j B(y^*)^{-1}\mathcal{N}_j^* = B(y^*)^{-1}G \quad \text{in } W_0^{1,q}(\Omega). \tag{58}$$

In view of the Riesz decomposition (53), the right hand side of (58) can be uniquely decomposed into

$$B(y^*)^{-1}G = r + g \tag{59}$$

with $r \in \operatorname{ran}\big(I - \Psi(y^*, A^*)\big)^l$ and $g \in \ker\big(I - \Psi(y^*, A^*)\big)^l$. On the one hand, we have $r \in \operatorname{ran}\big(I - \Psi(y^*, A^*)\big)^l \subset \operatorname{ran}\big(I - \Psi(y^*, A^*)\big)$ and hence there exists a $y^{(r)} \in W_0^{1,q}(\Omega)$ such that

$$(I - \Psi(y^*, A^*))y^{(r)} = r. \tag{60}$$

On the other hand, since $g \in \ker\big(I - \Psi(y^*, A^*)\big)^l$, Assumption 5 ensures the existence of a $u^{(g)} \in \mathbb{R}^n$ such that

$$g = \sum_{j=1}^{n} u_j^{(g)} B(y^*)^{-1}\mathcal{N}_j^*. \tag{61}$$

In conclusion, (58)–(61) imply that $(u^{(g)}, y^{(r)})$ is a solution to (57) and hence the assertion immediately follows. ∎

5 Optimal control

Having established the theoretical framework for the state equation and its linearization, we now turn to the optimal control problem (P) (see p. 50). Let us first define the convex set of all points satisfying the control constraints associated with (P) by

$$\mathcal{U}_{ad} := \{ (u, y) \in \mathbb{R}^n \times W_0^{1,q}(\Omega) \mid u_j^a \le u_j \le u_j^b \text{ for all } j = 1, \dots, n \}. \quad (62)$$

Using this set, (P) can also be equivalently written as

$$\begin{cases} \min\limits_{(u,y)\in\mathcal{U}_{ad}} J(u, y) \\ \text{subject to } C(u, y) = 0 \text{ in } W^{-1,q}(\Omega) \\ \qquad\qquad y_a(x) \le y(x) \le y_b(x) \quad \text{for a.a. } x \in \Omega. \end{cases} \quad \text{(P)}$$

For the remainder of the presentation, a pair $(u, y) \in \mathbb{R}^n \times W_0^{1,q}(\Omega)$ is said to be feasible if and only if $(u, y) \in \mathcal{U}_{ad}$ and it satisfies the equality constraint $C(u, y) = 0$ in $W^{-1,q}(\Omega)$ as well as the inequality constraints $y_a(x) \le y(x) \le y_b(x)$ for a.a. $x \in \Omega$. The set of all feasible pairs associated with (P) is then given by

$$U := \{(u, y) \in \mathcal{U}_{ad} \mid C(u, y) = 0 \text{ and } y_a(x) \le y(x) \le y_b(x) \text{ for a.a. } x \in \Omega \}.$$

By classical arguments (cf. [23]), (P) admits a solution if $U \ne \emptyset$. We summarize the existence result in the following theorem:

Theorem 4. *Let Assumptions 1 and 2 be satisfied. Further, suppose that $U \ne \emptyset$. Then,* (P) *admits a solution $(u^*, y^*) \in \mathbb{R}^n \times W_0^{1,q}(\Omega)$.*

Notice that the solution to (P) is not necessarily unique due to the nonlinearities involved in the state equation. We therefore concentrate in our analysis on local solutions in the following sense: A feasible pair (u^*, y^*) is called a local solution to (P) if there exists some $r > 0$ such that

$$J(u^*, y^*) \le J(u, y)$$

for all feasible pairs (u, y) satisfying $|u - u^*| \le r$ and $\|y - y^*\|_{W_0^{1,q}(\Omega)} \le r$. Next, by $\mathcal{M}(\overline{\Omega})$, we denote the space of all regular Borel measures on the compact set $\overline{\Omega}$. According to the Riesz-Radon theorem, the space $\mathcal{M}(\overline{\Omega})$

can be isometrically identified with the dual space $\mathcal{C}(\overline{\Omega})^*$ with respect to the duality pairing

$$\langle \mu, \eta \rangle_{\mathcal{C}(\overline{\Omega})^*, \mathcal{C}(\overline{\Omega})} := \int_{\overline{\Omega}} \eta \, d\mu, \quad \eta \in \mathcal{C}(\overline{\Omega}),\, \mu \in \mathcal{M}(\overline{\Omega}).$$

Now we are about to derive the first-order necessary optimality conditions of (P). Let us now introduce the notion of the Lagrange functional associated with (P).

Definition 5 (Lagrange functional)**.** *The Lagrange functional associated with* (P) $\mathscr{L} : \mathbb{R}^n \times W_0^{1,q}(\Omega) \times W_0^{1,q'}(\Omega) \times \mathcal{M}(\overline{\Omega}) \times \mathcal{M}(\overline{\Omega}) \to \mathbb{R}$ *is defined by*

$$\mathscr{L}(u, y, \varphi, \mu_a, \mu_b) := J(u, y) - \langle C(u, y), \varphi \rangle_{W^{-1,q}(\Omega), W_0^{1,q'}(\Omega)} + \int_{\overline{\Omega}} (y_a - y) d\mu_a + \int_{\overline{\Omega}} (y - y_b) d\mu_b.$$

Definition 6. *Let* (u^*, y^*) *be a local solution to* (P)*. We say* $(\mu_a, \mu_b) \in \mathcal{M}(\overline{\Omega}) \times \mathcal{M}(\overline{\Omega})$ *and* $\varphi \in W_0^{1,q'}(\Omega)$ *a pair of Lagrange multipliers and an adjoint state associated with the local solution* (u^*, y^*) *if and only if*

$$\frac{\partial \mathscr{L}}{\partial (u, y)}(u^*, y^*, \varphi, \mu_a, \mu_b)(u - u^*, y - y^*) \geq 0 \quad \forall\, (u, y) \in \mathcal{U}_{ad} \tag{63}$$

$$\mu_a, \mu_b \geq 0 \qquad \int_{\overline{\Omega}} (y_a - y^*) \, d\mu_a = \int_{\overline{\Omega}} (y^* - y_b) \, d\mu_b = 0. \tag{64}$$

Note that if $\mu \in \mathcal{M}(\overline{\Omega})$, then we write

$$\mu \geq 0 \;\Leftrightarrow\; \int_{\overline{\Omega}} y \, d\mu \geq 0 \quad \forall y \in \{ y \in \mathcal{C}(\overline{\Omega}) \mid y(x) \geq 0 \; \forall x \in \overline{\Omega} \}.$$

We observe that the adjoint state φ belongs only to $W_0^{1,q'}(\Omega)$ with $1 \leq q' < \frac{3}{2}$ since $q > 3$. Such weak regularity is typical when dealing with state-constrained optimal control problems (cf. Casas [7]).

Definition 7 (Constraint qualification)**.** *We say that* $(u^*, y^*) \in \mathcal{U}_{ad}$ *satisfies the constraint qualification if there exists* $(\tilde{u}, \tilde{y}) \in \mathcal{U}_{ad}$ *and some constant* $\rho > 0$ *such that*

$$C'(u^*, y^*)(\tilde{u}, \tilde{y}) = 0 \quad \text{in } W^{-1,q}(\Omega) \quad y_a(x) + \rho \leq \tilde{y}(x) \leq y_b(x) - \rho \quad \forall x \in \overline{\Omega}.$$

Theorem 5. *Let Assumptions 1, 2 and 3 be satisfied. Further, let* (u^*, y^*) *be a local solution to* (P) *and* $A^* = \mathscr{A}(u^*, y^*)$. *Suppose that* (u^*, y^*) *satisfies the constraint qualification in the sense of Definition 7 and either Assumption 4 or Assumption 5 is satisfied. Then, there exist* $\varphi \in W_0^{1,q'}(\Omega)$ *and* $(\mu_a, \mu_b) \in \mathcal{M}(\overline{\Omega}) \times \mathcal{M}(\overline{\Omega})$ *such that*

$$\begin{cases} -\nabla \cdot (\kappa(\cdot, y^*)\nabla\varphi) + \dfrac{\partial \kappa}{\partial y}(\cdot, y^*)\nabla y^* \cdot \nabla\varphi + \dfrac{\partial d}{\partial y}(\cdot, y^*)\varphi - \mathcal{T}(u^*, y^*)^*\varphi \\ \qquad -\dfrac{\omega^2}{2}\dfrac{\partial \sigma}{\partial y}(\cdot, y^*)|A^*|^2\varphi = y^* - y_d + \alpha(-\Delta y^* + \nabla \cdot z_d) + \\ \qquad\qquad + (\mu_b - \mu_a)_{|\Omega} \quad \text{in } \Omega \\ \qquad \varphi = (\mu_b - \mu_a)_{|\partial\Omega} + \alpha z_d \cdot \vec{n} \quad \text{on } \partial\Omega \end{cases} \tag{65}$$

$$\mu_a, \mu_b \geq 0 \qquad \int_{\overline{\Omega}} (y_a - y^*)\, d\mu_a = \int_{\overline{\Omega}} (y^* - y_b)\, d\mu_b = 0 \tag{66}$$

$$u^* = \mathbb{P}_{[u^a, u^b]}\left(\frac{1}{\beta} t^*(\varphi)\right) \tag{67}$$

$$t^*(\varphi)_j = \langle \mathcal{N}_j^*, \varphi \rangle_{W^{-1,q}(\Omega), W_0^{1,q'}(\Omega)} \quad j = 1, \ldots, n, \tag{68}$$

where $\mathbb{P}_{[u^a, u^b]}$ *in* (67) *denotes the projection from* $\mathbb{R}^n$ *onto* $[u_1^a, u_1^b] \times \ldots \times [u_n^a, u_n^b]$. *Further, for* $j = 1, \ldots, n$, $\mathcal{N}_j^* \in W^{-1,q}(\Omega)$ *is defined as in Definition 4.*

Proof. Either Assumption 4 or Assumption 5 implies that $C'(u^*, y^*) : \mathbb{R}^n \times W_0^{1,q}(\Omega) \to W^{-1,q}(\Omega)$ is surjective. This fact together with the constraint qualification guarantees the existence of Lagrange multipliers $(\mu_a, \mu_b) \in \mathcal{M}(\overline{\Omega}) \times \mathcal{M}(\overline{\Omega})$ and an adjoint state $\varphi \in W_0^{1,q'}(\Omega)$ in the sense of Definition 6 (see Kurcyusz and Zowe [24] or Tröltzsch [23, p. 251]). We demonstrate that (63)–(64) imply (65)–(67). First, by virtue of (62), (63) implies that

$$0 = \frac{\partial \mathscr{L}}{\partial y}(u^*, y^*, \varphi, \mu_a, \mu_b)y = \int_\Omega (y^* - y_d)y\, dx + \alpha \int_\Omega (\nabla y^* - z_d) \cdot \nabla y\, dx \\ - \langle \frac{\partial C}{\partial y}(u^*, y^*)y, \varphi \rangle + \langle \mu_b - \mu_a, y \rangle_{\mathcal{C}^*(\overline{\Omega}), \mathcal{C}(\overline{\Omega})} \tag{69}$$

for all $y \in W_0^{1,q}(\Omega)$. We recall from (38) that

$$\langle \frac{\partial C}{\partial y}(u^*,y^*)y,\varphi\rangle = \int_\Omega \big(\kappa(x,y^*)\,\nabla y + \frac{\partial \kappa}{\partial y}(x,y^*)y\nabla y^*\big)\cdot\nabla\varphi\,dx + \int_\Omega \frac{\partial d}{\partial y}(x,y^*)y\varphi\,dx - \omega^2\int_\Omega \sigma(x,y^*)\Big(\mathcal{R}e\,A^*\cdot\mathcal{R}e\,\big(\frac{\partial \mathscr{A}}{\partial y}(u^*,y^*)y\big) + \mathfrak{Im}\,A^*\cdot\mathfrak{Im}\,\big(\frac{\partial \mathscr{A}}{\partial y}(u^*,y^*)y\big)\Big)\varphi\,dx - \frac{\omega^2}{2}\int_\Omega \frac{\partial\sigma}{\partial y}(x,y^*)y|A^*|^2\varphi\,dx \qquad \forall y\in W_0^{1,q}(\Omega), \tag{70}$$

where $\frac{\partial \mathscr{A}}{\partial y}(u^*,y^*)y = A \in X_{N,0}(\mathcal{D})$ is given by the solution of

$$\int_{\mathcal{D}} \big(\frac{1}{\mu}\nabla\times A\big)\cdot\big(\nabla\times\overline{\psi}\big)\,dx + i\omega\Big(\int_\Omega \sigma(x,y^*)A\cdot\overline{\psi}\,dx + \int_R \sigma_R A\cdot\overline{\psi}\,dx\Big) = -i\omega\int_\Omega \frac{\partial\sigma}{\partial y}(x,y^*)yA^*\cdot\overline{\psi}\,dx \quad \forall\psi\in X_{N,0}(\mathcal{D}). \tag{71}$$

Using the operator $\mathcal{T}(u^*,y^*) : L^2(\Omega)\to W^{-1,q}(\Omega)$ defined in (47) on p. 65, we observe that (70) can be expressed as follows:

$$\langle \frac{\partial C}{\partial y}(u^*,y^*)y,\varphi\rangle = \int_\Omega \big(\kappa(x,y^*)\,\nabla y + \frac{\partial \kappa}{\partial y}(x,y^*)y\nabla y^*\big)\cdot\nabla\varphi\,dx + \int_\Omega \frac{\partial d}{\partial y}(x,y^*)y\varphi\,dx - \underbrace{\langle \mathcal{T}(u^*,y^*)y,\varphi\rangle_{W^{-1,q}(\Omega),W_0^{1,q'}(\Omega)}}_{=(y,\mathcal{T}(u^*,y^*)^*\varphi)_{L^2(\Omega)}} - \frac{\omega^2}{2}\int_\Omega \frac{\partial\sigma}{\partial y}(x,y^*)y|A^*|^2\varphi\,dx \qquad \forall y\in W_0^{1,q}(\Omega). \tag{72}$$

Setting (72) in (69) yields

$$\int_\Omega \big(\kappa(x,y^*)\nabla\varphi\cdot\nabla y\,dx + \int_\Omega \frac{\partial\kappa}{\partial y}(x,y^*)\nabla\varphi\cdot\nabla y^* y\,dx + \int_\Omega \frac{\partial d}{\partial y}(x,y^*)\varphi y\,dx - \int_\Omega \mathcal{T}(u^*,y^*)^*\varphi y\,dx - \frac{\omega^2}{2}\int_\Omega \frac{\partial\sigma}{\partial y}(x,y^*)|A^*|^2\varphi y\,dx = \int_\Omega (y^*-y_d)y\,dx + \alpha\int_\Omega (\nabla y^* - z_d)\cdot\nabla y\,dx + \langle \mu_b-\mu_a, y\rangle_{\mathcal{C}^*(\overline{\Omega}),\mathcal{C}(\overline{\Omega})} \quad \forall y\in W_0^{1,q}(\Omega).$$

The above variational form is exactly the weak formulation for (65).

To demonstrate the projection formula (67), we note that (62) yields

$$0 \le \frac{\partial \mathscr{L}}{\partial u}(u^*, y^*, \varphi, \mu_a, \mu_b)(u - u^*)$$
$$= \beta(u^*, u - u^*)_{\mathbb{R}^n} - \langle \frac{\partial C}{\partial u}(u^*, y^*)(u - u^*), \varphi \rangle \quad \forall u \in [u_1^a, u_1^b] \times \ldots \times [u_n^a, u_n^b]. \tag{73}$$

We recall from (56) that $\frac{\partial C}{\partial u}(u^*, y^*)(u - u^*) = \sum_{j=1}^n (u_j - u_j^*)\mathcal{N}_j^*$ where $\mathcal{N}_j^* \in W^{-1,q}(\Omega)$ is as defined in Definition 4. Using this identity in (73) results in

$$(-t^*(\varphi) + \beta u^*, u - u^*)_{\mathbb{R}^n} \ge 0 \quad \forall u \in [u_1^a, u_1^b] \times \ldots \times [u_n^a, u_n^b], \tag{74}$$

where $t^*(\varphi) \in \mathbb{R}^n$ as in (68). By classical arguments, cf. [23], a componentwise evaluation of (74) yields the desired projection formula (67). □

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