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VIABILITY FOR MULTI-VALUED SEMILINEAR REACTION-DIFFUSION SYSTEMS*

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Abstract

The aim of this paper is to prove some viability results for semilinear reaction-diffusion systems governed by multi-valued continuous perturbations of infinitesimal generators of C_0-semigroups.

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1 Introduction

The purpose of this paper is to prove some viability results referring to a class of semilinear reaction-diffusion systems, results announced without proofs in Burlică [1]. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real Banach spaces, $A : D(A) \subseteq X \to X$ and $B : D(B) \subseteq Y \to Y$ the infinitesimal generators of two C_0-semigroups, $\{S_A(t) : X \to X; t \geq 0\}$ and $\{S_B(t) : Y \to Y; t \geq 0\}$ respectively, $\mathcal{K}$ a nonempty and locally closed subset in $X \times Y$, $F : \mathcal{K} \to X$a

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given function and $G : \mathcal{K} \rightsquigarrow Y$ a given multi-function. We consider a semi-multi-valued reaction-diffusion system of the form:

$$\begin{cases} u'(t) = Au(t) + F(u(t), v(t)), & t \geq 0 \\ v'(t) \in Bv(t) + G(u(t), v(t)), & t \geq 0 \\ u(0) = \xi, \quad v(0) = \eta, \end{cases} \tag{1}$$

where $\xi \in X,\ \eta \in Y$.

Definition 1. *By a mild solution of the multi-valued Cauchy problem* (1) *on* $[\,0, T\,]$ *we mean a continuous function* $(u, v) : [\,0, T\,] \to \mathcal{K}$*, for which there exists* $g \in L^1(\,0, T; Y\,)$ *such that* $g(s) \in G(u(s), v(s))$ *a.e. for* $s \in [\,0, T\,]$ *and*

$$\begin{cases} u(t) = S_A(t)\xi + \displaystyle\int_0^t S_A(t-s)F(u(s), v(s))\, ds \\ v(t) = S_B(t)\eta + \displaystyle\int_0^t S_B(t-s)g(s)\, ds \end{cases} \tag{2}$$

for each $t \in [\,0, T\,]$.

Definition 2. *The set* $\mathcal{K}$ *is viable with respect to* $(A + F, B + G)$ *if for each* $(\xi, \eta) \in \mathcal{K}$ *there exists* $T > 0$ *such that the Cauchy problem* (1) *has at least one mild solution* $(u, v) : [\,0, T\,] \to \mathcal{K}$.

2 Preliminaries

We assume that the reader is familiar with the basic concepts and results concerning multi-functions, linear evolution and semilinear differential inclusions in Banach spaces and we refer to Cârjă [4] and Vrabie [9] for details.

In the sequel $(X, \|\cdot\|)$ will always be a Banach space. For $\xi \in X$ and $\rho > 0$, $D(\xi, \rho)$ denotes the closed ball in X of radius ρ centered in ξ and $dist(E, K)$ denotes the usual distance between the subsets E and K, i.e. $dist(E, K) = \inf_{(x,y) \in E \times K} \|x - y\|$.

We begin by recalling some definitions and basic results concerning u.s.c. multi-functions, the Hausdorff measure of noncompactness and uniqueness functions.

Let K be a subset in X and $F : K \rightsquigarrow X$ a given multi-function, i.e a function $F : K \to 2^X$.

Definition 3. *The multi-function $F : K \rightsquigarrow X$ is upper semicontinuous (u.s.c.) at $\xi \in K$ if for every open neighborhood V of $F(\xi)$ there exists an open neighborhood U of ξ such that $F(\eta) \subseteq V$ for each $\eta \in U \cap K$. We say that multi-function $F : K \rightsquigarrow X$ is upper semicontinuous (u.s.c.) on K if it is u.s.c. at each $\xi \in K$.*

In all that follows, strongly-weakly u.s.c. designates a multi-function which is u.s.c. if its domain is endowed with the strong (norm) topology and its range is endowed with the weak topology.

Lemma 1. *Let X be a Banach space, K a nonempty subset in X and $F : K \rightsquigarrow X$ a nonempty and (weakly) compact valued, (strongly-weakly) u.s.c. multi-function. Then, for each compact subset C of K, $\cup_{\xi \in C} F(\xi)$ is (weakly) compact and, in particular, there exists $M > 0$ such that $\|\eta\| \leq M$ for each $\xi \in C$ and each $\eta \in F(\xi)$.*

See Cârjă-Necula-Vrabie [6], Lemma 2.6.1, p.47.

Lemma 2. *Let X be a Banach space, K a nonempty subset in X and $F : K \rightsquigarrow X$ be a nonempty, closed and convex valued, strongly-weakly u.s.c. multi-function. Let $u_m : [0, T] \to X$ and $f_m \in L^1(0, T; X)$ be such that $f_m(t) \in F(u_m(t))$ for each $m \in \mathbf{N}$ and a.e. for $t \in [0, T]$. If $\lim_m u_m(t) = u(t)$ a.e. for $t \in [0, T]$ and $\lim_m f_m = f$ weakly in $L^1(0, T; X)$, then $f(t) \in F(u(t))$ a.e. for $t \in [0, T]$.*

See Cârjă-Necula-Vrabie [6], Lemma 2.6.2, p. 47-48.

Let $\mathcal{B}(X)$ be the family of all bounded subsets of X.

Definition 4. *The function $\beta : \mathcal{B}(X) \to \mathbf{R}_+$, defined by*

$$\beta(B) = \inf \left\{ \varepsilon > 0; \exists\, x_1, x_2, ...x_{n(\varepsilon)} \in X, B \subseteq \bigcup_{i=1}^{n(\varepsilon)} D(x_i, \varepsilon) \right\}$$

is called the Hausdorff-measure of noncompactness on X.

Remark 1. *We have $\beta(B) = 0$ if and only if B is a relatively compact set. If X is finite dimensional, the class of relatively compact subsets of X coincides with $\mathcal{B}(X)$, so, in this case, $\beta \equiv 0$.*

Lemma 3. *Let Y be a subspace in X, let $B \in \mathcal{B}(X)$ and let*

$$\beta_Y(B) = \inf\left\{\varepsilon > 0; \exists\, x_1, x_2, \ldots x_{n(\varepsilon)} \in Y,\ B \subseteq \bigcup_{i=1}^{n(\varepsilon)} D(x_i, \varepsilon)\right\}.$$

Then for each $B \in \mathcal{B}(Y)$ we have

$$\beta(B) \leq \beta_Y(B) \leq 2\,\beta(B).$$

For details, see Cârjă-Necula-Vrabie [6], Problem 2.7.2, p.49.

Definition 5. *A function $\omega : \mathbf{R}_+ \to \mathbf{R}_+$ which is continuous, nondecreasing and the only solution of the Cauchy problem*

$$\begin{cases} x'(t) = \omega(x(t)) \\ x(0) = 0 \end{cases}$$

is $x \equiv 0$ is called a uniqueness function.

Remark 2. *If $\omega : \mathbf{R}_+ \to \mathbf{R}_+$ is a uniqueness function, then, for each $k > 0$, $k\omega$ is a uniqueness function too.*

Next, we recall for easy reference the basic viability results, established in Cârjă-Necula-Vrabie [5] and [6], concerning the autonomous multi-valued semilinear Cauchy problem

$$\begin{cases} u'(t) \in Au(t) + F(u(t)), \quad t \geq 0 \\ u(0) = \xi, \end{cases} \tag{3}$$

where $A : D(A) \subseteq X \to X$ is the infinitesimal generator of C_0-semigroup $\{S(t) : X \to X; t \geq 0\}$, K is a nonempty subset in X and $F : K \rightsquigarrow X$ is a given multi-function.

Definition 6. *By a mild solution of the problem (3) on $[\,0, T\,]$ we mean a continuous function $u : [\,0, T\,] \to K$, for which there exists $f \in L^1(0, T; X)$ such that $f(s) \in F(u(s))$ a.e. for $s \in [\,0, T\,]$ and*

$$u(t) = S(t)\xi + \int_0^t S(t-s) f(s)\, ds \tag{4}$$

for each $t \in [\,0, T\,]$.

Definition 7. *The set $K \subseteq X$ is viable with respect to $A+F$ if for each $\xi \in K$, there exists $T>0$ such that the Cauchy problem* (3) *has at least one mild solution $u:[\,0,T\,]\to K$.*

In Cârjă-Necula-Vrabie [5] and [6] a new concept of tangent set is defined and used in order to prove necessary and sufficient conditions for viability with respect to $A+F$. We recall that the subset $K \subseteq X$ is *l*ocally closed if for each $\xi \in K$ there exists $\rho > 0$ such that $D(\xi,\rho)\cap K$ is closed. Each subset in X which is either open or closed is locally closed. Moreover, each subset K in X which is closed relative to some open subset D, i.e. for which there exists a closed subset $C \subset X$ such that $K = C \cap D$, is locally closed in X.

If E is a nonempty subset in X, we denote by

$$\mathcal{E} = \{f \in L^1(\mathbf{R}_+;X); f(s)\in E \ \text{ a.e. for } s\in \mathbf{R}_+\,\}.$$

Definition 8. *Let $A : D(A)\subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup, $\{S(t) : X \to X; t \geq 0\}$, K a subset in X and $\xi \in K$. We say that the set $E \subseteq X$ is $A-$ quasi-tangent to the set K at the point ξ if for each $\rho > 0$, we have*

$$\liminf_{h\downarrow 0} \frac{1}{h} dist\,(\mathcal{S}_{\mathcal{E}}(h)\xi; K\cap D(\xi,\rho)) = 0, \tag{5}$$

where

$$\mathcal{S}_{\mathcal{E}}(h)\xi = \Big\{ S(h)\xi + \int_0^h S(h-s)f(s)\,ds;\ f\in\mathcal{E}\,\Big\}.$$

We denote by $\mathcal{QTS}^A_K(\xi)$ the class of all A-quasi-tangent sets to K at $\xi \in K$.

Remark 3. *Let $K \subseteq X, \xi \in K$ and $E \subseteq X$. Then the following conditions are equivalent:*

(i) $E \in \mathcal{QTS}^A_K(\xi)$;

(ii) for each $\varepsilon>0,\ \rho>0$ and $\delta>0$ there exist $h\in(0,\delta),\ p\in D(0,\varepsilon)$ and $f \in \mathcal{E}$ such that

$$S(h)\xi + \int_0^h S(h-s)f(s)\,ds + hp \in K\cap D(\xi,\rho);$$

(iii) *there exist three sequences,* $(h_n)_n$ *in* $\mathbf{R}_+$ *with* $h_n \downarrow 0$, $(p_n)_n$ *in* X *with* $\lim_n p_n = 0$, *and* $(f_n)_n \in \mathcal{E}$, *with* $\lim_n \int_0^{h_n} S(h_n - s) f_n(s)\, ds = 0$ *and*

$$S(h_n)\xi + \int_0^{h_n} S(h_n - s) f_n(s)\, ds + h_n p_n \in K.$$

Before proceeding to the main results in this section,we introduce first:

Definition 9. *A set* $C \subseteq X$ *is quasi-weakly (relatively) compact if, for each* $r > 0$, $C \cap D(0, r)$ *is weakly (relatively) compact.*

We present now a necessary condition for mild viability.

Theorem 1. *Let* X *be a Banach space,* $A : D(A) \subseteq X \to X$ *the infinitesimal generator of a* C_0*-semigroup,* $\{S(t) : X \to X;\ t \geq 0\}$, K *a nonempty subset in* X *and* $F : K \rightsquigarrow X$ *a nonempty valued multi-function. If* K *is viable with respect to* $A + F$ *then, for each* $\xi \in K$ *at which* F *is u.s.c. and* $F(\xi)$ *is convex and quasi-weakly compact, we have*

$$F(\xi) \in \mathcal{QTS}_K^A(\xi). \tag{6}$$

The main sufficient condition for mild viability is:

Theorem 2. *Let* X *be a Banach space,* $A : D(A) \subseteq X \to X$ *the infinitesimal generator of a compact* C_0*-semigroup,* $\{S(t) : X \to X;\ t \geq 0\}$, K *a nonempty and locally closed subset in* X *and* $F : K \rightsquigarrow X$ *a nonempty, weakly compact and convex valued, strongly-weakly u.s.c. multi-function. If for each* $\xi \in K$, *the tangency condition* (6) *is satisfied, then* K *is viable with respect to* $A + F$.

3 The main results

We focus our attention to the main necessary and sufficient conditions for viability in the case of reaction diffusion systems of the form (1).

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ two real Banach spaces, $\mathcal{K} \subseteq X \times Y$, $F : \mathcal{K} \to X$, $G : \mathcal{K} \rightsquigarrow Y$, and $\xi \in X$, $\eta \in Y$. We assume that the operators $A : D(A) \subseteq X \to X$ and $B : D(B) \subseteq Y \to Y$ are the generators of two C_0-semigroups, $\{S_A(t) : X \to X; t \geq 0\}$ and $\{S_B(t) : Y \to Y; t \geq 0\}$ respectively.

The system (1) can be written as a multi-valued semilinear Cauchy problem in a product space. Let $\mathcal{X} = X \times Y$ be endowed with the norm $\| \cdot \|$, defined by $\|(x, y)\|_{\mathcal{X}} = \|x\|_X + \|y\|_Y$, for each $(x, y) \in \mathcal{X}$. Let $\mathcal{A} = (A, B) : D(\mathcal{A}) \subseteq \mathcal{X} \to \mathcal{X}$ be defined by $D(\mathcal{A}) = D(A) \times D(B)$, $\mathcal{A}(x, y) = (\, Ax, By\,)$ for each $(x, y) \in D(\mathcal{A})$ and let $\mathcal{F} : \mathcal{K} \rightsquigarrow \mathcal{X}$, $\mathcal{F}(z) = (\, F(z), G(z)\,)$ for each $z = (x, y) \in \mathcal{K}$, where $(\, F(z), G(z)\,) = \{F(z), \eta);\ \eta \in G(z)\}$. So, the system (1) can be written as

$$\begin{cases} z'(t) \in \mathcal{A}z(t) + \mathcal{F}(z(t)) \\ z(0) = \zeta, \end{cases} \tag{7}$$

where $\zeta = (\xi, \eta)$. We notice that, in the hypotheses above, $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $\{\, \mathcal{S}(t) : \mathcal{X} \to \mathcal{X}; t \geq 0\}$, given by

$$\mathcal{S}(t)(\xi, \eta) = (S_A(t)\xi, S_B(t)\eta)$$

for each $t \geq 0$ and $(\xi, \eta) \in \mathcal{X}$. Let us remark that $\mathcal{K}$ is viable with respect to $(A + F, B + G)$ in sense of Definition 2 if and only if $\mathcal{K}$ is viable with respect to $\mathcal{A} + \mathcal{F}$ in sense of Definition 7, which means that for each $\zeta \in \mathcal{K}$, there exists $T > 0$ such that the problem (7) has at least one mild solution $z : [\,0, T\,] \to \mathcal{K}$.

From Theorem 1 we deduce the necessary condition for viability.

Theorem 3. *Let X and Y be Banach spaces, $\mathcal{K}$ a nonempty subset in $X \times Y$, $A : D(A) \subseteq X \to X$, $B : D(B) \subseteq Y \to Y$ the infinitesimal generators of two C_0-semigroups, $\{S_A(t) : X \to X;\ t \geq 0\}$ and $\{S_B(t) : Y \to Y;\ t \geq 0\}$ respectively, $F : \mathcal{K} \to X$ a continuos function and $G : \mathcal{K} \rightsquigarrow Y$ a nonempty, convex and quasi-weakly compact valued, u.s.c. multi-function. If $\mathcal{K}$ is viable with respect to $(A + F, B + G)$ then, for each $\zeta \in \mathcal{K}$ we have:*

$$(F(\zeta), G(\zeta)) \in \mathcal{QTS}^{\mathcal{A}}_{\mathcal{K}}(\zeta). \tag{8}$$

In order to state and prove some sufficient conditions for viability, we need the hypotheses below.

(H_1) $A : D(A) \subseteq X \to X$, $B : D(B) \subseteq Y \to Y$ are the generators of two C_0-semigroups, $\{S_A(t) : X \to X;\ t \geq 0\}$ and $\{S_B(t) : Y \to Y;\ t \geq 0\}$ respectively;

(H_2) $\mathcal{K} \subseteq X \times Y$ is nonempty and locally closed;

(H_3) $F : X \times Y \to X$ is continuous and globally Lipschitz with respect to its first argument, i.e. there exists $L > 0$ such that

$$\|F(u,v) - F(\widetilde{u},v)\| \leq L\|u - \widetilde{u}\|$$

for each $(u,v), (\widetilde{u},v) \in X \times Y$;

(H_4) $A + F$ is $Y-$uniformly locally of β_X-compact type, which means that F is continuous and, for each $\zeta = (\xi, \eta) \in \mathcal{K}$, there exist $\rho > 0$, a continuous function $l : \mathbf{R}_+ \to \mathbf{R}_+$ and a uniqueness function $\omega : \mathbf{R}_+ \to \mathbf{R}_+$ such that, for each subset $C \subseteq D_{X\times Y}(\zeta, \rho) \cap \mathcal{K}$, with $\Pi_Y C$ relatively compact, and for each $t > 0$, we have

$$\beta_X(S_A(t)F(C)) \leq l(t)\omega(\beta_{X\times Y}(C));$$

(H_5) $\{S_B(t) : Y \to Y, t \geq 0\}$ is compact;

(H_6) $G : \mathcal{K} \rightsquigarrow Y$ is strongly-weakly u.s.c. multi-function with nonempty, convex and weakly compact values.

Theorem 4. *Assume that* (H_1), (H_2), (H_3), (H_5) *and* (H_6) *are satisfied. If, for each* $\zeta \in \mathcal{K}$ *the tangency condition* (8) *is satisfied, then* $\mathcal{K}$ *is viable with respect to* $(A + F, B + G)$.

Theorem 5. *Assume that* (H_1), (H_2), (H_4), (H_5) *and* (H_6) *are satisfied. If, for each* $\zeta \in \mathcal{K}$ *the tangency condition* (8) *is satisfied, then* $\mathcal{K}$ *is viable with respect to* $(A + F, B + G)$.

A nonautonomous variant of Theorem 4 is stated below. Let us consider the quasi-autonomous semilinear system

$$\begin{cases} u'(t) = Au(t) + F(t, u(t), v(t)), \quad t \geq \tau \\ v'(t) \in Bv(t) + G(t, u(t), v(t)), \quad t \geq \tau \\ u(\tau) = \xi, \quad v(\tau) = \eta \end{cases} \tag{9}$$

where $\mathcal{K} \subseteq \mathbf{R} \times X \times Y$, $F : \mathbf{R} \times X \times Y \to X$ and $G : \mathcal{K} \rightsquigarrow Y$.

Let $\mathcal{X} = \mathbf{R} \times X$ endowed with the norm $\|(t,x)\|_{\mathcal{X}} = |t| + \|x\|_X$, for each $(t,x) \in \mathcal{X}$. Let $\mathcal{A} = (0, A)$, $z(s) = (s + \tau, u(s + \tau))$, $w(s) = v(s + \tau)$ and let $\mathcal{F} : \mathcal{X} \times Y \to X$, $\mathcal{F}(z,w) = (\,1, F(z,w)\,)$ for each $(z,w) \in \mathcal{X} \times Y$. With the notation above the system (9) can be written as an autonomous one in the space $\mathcal{X} \times Y$

$$\begin{cases} z'(s) = \mathcal{A}z(s) + \mathcal{F}(z(s), w(s)), \quad s \geq 0 \\ w'(s) \in B(s) + G(z(s), w(s)), \quad s \geq 0 \\ z(0) = (\tau, \xi), \quad w(0) = \eta \end{cases} \tag{10}$$

From Theorem 4 we deduce:

Theorem 6. *Assume that X and Y are Banach spaces and (H_1), (H_5) are satisfied. Let $\mathcal{K} \subseteq \mathbf{R} \times X \times Y$ be a nonempty and locally closed set, $F : \mathbf{R} \times X \times Y \to X$ be continuous and $G : \mathcal{K} \rightsquigarrow Y$ be locally bounded, strongly-weakly u.s.c. multi-function with nonempty, convex and weakly compact values. Let us assume that F is globally Lipschitz with respect to its first and second arguments i.e. there exists $L > 0$ such that*

$$\|F(t, u, v) - F(\widetilde{t}, \widetilde{u}, v)\| \leq L(|t - \widetilde{t}| + \|u - \widetilde{u}\|).$$

If, for each $(\tau, \xi, \eta) \in \mathcal{K}$ the next tangency condition

$$(1, F(\tau, \xi, \eta), G(\tau, \xi, \eta)) \in \mathcal{QTS}_{\mathcal{K}}^{(\mathcal{A},B)}(\tau, \xi, \eta) \tag{11}$$

is satisfied, then $\mathcal{K}$ is viable with respect to $(A + F, B + G)$.

The nonautonomous case was studied in Necula-Vrabie [8] in the case when A and B are m-dissipative possibly nonlinear operators, while both F and G are single-valued, F is jointly continuous and locally Lipschitz with respect to its second variable, G is continuous and the semigroup generated by B is compact.

4 Proofs of Theorem 4 and Theorem 5

The proofs are essentially based on the construction of an $\varepsilon-$approximate solution for the Cauchy problem (3), i.e. a $5-$uple $(\sigma, \theta, g, f, u)$ given by lemma below.

Lemma 4. *Let X, Y be real Banach spaces, $\mathcal{A} : D(\mathcal{A}) \subseteq X \times Y \to X \times Y$ the infinitesimal generator of a C_0-semigroup, $\{\mathcal{S}(t) : X \times Y \to X \times Y; t \geq 0\}$, $\mathcal{K}$ a nonempty and locally closed subset in $X \times Y$ and $\mathcal{F} : \mathcal{K} \rightsquigarrow X \times Y$ a given nonempty-valued and locally bounded multi-function satisfying the tangency condition (6). Let $\zeta \in \mathcal{K}$ be arbitrary and let $r > 0$ be such that*

$D_{X\times Y}(\zeta,r)\cap\mathcal{K}$ is closed. Then, there exist $\rho\in(0,r\,]$ and $T>0$ such that, for each $\varepsilon\in(0,1)$, there exist $\sigma:[\,0,T\,]\to[\,0,T\,]$ nondecreasing, $\theta:\{\,(t,s);0\leq s<t\leq T\}\to[\,0,T\,]$ measurable, $\mathcal{G}:[\,0,\,T\,]\to X\times Y$, $\tilde{f}:[\,0,T\,]\to X\times Y$ Bochner integrable and $z:[\,0,T\,]\to X\times Y$ continuous such that:

(i) $s-\varepsilon\leq\sigma(s)\leq s$ *for each* $s\in[\,0,T\,]$;

(ii) $z(\sigma(s))\in D_{X\times Y}(\zeta,r)\cap\mathcal{K}$ *for each* $s\in[\,0,T\,]$ *and* $z(T)\in D_{X\times Y}(\zeta,r)\cap\mathcal{K}$;

(iii) $\|\mathcal{G}(s)\|\leq\varepsilon$ *for each* $s\in[\,0,T\,]$ *and* $\tilde{f}(s)\in\mathcal{F}(z(\sigma(s))$ *a.e. for* $s\in[\,0,T\,]$;

(iv) $\theta(t,s)\leq t$ *for each* $0\leq s<t\leq T$ *and* $t\mapsto\theta(t,s)$ *is nonexpansive on* $(\,s,T\,]$;

(v) $z(t)=\mathcal{S}(t)\zeta+\int_0^t\mathcal{S}(t-s)\tilde{f}(z(s))\,ds+\int_0^t\mathcal{S}(\theta(t,s))\mathcal{G}(s)\,ds$ *for each* $t\in[\,0,T\,]$;

(vi) $\|z(t)-z(\sigma(t))\,\|\leq\varepsilon$ *for each* $t\in[\,0,T\,]$.

See Cârjă-Necula-Vrabie [5] and [6], Lemma 9.3.1, p.185.

Remark 4. *Let $\mathcal{K}\subseteq X\times Y$ be a nonempty, locally closed set and $G:\mathcal{K}\rightsquigarrow Y$ be a strongly-weakly u.s.c. multi-function with nonempty, convex and weakly compact valued, then G is locally bounded.*

See Remark 7.1 in Cârjă-Necula-Vrabie [7].

Proof of Theorem 4 Let $\zeta=(\xi,\eta)\in\mathcal{K}$ and $r>0$ such that $D_{\mathcal{X}}(\zeta,r)\cap\mathcal{K}$ be closed. Let us choose $\rho\in(\,0,r\,]$, $N>0$, $M\geq1$ and $a\geq0$ such that $\|F(z)\|_X\leq N$ and $\|y\|_Y\leq N$ for every $z\in D_{\mathcal{X}}(\zeta,\rho)\cap\mathcal{K}$ and every $y\in G(z)$ and $\|\mathcal{S}(t)\|_{\mathcal{L}(\mathcal{X})}\leq Me^{at}$, for every $t\geq0$. Since $t\mapsto\mathcal{S}(t)\zeta$ is continuous in $t=0$ and $\mathcal{S}(0)\zeta=\zeta$, we may find a sufficiently small $T>0$ such that

$$\sup_{t\in[\,0,T\,]}\|\mathcal{S}(t)\zeta-\zeta\|_{\mathcal{X}}+TMe^{aT}(N+1)\leq\rho$$

and all the conclusions of Lemma 4, for the Cauchy problem (7), be satisfied.

Let $(\varepsilon_n)_n\downarrow0$ be a sequence in $(0,1)$ and let $((\sigma_n,\theta_n,\mathcal{G}_n,\widetilde{f_n},z_n))_n$ be a sequence of $(\varepsilon_n)_n-$ approximate solutions defined on $[\,0,T\,]$ whose existence is ensured by the Lemma 4. This means that $\widetilde{f_n}=(f_n,g_n)$ is Lebesque integrable, $f_n(s) = F(z_n(\sigma_n(s)))$ and $g_n(s) \in G(z_n(\sigma_n(s)))$ a.e. for $s \in [0,T]$, and $z_n(\sigma_n(t)) \in D_{\mathcal{X}}(\zeta,\rho) \cap \mathcal{K}$, for $n = 1,2,\dots$ and each $t \in [0,T]$, and

$$z_n(t) = \mathcal{S}(t)\zeta + \int_0^t \mathcal{S}(t-s)\widetilde{f}_n(s)ds + \int_0^t \mathcal{S}(\theta_n(t,s))\mathcal{G}_n(s)ds$$

for each $n \in \mathbf{N}$ and $t \in [0,T]$. Put $z_n = (u_n, v_n)$. So, (u_n, v_n) satisfies

$$\begin{cases} u_n(t) = S_A(t)\xi + \int_0^t S_A(t-s)F(z_n(\sigma_n(s)))ds + \int_0^t S_A(\theta_n(t,s))\mathcal{G}_n^X(s)ds \\ v_n(t) = S_B(t)\eta + \int_0^t S_B(t-s)g_n(s)ds + \int_0^t S_B(\theta_n(t,s))\mathcal{G}_n^Y(s)ds, \end{cases} \tag{12}$$

where $\mathcal{G}_n(t) = (\mathcal{G}_n^X(t), \mathcal{G}_n^Y(t))$ for each $n = 1,2,...$ and $t \in [0,T]$. Since $\|g_n(s)\|_Y \leq N$ for each $n = 1,2,...$ and for a.a. $s \in [0,T]$, the family $\{g_n(\cdot);\, n \in \mathbf{N}\}$ is uniformly integrable subset in $L^1(0,T;Y)$. Since the C_0-semigroup $\{S_B(t) : Y \to Y;\, t \geq 0\}$ is compact and $\|\mathcal{G}_n^Y(s)\|_Y \leq \varepsilon_n < 1$, from Theorem 8.4.2 in Vrabie [9] it follows that $\{v_n;\, n = 1,2,...\}$ is relatively compact in $C([0,T];Y)$. As $\lim_n \sigma_n(t) = t$ uniformly for $t \in [0,T]$ we deduce that there exists $v \in C([0,T];Y)$ such that, on a subsequence at least, $\lim_n v_n(t) = v(t)$ and $\lim_n v_n(\sigma_n(t)) = v(t)$ uniformly for $t \in [0,T]$.

At this point let us consider the problem:

$$\begin{cases} u'(t) = Au(t) + F(u(t),v(t)), & t \geq 0 \\ u(0) = \xi, \end{cases} \tag{13}$$

where $v \in C([0,T];Y)$ is as above. Since A is the infinitesimal generator of a C_0-semigroup and F is continuous on $X \times Y$ and globally Lipschitz with respect to its first argument, it follows that the problem (13) has an unique mild solution $u : [0,T] \to X$, i.e.

$$u(t) = S_A(t)\xi + \int_0^t S_A(t-s)F(u(s),v(s))ds \tag{14}$$

for each $t \in [0,T]$. We will prove next that $\lim_n u_n(t) = u(t)$ uniformly for $t \in [0,T]$. Indeed, we have

$$\|u_n(\sigma_n(t)) - u(t)\|_X \leq \|u_n(\sigma_n(t)) - u_n(t)\|_X + \|u_n(t) - u(t)\|_X. \tag{15}$$

From (12) and (14), it follows that

$$\|u_n(t) - u(t)\|_X \leq \int_0^t \|S_A(\theta_n(t,s))\|_{L(X)} \|\mathcal{G}_n^X(s)\|_X ds$$

$$+ \int_0^t \|S_A(t-s)\|_X \|F(u_n(\sigma_n(s)), v_n(\sigma_n(s))) - F(u(s), v(s))\|_X ds$$

.

for each $t \in [\,0, T\,]$. Using (iii)[1], (vi) and the Lipschitz's condition for F, from (15) we obtain

$$\|u_n(\sigma_n(t)) - u(t)\|_X \leq Me^{aT}\varepsilon_n$$

$$+Me^{aT}\int_0^t \|F(u(s), v_n(\sigma_n(s))) - F(u(s), v(s))\|_X ds \tag{16}$$

$$+LMe^{aT}\int_0^t \|u_n(\sigma_n(s)) - u(s)\|_X ds,$$

for each $t \in [\,0, T\,]$. On the other hand, $\varepsilon_n \downarrow 0$, $v_n(\sigma_n(s)) \to v(s)$ uniformly for $s \in [\,0, T\,]$ and F is continuous. So, for each $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that, for each $n \in \mathbf{N}$, $n \geq n_0$ and for each $t \in [\,0, T\,]$ we have:

$$Me^{aT}\varepsilon_n + Me^{aT}\int_0^t \|F(u(s), v_n(\sigma_n(s))) - F(u(s), v(s))\|_X ds \leq \varepsilon.$$

In view of (16) and the last inequality, we deduce

$$\|u_n(\sigma_n(t)) - u(t)\|_X \leq \varepsilon + LMe^{aT}\int_0^t \|u_n(\sigma_n(s)) - u(s)\|_X ds$$

for all $n \in \mathbf{N}$, $n \geq n_0$ and $t \in [\,0, T\,]$. Gronwall's Lemma implies

$$\|u_n(\sigma_n(t)) - u(t)\|_X \leq \varepsilon e^{LMTe^{aT}}$$

for each $n \in \mathbf{N}$, $n \geq n_0$ and each $t \in [\,0, T\,]$. Therefore $\lim_n u_n(\sigma_n(t)) = u(t)$ uniformly for $t \in [\,0, T\,]$. Taking into account (i) we deduce that $\lim_n u_n(t) =$

[1]Throughout this section, reference to (i)–(vi) are to the corresponding items in Lemma 4.

$u(t)$ uniformly for $t \in [0,T]$. Since $(u_n(\sigma_n(t)), v_n(\sigma_n(t))) \in D_{\mathcal{X}}(\zeta,\rho) \cap \mathcal{K}$, for $n = 1,2,\ldots$ and each $t \in [0,T]$ and $D_{\mathcal{X}}(\zeta,\rho) \cap \mathcal{K}$ is closed, it follows that $(u(t), v(t)) \in D_{\mathcal{X}}(\zeta,\rho) \cap \mathcal{K}$ for each $t \in [0,T]$.

Next, we will prove that $(g_n)_n$ is weakly convergent in $L^1(0,T;Y)$ to some function g. Indeed, since G is strongly-weakly u.s.c. with weakly compact values and since $\{(u_n(\sigma_n(s)), v_n(\sigma_n(s)));\ n = 1,2,\ldots,\ s \in [0,T]\}$ is relatively compact in $X \times Y$, from Lemma 1 and using by Theorem 1.3.2. in Cârjă-Necula-Vrabie [6] , it follows that the set

$$C = \overline{conv} \bigcup_{n=1}^{\infty} \bigcup_{s \in [0,T]} G(u_n(\sigma_n(s)), v_n(\sigma_n(s)))$$

is weakly compact. Since $g_n(s) \in C$ for $n = 1,2,\ldots$ and a.e. for $s \in [0,T]$, we obtain that $\{g_n(\cdot);\ n = 1,2,\ldots\}$ is weakly relatively compact in $L^1(0,T;Y)$. So, on a subsequence at least, $(g_n)_n$ is weakly convergent in $L^1(0,T;Y)$ to some function g. From Lemma 2 it follows that $g(s) \in G(u(s), v(s))$ a.e. for $s \in [0,T]$.

Now, let us consider the mild solution operator $Q : L^1(0,T;Y) \to C([0,T];Y)$, defined by

$$(Qg)(t) = S_B(t)\eta + \int_0^t S_B(t-s)g(s)\,ds,$$

for each $t \in [0,T]$ and for each $g \in L^1(0,T;Y)$. As the graph of Q is strongly$\times$strongly closed and convex, it is weakly$\times$strongly closed. So, we may pass to the limit in the second relation of (12) and, taking into account $\|\mathcal{G}_n^Y(s)\|_Y \le \varepsilon_n$, we obtain

$$v(t) = S_B(t)\eta + \int_0^t S_B(t-s)g(s)\,ds.$$

Thus (u,v) is a mild solution of (1). The proof is complete.
We prove now, that, in the hypotheses of Theorem 5, there exists at least one sequence $(\varepsilon_n)_n$, with $\varepsilon_n \downarrow 0$ such that the ε_n-approximate mild solutions sequence $((\sigma_n, \theta_n, \mathcal{G}_n, \widetilde{f}_n, z_n))_n$ has the property that $z_n = (u_n, v_n)$ is, on a subsequence at least, uniformly convergent on $[0,T]$ to some function $(u,v) : [0,T] \to \mathcal{K}$ which is the mild solution of (1).

Proof of Theorem 5 Let $\zeta = (\xi,\eta) \in \mathcal{K}$ be arbitrary and let $r > 0$, $\rho \in (0,r]$, $N > 0$, $M \ge 1$, $a \ge 0$ and $T > 0$ as in proof of Theorem 4.

Since $A+F$ is Y-uniformly locally of β_X-compact type, diminishing $\rho>0$ and $T>0$, if necessary, it follows that there exist a continuous function $l:\mathbf{R}_+\to\mathbf{R}_+$ and a uniqueness function $\omega:\mathbf{R}_+\to\mathbf{R}_+$ such that

$$\beta_X(S_A(t)F(C))\le l(t)\omega(\beta_X(\Pi_X C)) \tag{17}$$

for each subset $C\subseteq D_{\mathcal{X}}(\zeta,\rho)\cap\mathcal{K}$, with $\Pi_Y C$ relatively compact, and for each $t>0$. Let $(\varepsilon_n)_n$ with $\varepsilon_n\downarrow 0$ be a sequence in $(0,1)$ and let $((\sigma_n,\theta_n,\mathcal{G}_n,\widetilde{f}_n,z_n))_n$ be a sequence of $(\varepsilon_n)_n-$ approximate mild solutions for (7), defined on $[0,T]$. Put $\widetilde{f}_n=(f_n,g_n)$ and $z_n=(u_n,v_n)$. So $f_n(s)=F(z_n(\sigma_n(s)))$, $g_n(s)\in G(z_n(\sigma_n(s)))$ a.e. for $s\in[0,T]$, and $z_n(\sigma_n(t))\in D_{\mathcal{X}}(\zeta,\rho)\cap\mathcal{K}$, for $n=1,2,\dots$ and each $t\in[0,T]$. From (v), we have

$$z_n(t)=\mathcal{S}(t)\zeta+\int_0^t\mathcal{S}(t-s)\widetilde{f}_n(s)ds+\int_0^t\mathcal{S}(\theta_n(t,s))\mathcal{G}_n(s)ds, \tag{18}$$

for each $n=1,2,\dots$ and $t\in[0,T]$. This means that (u_n,v_n) satisfies

$$\begin{cases} u_n(t)=S_A(t)\xi+\int_0^t S_A(t-s)F(z_n(\sigma_n(s)))ds+\int_0^t S_A(\theta_n(t,s))\mathcal{G}_n^X(s)ds\\ v_n(t)=S_B(t)\eta+\int_0^t S_B(t-s)g_n(s)ds+\int_0^t S_B(\theta_n(t,s))\mathcal{G}_n^Y(s)ds, \end{cases} \tag{19}$$

where

$$\begin{cases} \mathcal{G}_n(s)=(\mathcal{G}_n^X(s),\mathcal{G}_n^Y(s))\\ \\ \mathcal{S}(\theta_n(t,s))\mathcal{G}_n(s)=(S_A(\theta_n(t,s))\mathcal{G}_n^X(s),S_B(\theta_n(t,s))\mathcal{G}_n^Y(s)), \end{cases}$$

for each $n\in\mathbf{N}$ and $0\le s<t\le T$.
Since the family $\{g_n(\cdot);\,n\in\mathbf{N}\}$ and $\{\mathcal{G}_n^Y(\cdot);\,n\in\mathbf{N}\}$ are uniformly integrable subsets in $L^1(0,T;Y)$ and the C_0-semigroup $\{S_B(t):Y\to Y;\,t\ge 0\}$ is compact, from Theorem 8.4.2 in Vrabie [9] it follows that $\{v_n;\,n=1,2,...\}$ is relatively compact in $C([0,T];Y)$. As $\lim_n\sigma_n(t)=t$ uniformly for $t\in[0,T]$ we deduce that there exists $v\in C([0,T];Y)$ such that, on a subsequence at least, $\lim_n v_n(t)=v(t)$ and $\lim_n v_n(\sigma_n(t))=v(t)$ uniformly for $t\in[0,T]$. We will prove that $\{u_n;\,n=1,2,\dots\}$ is relatively compact in $C([0,T];X)$.

We consider first the case when X is a separable space.
Since $\Pi_Y\{\,(u_n(\sigma_n(t)),v_n(\sigma_n(t)));n\ge k\,\}=\{v_n(\sigma_n(t));n\ge k\,\}$ is relatively compact in Y, $A+F$ is $Y-$uniformly locally of β_X-compact type, from (17),

we get

$$\begin{aligned}\beta_X(\{S_A(t-s)F(u_n(\sigma_n(s)),v_n(\sigma_n(s)));n\geq k\})\\ \leq l(t-s)\omega(\beta_X(\{u_n(\sigma_n(s)\,);n\geq k\})),\end{aligned}\tag{20}$$

for each $k=1,2,\dots$ and $0\leq s<t\leq T$. Since $\|\mathcal{G}_n^X(s)\|_X\leq\varepsilon_n$ for each $s\in[\,0,T\,]$ it follows that $\beta_X\left(\left\{\int_0^t S_A(\theta_n(t,s))\mathcal{G}_n^X(s)ds;n\geq k\right\}\right)=0$. Similarly, from (vi) we have that $\{u_n(\sigma_n(s))-u_n(s);n\geq k\}$ is relatively compact, for each $k\in\mathbf{N}$ and $s\in[\,0,T\,]$, and so $\beta(\{u_n(\sigma_n(s))-u_n(s);n\geq k\})=0$.

Using these arguments, the inequality (20), Lemma 2.7.2 and Problem 2.7.1 from Cârjă-Necula-Vrabie [6], we deduce that

$$\beta_X(\{u_n(t);n\geq k\}\,)\leq\beta_X\left(\left\{\int_0^t S_A(t-s)F(z_n(\sigma_n(s)))ds;n\geq k\right\}\right)$$

$$+\beta_X\left(\left\{\int_0^t S_A(\theta_n(t,s))\mathcal{G}_n^X(s)ds;n\geq k\right\}\right)$$

$$\leq\int_0^t\beta_X(\{S_A(t-s)F(z_n(\sigma_n(s)));n\geq k\})ds$$

$$\leq\int_0^t l(t-s)\omega(\beta_X(\{u_n(\sigma_n(s));n\geq k\}))ds$$

$$\leq\int_0^t l(t-s)\omega(\beta_X(\{u_n(s);n\geq k\}+\{u_n(\sigma_n(s))-u_n(s);n\geq k\}))ds$$

$$\leq\int_0^t m\omega\left(\beta_X(\{u_n(s);n\geq k\})+\beta_X(\{u_n(\sigma_n(s))-u_n(s);n\geq k\})\right)ds$$

where $m=\sup_{t\in[\,0,T\,]}l(t)$. Hence

$$\beta_X(\{u_n(t);n\geq k\}\,)\leq\int_0^t m\omega\left(\beta_X(\{u_n(s);n\geq k\})\right)ds,$$

for each $k=1,2,\dots$ and $t\in[\,0,T\,]$.

Since $\beta_X(\{u_n(t); n \geq k\}) = \beta_X(\{u_n(t); n \geq 1\})$ and we set $x(t) = \beta_X(\{u_n(t); n \geq 1\})$, $\omega_0 = m\omega$, we deduce that

$$x(t) \leq \int_0^t \omega_0(x(s))ds,$$

for all $t \in [\,0, T\,]$.

But ω_0 is a uniqueness function, so by Lemma 1.8.2 in Cârjă-Necula-Vrabie [6], we have $x(t) = 0$, for all $t \in [\,0, T\,]$, which means that

$$\beta_X(\{u_n(t); n \geq 1\}) = 0,$$

for all $t \in [\,0, T\,]$. It follows that for each $t \in [\,0, T\,]$, $\{u_n(t); n = 1, 2, \ldots\}$ is relatively compact in X. Since $(F(z_n))_n$ is bounded, it is uniformly integrable, so, by Theorem 8.4.1 in Vrabie [9], there exists $u \in C([\,0, T\,]; X)$ such that, on a subsequence at least,

$$\lim_n \left(u_n(t) - \int_0^t S_A(\theta_n(t, s))\mathcal{G}_n^X(s)ds \right) = u(t),$$

uniformly for $t \in [\,0, T\,]$. But, by (iii),

$$\lim_n \int_0^t S_A(\theta_n(t, s))\mathcal{G}_n^X(s)ds = 0,$$

uniformly for $t \in [\,0, T\,]$, so $\lim_n u_n(t) = u(t)$, uniformly for $t \in [\,0, T\,]$.

If X is not separable, in view of Theorem 1.1.3, p.3 and Remark 1.1.2, p.4 in Vrabie [9], there exists a separable and closed subspace Z of X such that

$$S_A(t)\xi,\ S_A(r)F(u_n(\sigma_n(s)), v_n(\sigma_n(s))),\ S_A(\theta_n(r, s))\mathcal{G}_n^X(s) \in Z$$

for $n = 1, 2, \ldots$ and a.e. for $t, r, s \in [\,0, T\,]$. Using Lemma 2.1 and the monotonicity of ω, we have

$$\beta_Z(S_A(t)F(C)) \leq 2\beta_X(S_A(t)F(C)) \leq 2l(t)\omega(\beta_X(\Pi_X C)) \leq 2l(t)\omega(\beta_Z(\Pi_X C)),$$

for each $t > 0$ and for each set $C \subseteq D_{\mathcal{X}}(\zeta, \rho) \cap \mathcal{K} \cap (Z \times Y)$ with $\Pi_Y C$ relatively compact. Since the restriction of β_Z to $\mathcal{B}(Z)$ is the Hausdorff measure of noncompactness on Z, we repeat now the arguments in the separable case with β_X replaced by β_Z and ω replaced by 2ω.

So, $\lim_n u_n(t) = u(t)$, uniformly for $t \in [\,0, T\,]$. From now on the proof follows the same lines as those of the proof of Theorem 4.

Remark 5. *We cannot deduce Theorem* 3.2 *from Theorem* 11.1 *in Cârjă-Necula-Vrabie [5] because the multi-function G is only strongly-weakly u.s.c., so, in this case, $\mathcal{A}+\mathcal{F}$ it cannot be locally of compact type.*

5 An example

Let $\Omega \subseteq \mathbf{R}^n$, $n = 1, 2, \ldots$ be a bounded domain with C^2 boundary Γ, let $\delta_1 \geq 0$, $\delta_2 > 0$, $p > 0$, $q > 0$, let $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}_+$, $g_i : \mathbf{R} \times \mathbf{R} \to \mathbf{R}_-$ for $i = 1, 2$ be given functions and let us consider the following general semilinear predator-pray system

$$\begin{cases} u_t = \delta_1 \Delta u - pu + f(u, v) & (t, x) \in Q_{\tau,T} \\ v_t \in \delta_2 \Delta v + qv + [g_1(u, v), g_2(u, v)] & (t, x) \in Q_{\tau,T} \\ u(t, x) = v(t, x) = 0 & (t, x) \in \Sigma_{\tau,T}, \\ u(\tau, x) = \xi(x), \quad v(\tau, x) = \eta(x) & x \in \Omega, \end{cases} \tag{21}$$

where $0 \leq \tau < T \leq \infty$, $Q_{\tau,T} = (\tau, T) \times \Omega$, $\Sigma_{\tau,T} = (\tau, T) \times \Gamma$, Δ is the usual Laplace operator, i.e. $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$ and $\xi, \eta \in L^2(\Omega)$. We assume that f is continuous function, g_1 is bounded and l.s.c. and g_2 is bounded and u.s.c. function on $\mathbf{R} \times \mathbf{R}$ and $g_1(u, v) \leq g_2(u, v)$ for each $(u, v) \in \mathbf{R} \times \mathbf{R}$.
Let $\widetilde{f} : \mathbf{R} \times \mathbf{R} \to \mathbf{R}_+$ and $\widetilde{g} : \mathbf{R} \times \mathbf{R} \to \mathbf{R}_-$ be two continuous functions such that

$$\begin{cases} f(u, v) \leq \widetilde{f}(u, v) \\ g_2(u, v) \geq \widetilde{g}(u, v) \end{cases} \tag{22}$$

for each $(u, v) \in \mathbf{R} \times \mathbf{R}$. Let us consider also the comparison predator-pray system

$$\begin{cases} u_t = \delta_1 \Delta u - pu + \widetilde{f}(u, v) & (t, x) \in Q_{0,\infty} \\ v_t = \delta_2 \Delta v + qv + \widetilde{g}(u, v) & (t, x) \in Q_{0,\infty} \\ u(t, x) = v(t, x) = 0 & (t, x) \in \Sigma_{0,\infty}, \\ u(0, x) = u_0(x), \quad v(0, x) = v_0(x) & x \in \Omega, \end{cases} \tag{23}$$

where $u_0, v_0 \in L^2(\Omega)$, $u_0(x) \geq 0, v_0(x) \geq 0$ a.e. for $x \in \Omega$. Let $(\widetilde{u}, \widetilde{v}) : \mathbf{R}_+ \times \Omega \to \mathbf{R}_+ \times \mathbf{R}_+$ be a mild solution of (23).

Using the viability result, we are interested to show that, in the specific hypotheses, for each $(\xi, \eta) \in L^2(\Omega) \times L^2(\Omega)$, with

$$\begin{cases} 0 \leq \xi(x) \leq \widetilde{u}(\tau, x) \\ \eta(x) \geq \widetilde{v}(\tau, x) \end{cases} \tag{24}$$

a.e. for $x \in \Omega$, the system (21) has at least one solution $(u, v) : \mathbf{R}_+ \times \Omega \to \mathbf{R}_+ \times \mathbf{R}_+$, such that, for each $t \in [\,\tau, \infty)$, we have

$$\begin{cases} 0 \le u(t,x) \le \widetilde{u}(t,x) \\ v(t,x) \ge \widetilde{v}(t,x) \end{cases} \tag{25}$$

a.e. for $x \in \Omega$.

Let $\mathcal{K} \subseteq \mathbf{R} \times L^2(\Omega) \times L^2(\Omega)$ be defined by

$$\mathcal{K} = \big\{(t, u, v) \in \mathbf{R}_+ \times L^2(\Omega) \times L^2(\Omega);\ (u, v) \text{ satisfies (27) below}\big\} \tag{26}$$

$$\begin{cases} 0 \le u(x) \le \widetilde{u}(t,x) \\ v(x) \ge \widetilde{v}(t,x) \end{cases} \tag{27}$$

a.e. for $x \in \Omega$.

Theorem 7. *Let $\Omega \subseteq \mathbf{R}^n$, $n = 1, 2, \ldots$, be a bounded domain with C^2 boundary Γ, $\delta_1 \ge 0$, $\delta_2 > 0$ and let $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}_+$ be continuous on $\mathbf{R} \times \mathbf{R}$ and globally Lipschitz with respect to its first argument, $g_1 : \mathbf{R} \times \mathbf{R} \to \mathbf{R}_-$ be bounded and l.s.c., $g_2 : \mathbf{R} \times \mathbf{R} \to \mathbf{R}_-$ be bounded and u.s.c. such that $g_1(u, v) \le g_2(u, v)$ for each $(u, v) \in \mathbf{R} \times \mathbf{R}$. Let $\widetilde{f} : \mathbf{R} \times \mathbf{R} \to \mathbf{R}_+$, $\widetilde{g} : \mathbf{R} \times \mathbf{R} \to \mathbf{R}_-$ be continuous such that (22) are satisfied. Assume that, for each $(u_0, v_0) \in \mathbf{R} \times \mathbf{R}$, $u \mapsto \widetilde{f}(u, v_0)$ and $v \mapsto \widetilde{g}(u_0, v)$ are nondecreasing, $u \mapsto \widetilde{g}(u, v_0)$ and $v \mapsto \widetilde{f}(u_0, v)$ are nonincreasing and there exist the constants $c_i \ge 0$, $i = 1, \ldots, 5$ such that $|\widetilde{f}(u, v)| \le c_1|u| + c_2$, and $|\widetilde{g}(u, v)| \le c_3|u| + c_4|v| + c_5$ for each $(u, v) \in \mathbf{R} \times \mathbf{R}$. Let $(u_0, v_0) \in L^2(\Omega) \times L^2(\Omega)$ with $u_0(x) \ge 0$ and $v_0(x) \ge 0$ a.e. for $x \in \Omega$ and let $(\widetilde{u}, \widetilde{v}) : \mathbf{R}_+ \to L^2(\Omega) \times L^2(\Omega)$ be a global mild solution of (23) with $\widetilde{u}(t, x) \ge 0$ for each $t \ge 0$ and a.e. for $x \in \Omega$. Let $\mathcal{K}$ be defined by (26). Then, for each $(\tau, \xi, \eta) \in \mathcal{K}$, the problem (21) has at least one global mild solution $(u, v) : [\,\tau, \infty) \to L^2(\Omega) \times L^2(\Omega)$ satisfying $(t, u(t), v(t)) \in \mathcal{K}$, for each $t \in [\,\tau, \infty)$.*

Proof. Let us denote by $X = L^2(\Omega)$. We rewrite (21) and (23) as an evolution systems in $X \times X$. Let us define $A : D(A) \subseteq X \to X$ and $B : D(B) \subseteq X \to X$ by

$$D(A) = H_0^1(\Omega) \cap H^2(\Omega) \quad \text{and} \quad Au = \delta_1 \Delta u - pu \quad \text{for } u \in D(A)$$

and respectively by

$$D(B) = H_0^1(\Omega) \cap H^2(\Omega) \quad \text{and} \quad Bv = \delta_2 \Delta v + qv \quad \text{for } v \in D(B).$$

Let us define $F : X \times X \to X$ and $G : X \times X \rightsquigarrow X$ by

$$F(u,v)(x) = f(u(x), v(x)) \text{ for each } (u,v) \in X \times X \text{ and a.e. for } x \in \Omega$$

and respectively by

$$G(u,v) = \{g \in L^2(\Omega);\ g_1(u(x), v(x)) \le g(x) \le g_2(u(x), v(x)) \text{ a.e. for } x \in \Omega\}$$

for each $(u,v) \in X \times X$. Let us observe that F is well-defined, continuous on $X \times X$ and is globally Lipschitz with respect to its first argument. Since g_1 is l.s.c., g_2 is u.s.c. and both are bounded, we conclude that G is strongly-weakly u.s.c. with nonempty, convex and weakly compact values. Let us define $\widetilde{F} : X \times X \to X$ and $\widetilde{G} : X \times X \to X$ by

$$\widetilde{F}(u,v)(x) = \widetilde{f}(u(x), v(x)) \quad \text{and} \quad \widetilde{G}(u,v)(x) = \widetilde{g}(u(x), v(x))$$

for each $(u,v) \in X \times X$ and a.e. for $x \in \Omega$. Since $\widetilde{f}$ and $\widetilde{g}$ are continuous and have sublinear growth, $\widetilde{F}$ and $\widetilde{G}$ are well-defined, continuous and have sublinear growth. With the notations above, the problem (21) can be rewritten as the abstract system

$$\begin{cases} u'(t) = Au(t) + F(u(t), v(t)) \\ v'(t) \in Bv(t) + G(u(t), v(t)) \\ u(\tau) = \xi,\ v(\tau) = \eta, \end{cases} \tag{28}$$

while (23) takes the abstract form

$$\begin{cases} u'(t) = Au(t) + \widetilde{F}(u(t), v(t)) \\ v'(t) = Bv(t) + \widetilde{G}(u(t), v(t)) \\ u(0) = u_0,\ v(0) = v_0. \end{cases} \tag{29}$$

We have to show first that $\mathcal{K}$ is viable with respect to $(A + F, B + G)$ and second that every mild solution $(u,v) : [\tau, T) \to X \times X$, satisfying $(t, u(t), v(t)) \in \mathcal{K}$ for each $t \in [\tau, T)$, can be extended to a global one obeying the very same constraints. Let $\widetilde{\mathcal{K}} \subseteq \mathbf{R}_+ \times L^2(\Omega) \times L^2(\Omega)$ be defined by

$$\widetilde{\mathcal{K}} = \left\{(t,u,v) \in \mathbf{R}_+ \times L^2(\Omega) \times L^2(\Omega);\ (u,v) \text{ satisfy (31) below}\right\}. \tag{30}$$

$$u(x) \le \widetilde{u}(t,x), \quad v(x) \ge \widetilde{v}(t,x) \tag{31}$$

a.e. for $x \in \Omega$. To prove that $\mathcal{K}$ is viable with respect to $(A + F, B + G)$ it suffices to show that $\widetilde{\mathcal{K}}$ is viable with respect to $(A + F, B + G)$. This is a direct consequence of the maximum principle for parabolic equations–see Theorem 1.7.5. in Cârjă-Necula-Vrabie [6]– combined with the fact that F and $\widetilde{u}$ are nonnegative. In view of Theorem 6, to show that $\widetilde{\mathcal{K}}$ is viable with respect to $(A+F, B+G)$, we have merely to check the tangency condition

$$((\tau+h, S_A(h)\xi + hF(\xi,\eta), S_B(h)\eta + hG(\xi,\eta)) \in \mathcal{QTS}^{\mathcal{A}}_{\widetilde{\mathcal{K}}}(\tau,\xi,\eta) \tag{32}$$

for each $(\tau,\xi,\eta) \in \widetilde{\mathcal{K}}$, where $\{S_A(t) : X \to X,\ t \geq 0\}$ is the C_0-semigroup generated by A and $\{S_B(t) : X \to X,\ t \geq 0\}$ is the compact C_0-semigroups generated by B and $\mathcal{A} = (A, B)$.

Let $(\tau,\xi,\eta) \in \widetilde{\mathcal{K}}$. To prove (32) it suffices that for each $h > 0$ there exist $(u_h, v_h) \in X \times X$ and $g_h \in G(\xi,\eta)$ with $(\tau+h, u_h, v_h) \in \widetilde{\mathcal{K}}$ and

$$\begin{cases} \liminf\limits_{h\downarrow 0} \dfrac{1}{h}\|S_A(h)\xi + hF(\xi,\eta) - u_h\| = 0 \\ \liminf\limits_{h\downarrow 0} \dfrac{1}{h}\|S_B(h)\eta + hg_h - v_h\| = 0. \end{cases} \tag{33}$$

Let us define $g_h(x) = g_2(\xi(x), \eta(x))$ a.e. for $x \in \Omega$ and u_h and v_h by

$$u_h = S_A(h)\xi + \int_\tau^{\tau+h} S_A(\tau+h-s)F(\xi,\eta)\,ds$$

$$+ \int_\tau^{\tau+h} S_A(\tau+h-s)[\,\widetilde{F}(\widetilde{u}(s),\widetilde{v}(s)) - \widetilde{F}(\widetilde{u}(\tau),\widetilde{v}(\tau))\,]\,ds$$

and respectively by

$$v_h = S_B(h)\eta + \int_\tau^{\tau+h} S_B(\tau+h-s)g_h\,ds$$

$$+ \int_\tau^{\tau+h} S_B(\tau+h-s)[\,\widetilde{G}(\widetilde{u}(s),\widetilde{v}(s)) - \widetilde{G}(\widetilde{u}(\tau),\widetilde{v}(\tau))\,]\,ds.$$

Now let us observe that, inasmuch as $\xi \leq \widetilde{u}(\tau)$ and $\eta \geq \widetilde{v}(\tau)$ a.e. on Ω, we have both

$$S_A(h)\xi \leq S_A(h)\widetilde{u}(\tau) \quad \text{and} \quad S_B(h)\eta \geq S_B(h)\widetilde{v}(\tau).$$

See Theorem 1.7.5 in Cârjă-Necula-Vrabie [6]. Recalling that $f \leq \widetilde{f}$ and taking into account of the monotonicity properties of $\widetilde{f}$, we get

$$F(\xi,\eta) \leq \widetilde{F}(\xi,\eta) \leq \widetilde{F}(\widetilde{u}(\tau),\widetilde{v}(\tau)).$$

Using the fact that $g_2 \geq \widetilde{g}$ and the monotonicity properties of $\widetilde{g}$, we deduce

$$g_h \geq \widetilde{G}(\xi,\eta) \geq \widetilde{G}(\widetilde{u}(\tau),\widetilde{v}(\tau)).$$

So, we get both $u_h \leq \widetilde{u}(\tau+h)$ and $v_h \geq \widetilde{v}(\tau+h)$ and thus $(\tau+h, u_h, v_h) \in \widetilde{\mathcal{K}}$. On the other hand

$$\|S_A(h)\xi + hF(\xi,\eta) - u_h\| \leq \int_\tau^{\tau+h} \|S_A(\tau+h-s)F(\xi,\eta) - F(\xi,\eta)\|\, ds$$
$$+Me^{ah} \int_\tau^{\tau+h} \|\widetilde{F}(\widetilde{u}(s),\widetilde{v}(s)) - \widetilde{F}(\widetilde{u}(\tau),\widetilde{v}(\tau))\|\, ds,$$

where $M \geq 1$ and $a > 0$ are the growth constants of the C_0-semigroups $\{S_A(t) : X \to X,\ t \geq 0\}$ and $\{S_B(t) : X \to X,\ t \geq 0\}$. Since $\widetilde{F}$, $\widetilde{u}$ and $\widetilde{v}$ are continuous we conclude that the first equality in (33) holds. Similarly, we have

$$\|S_B(h)\eta + hg_h - v_h\| \leq \int_\tau^{\tau+h} \|S_B(\tau+h-s)g_h - g_h\|\, ds$$
$$+Me^{ah} \int_\tau^{\tau+h} \|\widetilde{G}(\widetilde{u}(s),\widetilde{v}(s)) - \widetilde{G}(\widetilde{u}(\tau),\widetilde{v}(\tau))\|\, ds,$$

and we get the second equality from (33). This completes the proof of the viability of $\widetilde{\mathcal{K}}$ and consequently the viability of $\mathcal{K}$. Let us observe that $\mathcal{K}$ satisfies the next property: for each sequence $((t_n,\xi_n,\eta_n))_n$ in $\mathcal{K}$ with $\lim_n(t_n,\xi_n\eta_n) = (t,\xi,\eta)$ and $t < T_{\mathcal{K}}$, where $T_{\mathcal{K}}$ is given by (34) below, it follows that $(t,\xi,\eta) \in \mathcal{K}$.

$$T_{\mathcal{K}} = \sup\{t \in \mathbf{R};\ \text{there exist}\,(\xi,\eta) \in X \times X,\ \text{with}\,(t,\xi,\eta) \in \mathcal{K}\}. \tag{34}$$

Then it follows that each mild solution $(u,v) : [\,\tau,T\,] \to X \times X$ of (21) satisfying $(t,u(t),v(t)) \in \mathcal{K}$ for each $t \in [\,\tau,T\,]$ can be continued up to a global one $(u^*,v^*) : [\,\tau,T_{\mathcal{K}}) \to X \times X$ satisfying the very same condition on $[\,\tau,T_{\mathcal{K}})$. Since $(\widetilde{u},\widetilde{v})$ is defined on $\mathbf{R}_+$ and $(t,\widetilde{u}(t),\widetilde{v}(t)) \in \mathcal{K}$ for each $t \in [\,0,\infty)$, we conclude that $T_{\mathcal{K}} = \infty$ and this completes the proof.

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