

DYNAMIC ANALYSIS OF TWO ADHESIVELY BONDED RODS*

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Abstract

This work presents two models for the dynamic analysis of two rods that are adhesively bonded. The first model assumes that the adhesive is an elasto-plastic material and that complete debonding occurs when the stress reaches the yield limit. In the second model the degradation of the adhesive is described by the introduction of material damage. Failure occurs when the material is completely damaged, or the damage reaches a critical floor value. Both models are analyzed and the existence of a weak solution is established for the model with damage. In the quasistatic case, a new condition for adhesion is found as the limit of the adhesive thickness tends to zero.

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1 Introduction

We study two different models for the dynamic process of debonding of two slender rods that are adhesively bonded. In the first model, the adhesive

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is treated as a short rod made of a softer elasto-plastic material. System failure, i.e., complete debonding, occurs when the stress reaches the yield limit of the adhesive material. In the second model, the adhesive is treated as a damageable rod via the use of a damage function. In this case, there is a continuous decrease in the adhesive strength as cycles of tension and compression progress. The adhesive undergoes cumulative damage, similar to fatigue, and may completely fail, even if the cyclic stress never reaches the yield limit.

There is considerable interest in the engineering literature in models for material damage and metal fatigue, since predicting damage failure is of paramount concern to the design engineer.

Recent mathematical models for material damage, following the fundamental idea of Kachanov in the 1960s (see [11] for details) of introducing an internal variable, the *damage function* that measures the damage of the material, can be found in the monographs [10, 18, 22, 25], as well as in the recent papers [6, 7, 13, 17] and in the references therein. The various aspects of general models of material damage were studied in these references. Models of damage in specialized settings, similar to the one in this paper, can be found in [2, 3, 4]. Related mathematical models are those of adhesion, where a surface internal variable, the *bonding function*, was introduced by Frémond [10] and has a similar interpretation, namely, it measures the damage of the surface bonds.

Mathematical models for adhesive contact can be found in the monographs [22, 25] and in recent papers [1, 8, 9, 15, 20, 21] (see also the references therein).

In this paper we combine the two concepts of a damage function and a bonding function, and use the first to derive the source function for the debonding process. We consider a simplified one-dimensional model of two rods glued together. In this model we obtain an evolution equation for the bonding function by considering the evolution of the damage of the glue as the glue layer becomes relatively thin.

This work is the continuation of [21], where the quasistatic model was studied and numerically simulated. However, there the model did not allow for complete debonding in finite time. Models which allow complete debonding can be found in [15, 20] and here. We note that some of the models proposed and used in the above literature do not allow for complete debonding, and the issue is under current study.

As noted above, we consider a setting in which two thin rods are glued, and the glue is considered as a third (shorter) rod. In one of the models, the adhesive layer is considered as a damageable material. System failure happens when the adhesive reaches complete damage, and then the rods completely debond. The main interest in this work is in the models, and in the limit when the thickness of the adhesive layer approaches zero.

We present the two dynamic models in Section 2: one without, and the other one with material damage. We establish the existence of a weak solution for the second model in Section 4, and obtain interesting estimates on the strain in Section 5. For the first model the existence of the unique solution is straightforward to show. Then, in Section 3, we study the quasistatic problem, which reduces to a nonlinear ordinary differential equation for the damage function, since the equations of motion for the displacements can be integrated. Thus, we obtain expressions for the time to failure, i.e., the time to complete debonding. We also pass to the limit when the glue thickness is very small, and obtain an evolution equation for the adhesive as a limit of the damage equation, Problem $P_{\zeta 0}$. In this way, we obtain a new expression (unlike any in the above references) for the debonding source function, in the limit of the damage source function. This is the main modeling novelty in the paper. Some of the estimates in Section 5 are new, too.

The paper concludes with Section 6, where some future research suggestions can be found.

2 The model

Figure 1 depicts the setting of the two bonded rods. The left end of the first rod is attached to a movable device. The reference configuration of the rods are $0 \leq x \leq l_1$ and $l_2 \leq x \leq L$ ($l_1 < l_2$), and the interval $[l_1, l_2]$ is occupied by the adhesive, assumed to be a softer deformable material.

The horizontal displacements of the rods are $u_i = u_i(x, t)$, where $i = 1, 2$ for rod 1 and rod 2, respectively. The displacement of the adhesive is $u_0 = u_0(x, t)$. Below, we use the subscripts 1 and 2 for the rods, and 0 for the adhesive.

We are also interested in the limit case when the thickness of the adhesive layer vanishes, i.e., $|l_2 - l_1| \rightarrow 0$.

A body force of density $f_B = f_B(x, t)$ (per unit length) is acting on the rods, and on the adhesive segment. The left end ($x = 0$) of rod 1 is subjected

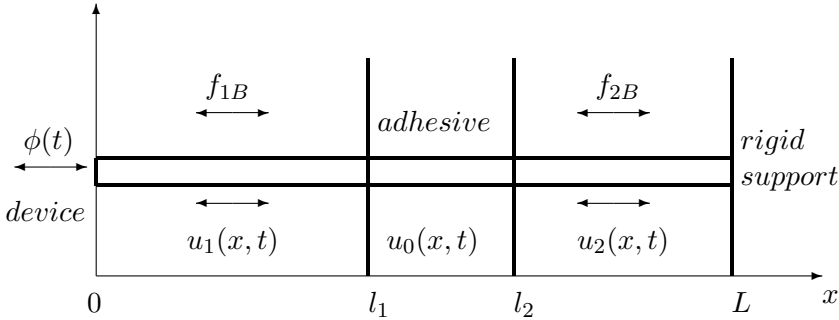


Figure 1. Two rods in adhesive contact

to a dynamic axial displacement $\phi = \phi(t)$. Thus $u_1(0, t) = \phi(t)$. The right end of rod 2 is fixed, so $u_2(L, t) = 0$. When ϕ is negative, the rods are in tension, and when ϕ is positive, the are in compression.

The dynamic motion of each one of the three rods is described by the wave equation and the displacements are assumed to be continuous at the interfaces $x = l_1, l_2$ where the tractions are equal, too.

We consider two different scenarios, which result in two different models. In the first scenario, the adhesive is considered as an elasto-plastic material with lower modulus of elasticity, as compared to the rods. The adhesion between the two rods is assumed to break down, or completely debond, when the stress in the adhesive region reaches the yield limit.

In the second model we assume that the adhesive material undergoes damage as a result of the strains. Then, complete debonding occurs when the damage reaches the threshold limit.

We denote by ρ_i and E_i , for $i = 0, 1, 2$, the density (per unit length) and the elasticity modulus of the material in each region.

The classical formulation of the first model for *the vibrations of two rods in adhesive contact* is:

Problem P_{cl} . Find a triple of functions (u_1, u_0, u_2) such that, for $0 < t \leq T$:

$$\rho_1 u_{1tt}(x, t) - E_1 u_{1xx}(x, t) = \rho_1 f_B(x, t), \quad x \in (0, l_1), \quad (1)$$

$$\rho_0 u_{0tt}(x, t) - E_0 u_{0xx}(x, t) = \rho_0 f_B(x, t), \quad x \in (l_1, l_2), \quad (2)$$

$$\rho_2 u_{2tt}(x, t) - E_2 u_{2xx}(x, t) = \rho_2 f_B(x, t), \quad x \in (l_2, L), \quad (3)$$

$$u_1(0, t) = \phi(t), \quad u_2(L, t) = 0, \quad (4)$$

$$u_1(l_1, t) = u_0(l_1, t), \quad E_1 u_{1x}(l_1, t) = E_0 u_{0x}(l_1, t), \quad (5)$$

$$u_2(l_2, t) = u_0(l_2, t), \quad E_2 u_{2x}(l_2, t) = E_0 u_{0x}(l_2, t), \quad (6)$$

$$u(x, 0) = u_{in}(x), \quad (7)$$

$$u_t(x, 0) = v_{in}(x). \quad (8)$$

Here, u_{in} and v_{in} are the (prescribed) initial displacements and velocities, respectively, with the understanding that $u_1(x, 0) = u_{in}(x)$ and $u_{1t}(x, 0) = v_{in}(x)$ for $x \in [0, l_1]$, and similarly for the other two rods.

The problem consists of three coupled wave equations for the displacements $u_1(x, t)$, $u_2(x, t)$, and $u_0(x, t)$.

To describe the second model, we follow [11] (see also [10, 18, 22, 25] and the references therein) and introduce the *damage function* $\zeta = \zeta(x, t)$, which measures the pointwise fractional decrease in the strength of the adhesive material. To describe the damage process of the material the damage-free adhesive modulus of elasticity E_0 is replaced with the effective modulus

$$E_{eff} = \zeta E_0.$$

Then, it follows that

$$0 \leq \zeta(x, t) \leq 1, \quad (9)$$

and when $\zeta = 1$ the material is damage-free; when $\zeta = 0$ the damage is complete and the system breaks at the point; and when $0 < \zeta(x, t) < 1$ the material is partially damaged and has a decreased load carrying capacity.

Next, we need to describe the evolution of the damage function ζ . Following [10, 11, 22, 25] (see also the other references mentioned above), we assume that the evolution of damage is caused by the growth of micro-cracks and micro-cavities caused by the cyclic stress. The damage function has to satisfy the growth equation

$$\zeta_t - \kappa \zeta_{xx} = \Phi(\zeta, u_{0x}) + \xi,$$

where, $\Phi = \Phi(\zeta, u_{0x})$ is the *damage source function*, which is described shortly in (10), κ is the damage diffusion coefficient, and ξ is a ‘force’ that prevents ζ from violating (9). To describe the latter, we let $I_{[0,1]}$ denote the indicator function of the interval $[0, 1]$, and then its subdifferential is the set-valued mapping denoted by $\partial I_{[0,1]}(z)$. To enforce the condition $0 \leq \zeta \leq 1$, we require that $-\xi \in \partial I_{[0,1]}(\zeta)$. Indeed, when $0 < \zeta < 1$ then $\xi = 0$; when $\zeta = 0$ then $\xi > 0$ has the exact value that prevents ζ from becoming negative; and when $\zeta = 1$ then $\xi < 0$ has the exact value that prevents ζ from exceeding the value one.

General damage source functions can be found in [10, 22, 25]; here, we use a somewhat simple function which depends only on the mechanical energy $E_0 \zeta u_{0x}^2$ and the damage process is assumed to be irreversible so that once micro-cavities or micro-crack are formed, they do not mend, thus

$$\Phi(\zeta, u_{ax}) = -d(\zeta u_{0x}^2 - \epsilon_0)_+. \quad (10)$$

Here, d is the damage rate coefficient, ϵ_0 is the scaled damage threshold energy, below which there is no damage change, and $(r)_+$ is the positive part function, i.e., $(r)_+ = r$ if $0 \leq r$ and $(r)_+ = 0$ if $r < 0$. The negative sign makes the process irreversible. With this choice, the parabolic equation for ζ (with $\xi = 0$) predicts that if initially $\zeta_{in} \leq 1$, then $\zeta \leq 1$ for $0 < t$.

For the sake of generality, we also assume that the adhesive has viscosity which we model with $\nu(\zeta u_{0tx})_x$, where ν is the viscosity coefficient, assumed to be small.

Problem P_ζ . Find a quadruple of functions (u_1, u_0, ζ, u_2) such that, for $0 < t \leq T$ (1), (3), (4), (7), and (8) hold, together with

$$\rho_0 u_{0tt}(x, t) - E_0(\zeta u_{0x})_x(x, t) - \nu(\zeta u_{0tx})_x(x, t) = \rho_0 f_B(x, t), \quad x \in (l_1, l_2), \quad (11)$$

$$\zeta_t - \kappa \zeta_{xx} + d(\zeta u_{0x}^2 - \epsilon_0)_+ \in -\partial I_{[0,1]}(\zeta), \quad x \in (l_1, l_2), \quad (12)$$

$$u_1(l_1, t) = u_0(l_1, t), \quad E_1 u_{1x}(l_1, t) = E_0(\zeta u_{0x})(l_1, t), \quad (13)$$

$$u_2(l_2, t) = u_0(l_2, t), \quad E_2 u_{2x}(l_2, t) = E_0(\zeta u_{0x})(l_2, t), \quad (14)$$

$$\zeta_x(l_1, t) = 0 = \zeta_x(l_2, t), \quad \zeta(x, 0) = \zeta_{in}(x). \quad (15)$$

Here, ζ_{in} is the initial damage, which has the value one in a damage-free material.

The analysis of problems P_{cl} and P_ζ will be done in Section 4. Next, we study the equations for the problems when the process is quasistatic and the adhesive layer is thin.

3 Quasistatic problems

We study three problems which model the process when it is quasistatic, i.e., slow enough so that the acceleration terms may be neglected, and in the absence of body forces ($f_B = 0$).

3.1 Quasistatic version of P_{cl}

We begin with the quasistatic version of Problem P_{cl} . Since there are no body forces and the second time derivatives are neglected, the displacements are linear. Writing

$$u_0(x, t) = \alpha(t)x + \beta(t), \quad (16)$$

straightforward manipulations, using the facts that the displacements u_1 and u_2 are linear and the boundary conditions (4)–(6), yield

$$\alpha(t) = \frac{-\phi(t)}{(l_2 - l_1) + \frac{E_0}{E_2}(L - l_2) + \frac{E_0}{E_1}l_1}, \quad (17)$$

and

$$\beta(t) = -\alpha \left(l_2 + (L - l_2) \frac{E_0}{E_2} \right). \quad (18)$$

Moreover,

$$u_1(x, t) = \frac{E_0}{E_1} \alpha(t)x + \phi(t), \quad u_2(x, t) = -\frac{E_0}{E_2} \alpha(t)(L - x). \quad (19)$$

We note that when the displacement ϕ is negative the system is under tension and when it is positive the system is under compression.

In the limit when the thickness of the layer of glue tends to zero, $l_2 \rightarrow l_1 = l$, we find that

$$\alpha(t) = \frac{-\phi(t)}{\frac{E_0}{E_2}(L - l) + \frac{E_0}{E_1}l}, \quad \beta(t) = -\alpha \left(l + (L - l) \frac{E_0}{E_2} \right).$$

Thus, the influence of the adhesive enters via its stiffness E_0 . The displacement at $x = l$ is given by

$$u_1(l, t) = u_2(l, t) = \frac{\phi(t)(L - l)E_1}{E_1(L - l) + E_2l}.$$

The stress by $p(t) = E_1 u_{1x}(l, t) = E_0 \alpha(t) = E_2 u_{2x}(l, t)$. Therefore, this system will debond (completely) only when the stress reaches the plasticity yield or the debonding limit σ^* ,

$$E_0 \alpha(t) = \sigma^*.$$

Clearly, this formulation cannot take into account gradual degradation of the strength of the bonds as a result of cycles in ϕ .

The quasistatic problem with a prescribe traction boundary condition at $x = 0$ is straightforward to study, and is not very interesting, since in a one-dimensional system the stress is uniform.

3.2 Quasistatic version of P_ζ

We turn to the quasistatic version of Problem P_ζ , which turns out to be more interesting. In particular, it accounts for degradation of the strength of the bonds as a result of cycles in ϕ . Since there are no body forces and the second time derivatives are neglected, the displacements u_1 and u_2 are linear. In equation (11) for u_0 we neglect the viscosity term, and obtain $(\zeta u_{0x})_x = 0$. Therefore,

$$\zeta(x, t) u_{0x}(x, t) = \gamma(t), \quad l_1 \leq x \leq l_2, \quad (20)$$

where $\gamma(t)$ is to be determined. Then, the boundary conditions (13) and (14) yield

$$E_1 u_{1x}(l_1, t) = E_0 \gamma, \quad E_2 u_{2x}(l_2, t) = E_0 \gamma.$$

Thus,

$$u_{1x}(l_1, t) = \frac{E_0}{E_1} \gamma(t), \quad u_{2x}(l_2, t) = \frac{E_0}{E_2} \gamma(t),$$

and then,

$$u_1(x, t) = \frac{E_0}{E_1} \gamma(t) x + \phi(t), \quad u_2(x, t) = -\frac{E_0}{E_2} \gamma(t) (L - x).$$

Next, integration in (20) yields

$$u_0(x, t) = \gamma(t) \int_{l_1}^x \frac{1}{\zeta(x, t)} dx + \delta(t), \quad (21)$$

for $l_1 \leq x \leq l_2$, where δ is a constant of integration. It follows from the continuity of the displacements that

$$u_1(l_1, t) = \frac{E_0}{E_1} \gamma(t) l_1 + \phi(t) = \delta(t),$$

$$u_2(l_2, t) = -\frac{E_0}{E_2} \gamma(t) (L - l_2) = \gamma(t) \int_{l_1}^{l_2} \frac{1}{\zeta(x, t)} dx + \delta(t).$$

Let

$$c_{12} = \frac{E_0}{E_2} (L - l_2) + \frac{E_0}{E_1} l_1.$$

Substituting δ from the first equation and rearranging yields

$$\gamma(t) = \frac{-\phi(t)}{c_{12} + \int_{l_1}^{l_2} \frac{1}{\zeta(x, t)} dx}. \quad (22)$$

Then,

$$\delta(t) = \frac{E_0}{E_1} \gamma(t) l_1 + \phi(t) = \phi(t) - \frac{E_0 l_1 \phi(t)}{E_1 c_{12} + E_1 \int_{l_1}^{l_2} \frac{1}{\zeta(x, t)} dx}. \quad (23)$$

It follows that once ζ is found, the problem is solved. To obtain ζ , we note that $u_{0x} = \gamma/\zeta$, hence

$$\Phi(u_{ax}) = -d(\zeta u_{ax}^2 - \epsilon_0)_+ = -d\left(\frac{\gamma^2}{\zeta} - \epsilon_0\right)_+ = -d(\Theta(\phi; \zeta, t) - \epsilon_0)_+,$$

where we defined

$$\Theta(\phi; \zeta, t) = \frac{\phi^2(t)}{\zeta \left(c_{12} + \int_{l_1}^{l_2} \frac{1}{\zeta(x, t)} dx \right)^2}.$$

Now, the problem for ζ is the following.

Problem $P_{quas-\zeta}$. Given ϕ , find a function $\zeta = \zeta(x, t)$ such that, for $0 < t \leq T$,

$$\zeta_t - \kappa \zeta_{xx} = -d(\Theta(\phi; \zeta, t) - \epsilon_0)_+, \quad x \in (l_1, l_2), \quad (24)$$

$$\zeta(x, 0) = \zeta_{in}, \quad \zeta_x(l_1, t) = \zeta_x(l_2, t) = 0. \quad (25)$$

We note that the problem is nonlocal, since the source term on the right-hand side of (24) depends on $\int_{l_1}^{l_2} \frac{1}{\zeta(x,t)} dx$. It is somewhat unusual and has mathematical interest in and of itself, and will be analyzed elsewhere.

Next, we consider the limit $\lim l_1 = \lim l_2 = l$. It follows from the boundary conditions (25) that $\zeta = \zeta(t)$ only, as it does not depend on x . Also,

$$\lim_{|l_2-l_1| \rightarrow 0} \Theta(\phi; \zeta, t) = \Theta_0(\zeta, t) = \frac{\phi^2(t)}{c_{12}^2 \zeta}.$$

Therefore, the limit problem is as follows.

Problem P_{ζ_0} . Find a function $\zeta = \zeta(t)$ such that, for $0 < t \leq T$,

$$\zeta' = -d \left(\frac{\phi^2(\tau)}{c_{12}^2 \zeta} - \epsilon_0 \right)_+, \quad (26)$$

$$\zeta(0) = \zeta_{in}. \quad (27)$$

The problem is a nonlinear ordinary differential equation with non-Lipschitz right-hand side. We study it in Section 4.

We note that when ϵ_0 is negligible, as compared to the average of $\phi^2(t)/c_{12}^2$, the equation for ζ becomes

$$\zeta' = -d\Theta_0(\zeta, t) = -\frac{d\phi^2(t)}{c_{12}^2 \zeta}.$$

Using the initial condition, we obtain

$$\zeta^2(t) = \zeta_{in}^2 - \frac{2d}{c_{12}^2} \int_0^t \phi^2(\tau) d\tau.$$

It follows that the time to failure t_0^* is given in this case implicitly by

$$\int_0^{t_0^*} \phi^2(\tau) d\tau = \frac{c_{12}^2 \zeta_{in}^2}{2d}.$$

A simple comparison argument shows that if t^* is the time to failure of the solution of (26) and (27), then $t_0^* \leq t^*$, as one would expect.

Problem P_{ζ_0} connects material damage and adhesion at the joint point and it has a very different structure from the usual bonding conditions used in the literature (see, e.g., [21]). Indeed, there, the bonding was assumed to be of the form

$$\zeta' = -d\zeta(u_x^2 - \epsilon_0)_+,$$

which doesn't allow for failure, i.e., complete debonding in finite time, or a more recent condition ([15])

$$\zeta' = -d\zeta^\alpha(u_x^2 - \epsilon_0)_+,$$

which allows for failure when $0 \leq \alpha < 1$. Here, we find that $\alpha = -2$, and this makes the analysis quite different.

3.3 Quasistatic version of P_ζ with traction condition

We describe briefly the case when instead of the displacement ϕ , a traction $q = q(t)$ is applied at the left end ($x = 0$). This is often the case in experimental settings. Thus, we replace the first condition in (4) with $E_1 u_{1x}(0, t) = q(t)$. Then,

$$u_1(x, t) = \frac{1}{E_1} q(t)x + b(t),$$

where $b(t)$ is to be determined. At $x = l_1$ we have $E_1 u_{1x}(l_1, t) = q(t) = E_0 \gamma(t)$, hence

$$\gamma(t) = \frac{1}{E_0} q(t).$$

Moreover, $u_2(x, t) = (q(t)/E_2)(x - L)$. It follows from (21) that

$$u_0(x, t) = \frac{1}{E_0} q(t) \int_{l_1}^x \frac{1}{\zeta(x, t)} dx + \delta(t), \quad (28)$$

for $l_1 \leq x \leq l_2$, and δ is a constant. The displacements' continuity implies

$$\frac{1}{E_1} q(t) l_1 + b(t) = \delta(t), \quad \frac{1}{E_0} q(t) \int_{l_1}^{l_2} \frac{1}{\zeta(x, t)} dx + \delta(t) = \frac{1}{E_2} q(t) (l_2 - L).$$

It follows that

$$\delta(t) = -q(t) \left(\frac{1}{E_2} (L - l_2) + \frac{1}{E_0} \int_{l_1}^{l_2} \frac{1}{\zeta(x, t)} dx \right). \quad (29)$$

Also,

$$b(t) = -q(t) \left(\frac{1}{E_1} l_1 + \frac{1}{E_2} (L - l_2) + \frac{1}{E_0} \int_{l_1}^{l_2} \frac{1}{\zeta(x, t)} dx \right). \quad (30)$$

It is seen that once ζ is found, the displacements u_1 , u_2 , and u_0 are given by the expressions above. It remains to obtain an equation for ζ . We have

$$\Phi(u_{0x}) = -d(\zeta u_{0x}^2 - \epsilon_0)_+ = -d \left(\frac{q^2(t)}{E_0^2 \zeta} - \epsilon_0 \right)_+.$$

We conclude that the quasistatic problem for ζ , when a traction q is prescribed at $x = 0$, is the following.

Problem $P_{\zeta q}$. Given $q(t)$, find a function $\zeta = \zeta(x, t)$ such that, for $0 < t \leq T$,

$$\zeta_t - \kappa \zeta_{xx} = -d \left(\frac{q^2(t)}{E_0^2 \zeta} - \epsilon_0 \right)_+, \quad x \in (l_1, l_2), \quad (31)$$

$$\zeta(x, 0) = \zeta_{in}, \quad \zeta_x(l_1, t) = \zeta_x(l_2, t) = 0. \quad (32)$$

We note that this problem is local, but is also somewhat unusual and has mathematical interest in and of itself, and will be analyzed elsewhere.

The problem for a thin layer of glue is obtained in the limit $\lim l_2 = l_1 = l$.

Problem $P_{\zeta q_0}$. Given $q(t)$, find a function $\zeta = \zeta(t)$ such that, for $0 < t \leq T$,

$$\zeta' = -d \left(\frac{q^2(t)}{E_0^2 \zeta} - \epsilon_0 \right)_+, \quad (33)$$

$$\zeta(0) = \zeta_{in}. \quad (34)$$

We note that whereas problems $P_{quas-\zeta}$ and $P_{\zeta q}$ are substantially different, the limit problems P_{ζ_0} and $P_{\zeta q_0}$ are very similar, with q^2/E_0^2 replacing ϕ^2/c_{12}^2 . Therefore, the existence of the unique solution of Problem $P_{\zeta q_0}$ follows from Theorem 1 below.

In this case, if we neglect the Dupré energy ϵ_0 , we find that the time to complete debonding t_0^* is given implicitly by

$$\int_0^{t_0^*} q^2(\tau) d\tau = \frac{E_0^2 \zeta_0^2}{2d}.$$

4 Analysis

We first study Problem P_{ζ_0} , (26) and (27), and establish the existence of a unique local (in time) solution. Then, we prove the existence of a weak solution to the dynamic problem with damage, Problem P_{ζ} .

4.1 Problem P_{ζ_0}

For the sake of generality, we replace the function $\phi^2(t)/c_{12}^2$ in (26) with a more general nonnegative smooth and bounded function $\psi = \psi(t)$. Then, the problem is as follows.

Problem $P_{\zeta\psi}$. Given a function ψ , find a function $\zeta = \zeta(t) \geq 0$, such that, for $0 < t \leq T$,

$$\zeta' = -d \left(\frac{\psi(t)}{\zeta} - \epsilon_0 \right)_+, \quad (35)$$

$$\zeta(0) = \zeta_{in}. \quad (36)$$

We make the following assumptions on the problem data.

H_1 . The function $\psi : [0, T] \rightarrow [0, \infty)$ is continuous and bounded.

H_2 . The constants d and ϵ_0 are positive and $\zeta_{in} \in (0, 1]$.

Theorem 1. *Assume that H_1 and H_2 hold. Then there exists $T^* > 0$ such that there exists a unique solution ζ of Problem $P_{\zeta\psi}$ on the time interval $[0, T^*)$. Moreover,*

$$\zeta \in C^1([0, T^*)). \quad (37)$$

Proof. Let $0 < a < \zeta_{in}$ and let $g_a(\zeta, t)$ be a function with the graph of a straight line through $(0, 0)$ and $-d \left(\frac{\psi(t)}{a} - \epsilon_0 \right)$, and let

$$F(\zeta, t) \equiv \max \left(-d \left(\frac{\psi(t)}{\zeta} - \epsilon_0 \right)_+, g_a(\zeta, t) \right).$$

Then, $F(\zeta, t)$ is Lipschitz in ζ and so there exists a unique solution to

$$\zeta' = F(\zeta, t), \quad \zeta(0) = \zeta_{in}.$$

Letting t_a^* be the value of t at which $\zeta(t)$ first equals a , then, since $0 < a$ is arbitrary, the theorem follows when we choose $T^* = \sup(t_a^*)$, for $a \in (0, \zeta_{in})$.

4.2 Problem P_ζ

We turn to Problem P_ζ , and establish the existence of its weak solution. The weak formulation is obtained in the usual manner, and we use the following notation: u represents the displacements, and is such that $u = u_1$ on $[0, l_1]$, $u = u_0$ on $[l_1, l_2]$, and $u = u_2$ on $[l_2, L]$. Similarly, we define the functions

$\rho(x)$ and $c(x)$ as $\rho = \rho_1, c = E_1$ on $[0, l_1]$, $\rho = \rho_0, c = E_0$ on $[l_1, l_2]$, and $\rho = \rho_2, c = E_2$ on $[l_2, L]$. Finally, for the sake of generality we add a viscosity term in (1) and (3), and let the viscosity $\nu(x)$ be defined in the same way. We also extend the definition of the unknown function ζ as 1 outside of the interval $[l_1, l_2]$, and replace c with $c\zeta$ in (1)–(3).

We now multiply equations (1)–(3) by a test function φ , integrate by parts and use the boundary conditions to obtain the following weak formulation for u , for a.a. $t \in (0, T)$,

$$\begin{aligned} & \int_0^L \rho(x) u_{tt}(x, t) \varphi(x) dx + \int_0^L c(x) \zeta(x, t) u_x(x, t) \varphi_x(x) dx \\ & + \int_0^L \nu(x) \zeta(x, t) u_{xt}(x, t) \varphi_x(x) dx = \int_0^L \rho(x) f_B(x, t) \varphi(x) dx. \end{aligned}$$

Similarly, using θ as a test function, we obtain from (12),

$$\begin{aligned} & \int_{l_1}^{l_2} \zeta_t(x, t) \theta(x) dx + \kappa \int_{l_1}^{l_2} \zeta_x(x, t) \theta_x(x) dx \\ & \geq -d \int_{l_1}^{l_2} (\zeta(x, t) u_x^2(x, t) - \epsilon_0)_+ \theta(x) dx. \end{aligned}$$

Actually, as explained below, we can eliminate the subgradient term because the source term for damage is sufficient to keep the damage parameter in the interval of interest.

We regard the adhesive and the two rods as a single continuum, as described above, but damage is assumed to affect only the adhesive.

To proceed with the analysis we need the following spaces.

$$V \equiv H_0^1(0, L), \quad H \equiv L^2(0, L),$$

and

$$\mathcal{V} \equiv \mathcal{L}^2(0, T; V), \quad \mathcal{H} \equiv \mathcal{L}^2(0, T; H).$$

We use on V and \mathcal{V} the (equivalent) norms

$$\|w\|_V^2 = \int_0^L w_x^2 ds, \quad \|w\|_{\mathcal{V}}^2 = \int_0^T \int_0^L w_x^2 ds dt.$$

We need to introduce a truncation to preserve the coercivity of the problem, which becomes noncoercive in the limit $\zeta \rightarrow 0$. To that end we let η

be a truncation function, assumed to be smooth and nondecreasing with the following properties:

$$\eta(r) \leq 2 \text{ if } r \geq 1, \quad \eta(r) = \delta \text{ if } r < \delta, \quad \eta(r) = r \text{ if } r \in (2\delta, 1],$$

where δ is assumed to be very small, in particular, $\delta \ll \varepsilon_0$. We note that these problems, typically, possess only local solutions, so this is not a serious restriction. Moreover, we show below that η is not active (i.e., $\eta(\zeta) = \zeta$) on some interval of time.

Now, we define the operator $A : V \rightarrow V'$ as follows: for ζ in H let

$$\langle A(\zeta, u), v \rangle \equiv \int_0^L c(x) \eta(\zeta(x)) u_x(x) v_x(x) dx.$$

We note that $c(x)$ is discontinuous, and bounded away from zero, as it takes the values E_1, E_0, E_2 in the different intervals. We also assume that

$$\phi \in C^2([0, T]).$$

To obtain homogeneous boundary conditions at $x = 0, x = L$ we define a new variable $w(x, t) = u(x, t) - \phi(t)(1 - x/L)$ and obtain a similar equation for w involving only a change in $f_B(x, t)$, but with w satisfying zero boundary conditions at $x = 0$ and $x = L$. Therefore, we assume at the outset that $\phi(t) = 0$ to make the presentation simpler. To slightly simplify the presentation we also assume that the density $\rho(x)$ is a constant, rescaled as $\rho = 1$. In addition, we let

$$v(t) \equiv u'(t), \quad v(t) \in \mathcal{V}, \quad u(t) \equiv u_0 + \int_0^t v(s) ds.$$

The truncated problem is as follows. Find $v \in \mathcal{V}$ such that,

$$v' + A(\zeta, v) + A(\zeta, u) = f, \tag{38}$$

$$v(0) = v_0, \tag{39}$$

$$u(t) \equiv u_0 + \int_0^t v(s) ds, \quad u_0 \in V. \tag{40}$$

Here f is a body force, assumed in \mathcal{H} . The problem for the damage is to find $\zeta \in W^{1,2}((0, L) \times (0, T))$ such that,

$$\zeta' - \Delta \zeta = -d(\eta(\zeta) \mathcal{X}_{[l_1, l_2]} Q_M(u_x) - \varepsilon_0)_+, \tag{41}$$

$$\zeta(0) = \zeta_0 \in H^1(0, L), \quad \zeta_0(x) \in (3\delta, 1]. \tag{42}$$

We let $\zeta_0(x) = 1$ for $x \notin [l_1, l_2]$. This forces the extension of ζ to the rest of $[0, L]$ to equal 1. Then, the requirement $\zeta \in H^1(0, L)$ guarantees that $\zeta = 1$ at the end points $x = l_1, l_2$, so damage is happening in the interior of this interval but not at the ends. Also, we obtain the natural boundary conditions $\zeta_x = 0$ at the endpoints of the adhesion interval.

Moreover, $Q_M(r)$ is a truncation of u_x , making it easier to obtain some of the estimates below. It is a bounded Lipschitz continuous function which equals r^2 whenever $|r| < M$, say

$$Q_M \in C^1(\mathbb{R}), \quad 0 \leq Q(r) \leq M^2. \quad (43)$$

The characteristic function $\mathcal{X}_{[l_1, l_2]}$ of the middle interval is used to guarantee that the damage process is taking place only in the glue layer.

We show below that on a suitable interval the truncation is inactive but, to begin with, it is convenient to include it. The source term for damage in (41) is such that together with the assumptions on ζ_0 , it implies that $\zeta(x, t) \in (\delta, 1]$ a.e. x for all t . It is a consequence of maximum principle arguments and a proof can be found in [13].

We begin with the study of the mechanical part of the problem.

Lemma 1. *Let $\zeta \in \mathcal{H}$. Then there exists a unique solution to (38) – (40). Also, if v_ζ is the solution corresponding to ζ then the map $\zeta \rightarrow v_\zeta$ is continuous from \mathcal{H} to \mathcal{V} .*

Proof. We consider the existence part first. It follows from standard theorems in Lions, [19], that there exists a unique solution v_u to (38) for each $u \in \mathcal{V}$. Also, the operator $Av(t) \equiv A(\zeta(t), v(t))$ is monotone, hemi-continuous, bounded, and coercive as a map from \mathcal{V} to \mathcal{V}' , so the the main existence theorem in [16] is applicable. Consider now the map $\Psi : \mathcal{V} \rightarrow \mathcal{V}$, given by

$$\Psi(u(t)) \equiv u_0 + \int_0^t v_u(s) ds.$$

Then,

$$\Psi(u(t)) - \Psi(w(t)) = \int_0^t (v_u(s) - v_w(s)) ds.$$

Next, simple manipulations, using (38), yield

$$\frac{1}{2} \|v_u(t) - v_w(t)\|_H^2 + \frac{\delta}{2} a \int_0^t \|v_u - v_w\|_V^2 ds \leq C_\delta \int_0^t \|w - u\|_V^2 ds.$$

It follows that

$$\begin{aligned} \|\Psi(u(t)) - \Psi(w(t))\|_V^2 &\leq C_T \int_0^t \|v_u(s) - v_w(s)\|_V^2 ds \\ &\leq C_T C_\delta \int_0^t \|u(s) - w(s)\|_V^2 ds, \end{aligned}$$

and this implies that a large enough power of Ψ is a contraction mapping on \mathcal{V} , so there exists a unique solution (v, u) to (38)–(40).

Let (v, u) be a solution of this initial value problem. Then, it follows from the equation that

$$\begin{aligned} &\frac{1}{2} \|v(t)\|_H^2 + \frac{\delta}{2} a \int_0^t \|v\|_V^2 ds \\ &\leq \frac{1}{2} \|v_0\|_H^2 + C_\delta \int_0^t \|u\|_V^2 ds + C(f) + \int_0^t \|v\|_H^2 ds \\ &\leq C_{\delta T} \int_0^t \int_0^s \|v\|^2 dr ds + C(f, \|u_0\|_V) + \int_0^t \|v\|_H^2 ds. \end{aligned}$$

Here and below, we denote by $C = C(\dots)$ a constant that depends only on the argument and the problem constants. It follows from Gronwall's inequality that there exists a constant, depending on the indicated quantities, such that

$$\|v(t)\|_H^2 + \int_0^t \|v\|_V^2 ds \leq C \left(\|v_0\|_H^2, f, \|u_0\|_V, \delta \right). \quad (44)$$

Next, we show the continuous dependence of the solution (v, u) on ζ . Let v_i correspond to $\zeta_i, i = 1, 2$. Then, from the initial value problem (38)–(40), together with routine manipulations, we obtain

$$\begin{aligned} &\frac{1}{2} \|v_1(t) - v_2(t)\|_H^2 + \frac{\delta}{2} a \int_0^t \|v_1(s) - v_2(s)\|_V^2 ds \\ &\leq C_\delta \int_0^t \int_\Omega |\eta(\zeta_1) - \eta(\zeta_2)|^2 |v_{1x}|^2 dx ds \\ &+ C_\delta \int_0^t \int_\Omega |\eta(\zeta_1) - \eta(\zeta_2)|^2 |u_{1x}|^2 dx ds. \end{aligned} \quad (45)$$

Assume that the map $\zeta \rightarrow v_\zeta$ is not continuous. Then, there exists $\zeta \in \mathcal{H}$ and a sequence $\{\zeta_n\}$ such that $\zeta_n \rightarrow \zeta$ pointwise, as well as in \mathcal{H} , but for some $\varepsilon > 0$,

$$\int_0^T \|v_n(s) - v(s)\|_V^2 ds \geq \varepsilon,$$

where v is the solution of (38)–(40) that corresponds to ζ and v_n corresponds to ζ_n . Now, let $t = T$ and $v_2 = v_n, v_1 = v$ in (45). Since η is a bounded function, the dominated convergence theorem applies and the right-hand side of (45) converges to zero, which is a contradiction. This proves the lemma.

The next two theorems are used below, and can be found in Lions [19] and Simon [24], respectively.

Theorem 2. *Assume $p \geq 1, q > 1$, and $W \subseteq U \subseteq Y$, where the inclusion map $W \rightarrow U$ is compact and the inclusion map $U \rightarrow Y$ is continuous. Let*

$$S_R = \{\mathbf{u} \in L^p(0, T; W) : \mathbf{u}' \in L^q(0, T; Y), \|\mathbf{u}\|_{L^p(0, T; W)} + \|\mathbf{u}'\|_{L^q(0, T; Y)} < R\}.$$

Then S_R is precompact in $L^p(0, T; U)$.

Theorem 3. *Let W, U , and Y be as in Theorem 2, $q > 1$, and let*

$$S_{RT} = \{\mathbf{u} : \|\mathbf{u}(t)\|_W + \|\mathbf{u}'\|_{L^q(0, T; Y)} \leq R, \quad t \in [0, T]\}.$$

Then S_{RT} is precompact in $C(0, T; U)$.

We now consider the question of existence for a solution (v, ζ) of (38)–(42). To that end let $\zeta \in \mathcal{H}$ be given. Then, let (v_ζ, u_ζ) denote the unique solution of problem (38)–(40). Using ζ and u_ζ in the right side of (41) and (42), it follows from a well known results of Brezis ([5]), see also Showalter ([23]), since the differential operator $-\Delta$ is a subgradient of a proper lower semicontinuous functional, that there exists a unique function $\xi \in L^2(0, T; H^2(0, L)), \xi' \in \mathcal{H}, \zeta_x = 0$ at $x = 0$ and L , which satisfies (41) and (42). Let $\Phi(\zeta) \equiv \xi$. Thus, this Φ is a map from \mathcal{H} to \mathcal{H} . It was shown in Lemma 1 that the map $\zeta \rightarrow v_\zeta$ is continuous from \mathcal{H} to \mathcal{V} . From the definition of u_ζ as an integral of v_ζ given in (40), it follows that $\zeta \rightarrow u_\zeta$ is continuous from \mathcal{H} to $C([0, T]; V)$. Therefore, since all the truncation functions in the source term for damage in (41) are bounded and Lipschitz continuous, it follows from simple manipulations, such as those above, that

$\zeta \rightarrow \Phi(\zeta)$ is continuous as a map from \mathcal{H} to \mathcal{H} . In fact, more can be said, but this is enough for our purposes. We note the fact that Φ is not only continuous, but maps \mathcal{H} into a compact subset of \mathcal{H} . This follows from Theorem 2 and the following interesting lemma which is stated in more generality than needed here.

Lemma 2. *Assume that the boundary of Ω is in $C^{1,1}$. Let $y, y' \in L^2(0, T; L^2(\Omega))$, $y(0) = y_0 \in H^1(\Omega)$, assume also that $y \in L^2(0, T; H^2(\Omega))$ and it satisfies $\partial y / \partial n = 0$ on $\partial\Omega$. Then,*

$$\int_0^t (y', -\Delta y)_{L^2(\Omega)} ds = \frac{1}{2} \|\nabla y(t)\|_{L^2(\Omega)^d}^2 - \frac{1}{2} \|\nabla y_0\|_{L^2(\Omega)^d}^2.$$

Proof. Let $Ly \equiv -\Delta y$, where $y \in D(L)$ is given by

$$\{y \in L^2(0, T; L^2(\Omega)); \Delta y \in L^2(0, T; L^2(\Omega)), \partial y / \partial n = 0 \text{ on } \partial\Omega\}.$$

Then, L is a maximal monotone operator. Also, since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, it follows that $D(L)$ is dense in $L^2(0, T; L^2(\Omega))$. Let

$$y_\varepsilon \equiv (I + \varepsilon L)^{-1} y,$$

for a small $\varepsilon > 0$. Thus, $y'_\varepsilon = (I + \varepsilon L)^{-1} y' \in D(L)$ and it is routine to verify that

$$\int_0^t (y'_\varepsilon, (-\Delta y_\varepsilon))_{L^2(\Omega)} ds = \frac{1}{2} \|\nabla y_\varepsilon(t)\|_{L^2(\Omega)^d}^2 - \frac{1}{2} \|\nabla y_\varepsilon(0)\|_{L^2(\Omega)^d}^2.$$

Moreover, since $D(L)$ is dense in $L^2(0, T; L^2(\Omega))$, it follows from standard results on maximal monotone operators (see, e.g., [5]) that, as $\varepsilon \rightarrow 0$,

$$\begin{aligned} -\Delta y_\varepsilon = Ly_\varepsilon = L(I + \varepsilon L)^{-1} y &= (I + \varepsilon L)^{-1} Ly \rightarrow Ly = -\Delta y, \\ (I + \varepsilon L)^{-1} y' &= y'_\varepsilon \rightarrow y' \quad \text{in } L^2(0, T; L^2(\Omega)). \end{aligned}$$

In addition,

$$\begin{aligned} \nabla y_\varepsilon &= \nabla (I + \varepsilon L)^{-1} y = (I + \varepsilon L)^{-1} \nabla y \rightarrow \nabla y, \\ \nabla y_\varepsilon(0) &= \nabla (I + \varepsilon L)^{-1} y_0 = (I + \varepsilon L)^{-1} \nabla y_0 \rightarrow \nabla y_0, \end{aligned}$$

and by using subsequences, if necessary, all these convergence results take place for a.a. t . Therefore, for a.a. t ,

$$\begin{aligned} & \frac{1}{2} \|\nabla y(t)\|_{L^2(\Omega)^d}^2 - \frac{1}{2} \|\nabla y_0\|_{L^2(\Omega)^d}^2 \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \|\nabla y_\varepsilon(t)\|_{L^2(\Omega)^d}^2 - \frac{1}{2} \|\nabla y_\varepsilon(0)\|_{L^2(\Omega)^d}^2 \\ &= \lim_{\varepsilon \rightarrow 0} \int_0^t (y'_\varepsilon, -\Delta y_\varepsilon)_{L^2(\Omega)} ds = \int_0^t (y', -\Delta y)_{L^2(\Omega)} ds. \end{aligned}$$

Now, using the fact the source term for damage in (41) is bounded independently of ζ and u_x , it follows from the lemma that

$$\frac{1}{2} \|\zeta_x(t)\|_H^2 + \frac{1}{2} \int_0^t \|\Delta \zeta(s)\|_H^2 ds \leq \frac{1}{2} \|\zeta_{0x}\|_H^2 + C(M).$$

This estimate, along with (41), shows that ζ' is bounded in \mathcal{H} . Thus, we obtain an estimate of the form,

$$\|\zeta'\|_{\mathcal{H}}^2 + \frac{1}{2} \|\zeta_x(t)\|_H^2 + \frac{1}{2} \int_0^t \|\Delta \zeta(s)\|_H^2 ds \leq \frac{1}{2} \|\nabla \zeta_0\|_H^2 + C(M).$$

Using now Theorem 3, it follows that the image $\Phi(\mathcal{H})$ belongs to a compact subset of $C([0, T]; U) \subseteq \mathcal{H}$, where $U \equiv H^\alpha(0, L)$, and $\alpha < 1$ is large enough so that the embedding of U into $C([0, L])$ is compact. We conclude by the Schauder fixed point theorem that there exists a fixed point of Φ in $C([0, T]; U)$. This proves the existence part of the following theorem, which is one of the the main results in this work.

Theorem 4. *There exists a unique solution (v, u, ζ) to problem (38)–(42) and it satisfies:*

$$v \in \mathcal{V}, \quad u \in C([0, T]; V), \quad v' \in \mathcal{V}',$$

$$\zeta' \in \mathcal{H}, \quad \zeta \in L^\infty(0, T; H^1(0, L)) \cap L^2(0, T; H^2(0, L)) \cap C([0, T]; U).$$

For each $t \in [0, T]$

$$\zeta(x, t) \in [\delta, 1] \quad \text{a.e. } x.$$

Proof. It only remains to verify the uniqueness of the solution. Suppose that (v_i, u_i, ζ_i) , for $i = 1, 2$, are two solutions. We find from (38), using simple manipulations involving the relation between u and v , that

$$\begin{aligned} & \frac{1}{2} \|v_1(t) - v_2(t)\|_H^2 + \frac{\delta}{2} \int_0^t \|v_1(s) - v_2(s)\|_V^2 ds \\ & \leq K_\delta \int_0^t \|\zeta_1(s) - \zeta_2(s)\|_{L^\infty(0,L)}^2 \left(\|v_1(s)\|_V^2 + 1 \right) ds. \end{aligned} \quad (46)$$

Now, using Lemma 2 again to the difference between the equations solved by ζ_i , we obtain

$$\begin{aligned} & \frac{1}{2} \|\zeta_{1x}(t) - \zeta_{2x}(t)\|_H^2 + \frac{1}{2} \int_0^t \|\Delta(\zeta_1 - \zeta_2)\|_H^2 ds \\ & \leq K(M) \int_0^t (\|\zeta_1 - \zeta_2\|_H^2 + \|u_{1x} - u_{2x}\|_H^2) ds. \end{aligned}$$

Therefore, there is a positive constant C , independent of the solutions, such that

$$\begin{aligned} & \|\zeta_{1x}(t) - \zeta_{2x}(t)\|_H^2 + \int_0^t \|\zeta_1 - \zeta_2\|_{H^2(0,L)}^2 ds \\ & \leq C \int_0^t \left(\|\zeta_1 - \zeta_2\|_H^2 + \int_0^s \|v_1 - v_2\|_V^2 dr \right) ds. \end{aligned}$$

Similar, but somewhat simpler, computations using (41) yield

$$\begin{aligned} & \frac{1}{2} \|\zeta_1(t) - \zeta_2(t)\|_H^2 + \frac{1}{2} \int_0^t \|\zeta_{1x} - \zeta_{2x}\|_H^2 ds \\ & \leq C \int_0^t \left(\|\zeta_1 - \zeta_2\|_H^2 + \int_0^s \|v_1 - v_2\|_V^2 dr \right) ds. \end{aligned}$$

Therefore,

$$\begin{aligned} & \|\zeta_1(t) - \zeta_2(t)\|_V^2 + \int_0^t \|\zeta_1 - \zeta_2\|_{H^2(0,L)}^2 ds \\ & \leq C \int_0^t \left(\|\zeta_1 - \zeta_2\|_H^2 + \int_0^s \|v_1 - v_2\|_V^2 dr \right) ds. \end{aligned}$$

We use (46) to substitute into this inequality and obtain

$$\begin{aligned} & \|\zeta_1(t) - \zeta_2(t)\|_V^2 + \int_0^t \|\zeta_1 - \zeta_2\|_{H^2(0,L)}^2 ds \\ & \leq C_\delta \int_0^t (\|\zeta_1(s) - \zeta_2(s)\|_H^2 \\ & + \int_0^s (\|\zeta_1(r) - \zeta_2(r)\|_{L^\infty(0,L)}^2 (\|v_1(r)\|_V^2 + 1)) dr) ds \end{aligned}$$

We let $r < 2$ be large enough so that H^r embeds continuously into L^∞ and by the compactness of the embedding of H^2 into H^r , if $\varepsilon > 0$ we find

$$\begin{aligned} & \|\zeta_1(t) - \zeta_2(t)\|_V^2 + \frac{1}{\varepsilon} \int_0^t \|\zeta_1 - \zeta_2\|_{H^r(0,L)}^2 ds \\ & \leq C_\varepsilon \int_0^t \|\zeta_1 - \zeta_2\|_H^2 ds + C_\delta \int_0^t (\|\zeta_1(s) - \zeta_2(s)\|_H^2 \\ & + \int_0^s (\|\zeta_1(r) - \zeta_2(r)\|_{H^r(0,L)}^2 (\|v_1(r)\|_V^2 + 1)) dr) ds. \end{aligned}$$

Now, choosing ε small enough,

$$\begin{aligned} & \|\zeta_1(t) - \zeta_2(t)\|_V^2 + \frac{1}{2\varepsilon} \int_0^t \|\zeta_1 - \zeta_2\|_{H^r(0,L)}^2 ds \\ & \leq C(\delta, \varepsilon) \int_0^t \int_0^s (\|\zeta_1(r) - \zeta_2(r)\|_{H^r(0,L)}^2 (\|v_1(r)\|_V^2 + 1)) dr ds, \end{aligned}$$

and by Gronwall's inequality $\zeta_1 = \zeta_2$, which implies by Lemma 1 that $v_1 = v_2$. This proves the theorem.

We note that the proof above implies the following corollary.

Corollary 1. *Consider problem (38) – (42), then there exists $T^* > 0$ such that for $t \in [0, T^*]$ the function ζ in the solution provided in Theorem 4 stays within the interval $(2\delta, 1]$ so that every occurrence of $\eta(\zeta)$ in (38 – 42) may be replaced with ζ .*

Proof. It follows from the fact that $\zeta \in C([0, T]; U)$, where U embeds continuously into $C([0, L])$, and ζ_0 .

5 Estimates on strain

In this section we remove the truncation Q_M . Since the problem is one-dimensional, it suffices to obtain an estimate for u in $L^\infty(0, T; H^2(0, L))$. We make additional assumptions on the problem data to obtain such an estimate, which involves pointwise bounds on u_x .

We assume the compatibility conditions on the initial data,

$$\begin{aligned} \Delta\zeta_0 - (\zeta_0 \mathcal{X}_{[l_1, l_2]} Q_M(u_{0x}) - \varepsilon_0)_+ &\in H^1(0, L), \\ (cu_{0x})_x &\in H. \end{aligned} \tag{47}$$

Let $\xi \equiv \zeta'$ and note that the time derivative of the source term in (41), $g(\zeta', v_x)$ is in \mathcal{H} . Therefore, there exists a unique solution to the problem

$$\begin{aligned} \xi' - \Delta\xi &= g(\zeta', v_x), \\ \xi(0) &= \Delta\zeta_0 - (\zeta_0 \mathcal{X}_{[l_1, l_2]} Q_M(u_{0x}) - \varepsilon_0)_+ \in H, \end{aligned}$$

which satisfies $\xi' \in \mathcal{H}$, $\xi \in L^2(0, T; H^2(0, L))$. Then using Lemma 2, again, we obtain, for a.a. t ,

$$\|\zeta'_x(t)\|_H^2 = \|\xi_x(t)\|_H^2 \leq C(\zeta_0, \Delta\zeta_0, u_{0x}). \tag{48}$$

Similarly, an easier estimate for $\|\xi(t)\|_H^2$ is

$$\xi = \zeta' \in L^\infty(0, T; H^1(0, L)). \tag{49}$$

Also, as above, we obtain an estimate on $\|\Delta\xi\|_{L^2(0, T; H^2(0, L))}$ which yields the pointwise estimate

$$\|\zeta\|_{L^\infty(0, T; H^2(0, L))} \leq C(\zeta_0, \Delta\zeta_0, u_{0x}),$$

which, in particular, implies that

$$\|\zeta_x\|_{L^\infty(0, T; L^\infty(0, L))} \leq C(\zeta_0, \Delta\zeta_0, u_{0x}), \tag{50}$$

since in one dimension $H^1(0, L)$ embeds continuously into $L^\infty(0, L)$. One would need to work much harder if the problem were in a higher dimension.

We define the following time dependent family of functionals on H , which are convex, proper, and lower semicontinuous,

$$\phi(t, u) \equiv \begin{cases} \frac{1}{2} \int_0^L c(x) \zeta(x, t) u_x^2(x) dx & \text{if } u \in V, \\ +\infty & \text{if } u \notin V \end{cases}$$

Here, $t \in [0, T]$, and $D(\phi(t, \cdot)) = V$ is independent of t because $\delta \leq \zeta \leq 1$. Also, for $u \in V$,

$$\begin{aligned} \|\phi(t, u) - \phi(s, u)\|_H &\leq \frac{1}{2} \int_0^L c(x) |\zeta(x, t) - \zeta(x, s)| u_x^2(x) dx \\ &\leq \frac{b}{2} \|\zeta(t) - \zeta(s)\|_{L^\infty(0, L)} \int_0^L u_x^2(x) dx \\ &\leq \frac{b}{2} \|\zeta(t) - \zeta(s)\|_{L^\infty(0, L)} \phi(r, u) \\ &\leq C \phi(r, u) \int_s^t \|\zeta'(\tau)\|_V d\tau \leq C \phi(r, u) |t - s|, \end{aligned} \quad (51)$$

where $r \in [0, T]$ is arbitrary and we used (49). Also, the subgradient of $\phi(t, \cdot)$ is given by $\partial\phi(t, \cdot) = -(c(\cdot)\zeta(\cdot, t)u_x)_x$, and its domain is

$$\{u \in V : (c(\cdot)\zeta(\cdot, t)u_x)_x \in H\}.$$

Now consider (38)–(40) in which ζ is the solution satisfying (49), thanks to the compatibility condition (47) made on ζ_0 . We have the following.

Lemma 3. *Assume that (47) holds and $v_0 \in V$. Then the solution to (38)–(40) satisfies $v' \in \mathcal{H}$ and $(\zeta v_x)_x \in \mathcal{H}$.*

Proof. Problem (38)–(40) is just an abstract form of the initial boundary value problem

$$v_t - (c\zeta v_x)_x - (c\zeta u_x)_x = f, \quad (52)$$

$$v(0, t) = v(L, t) = 0, \quad (53)$$

$$v(0) = v_0, \quad (54)$$

$$u(t) = u_0 + \int_0^t v(s) ds. \quad (55)$$

The partial differential equation is of the form

$$v_t - \zeta_x (cv_x) - \zeta (cv_x)_x - \zeta_x (cu_x) - \zeta (cu_x)_x = f,$$

and when $(\zeta cv_x)_x \in \mathcal{H}$, it follows from the regularity of ζ , established earlier, that $(cv_x)_x \in \mathcal{H}$. Let $W \equiv \{v \in V : (cv_x)_x \in H\}$ and

$$\mathcal{W} \equiv \{v \in \mathcal{V} : (cv_x)_x \in \mathcal{H}\}$$

with the norm $\|v\|_{\mathcal{W}} \equiv \|(cv_x)_x\|_{\mathcal{H}}$.

Let $v_1 \in \mathcal{W}$ and define $u_1(t) \equiv u_0 + \int_0^t v_1(s) ds$. Then, it follows from the main existence theorem in [14] that, given such ζ , there exists a unique solution v to the problem

$$\begin{aligned} v_t - (c\zeta v_x)_x - \zeta_x cu_{1x} - \zeta (cu_{1x})_x \\ = v_t - \zeta_x (cv_x) - \zeta (cv_x)_x - \zeta_x cu_{1x} - \zeta (cu_{1x})_x = f, \\ v(0) = v_0, \end{aligned} \quad (56)$$

which satisfies $v_t \in \mathcal{H}$ and $(\zeta cv_x)_x \in \mathcal{H}$. Denote this v by $\Phi(v_1)$. Then consider $v_1, v_2 \in \mathcal{W}$ with the corresponding u_1, u_2 . A similar argument as in Lemma 2 implies that we can multiply both sides of (56) by

$$-(c\Phi(v_1)_x)_x - (-(c\Phi(v_2)_x)_x)$$

and integrate by parts, eventually obtaining the estimate

$$\begin{aligned} \frac{1}{2} \|\sqrt{c}(\Phi(v_1)_x(t) - \Phi(v_2)_x(t))\|_H^2 + \frac{\delta}{2} \int_0^t \|(c\Phi(v_1)_x)_x - (c\Phi(v_2)_x)_x\|_H^2 ds \\ \leq C \int_0^t \|c\Phi(v_1)_x(s) - c\Phi(v_2)_x(s)\|_H^2 ds \\ + C \int_0^t \|cu_{1x}(s) - cu_{2x}(s)\|_H^2 + \|(cu_{1x})_x(s) - (cu_{2x})_x(s)\|_H^2 ds \end{aligned}$$

where here and below $C = C(\delta, \zeta_0, \Delta\zeta_0, u_{0x})$, and we used the fact $\zeta \geq \delta$ and the pointwise bound on ζ_x which follows from (50). After adjusting the constants, this simplifies to

$$\begin{aligned} \|\Phi(v_1)_x(t) - \Phi(v_2)_x(t)\|_H^2 + \int_0^t \|(c\Phi(v_1)_x)_x - (c\Phi(v_2)_x)_x\|_H^2 ds \\ \leq C \int_0^t \|c\Phi(v_1)_x(s) - c\Phi(v_2)_x(s)\|_H^2 ds \\ + C \int_0^t \int_0^s \|(cv_{1x})_x - (cv_{2x})_x\|_H^2 dr ds. \end{aligned}$$

Then,

$$\|\Phi(v_1)_x(t) - \Phi(v_2)_x(t)\|_H^2$$

$$\leq C \left[\int_0^t \|c\Phi(v_1)_x(s) - c\Phi(v_2)_x(s)\|_H^2 ds + \int_0^t \int_0^s \|v_1 - v_2\|_W^2 dr ds \right],$$

and by Gronwall's inequality and adjusting the constants, we obtain

$$\|\Phi(v_1)_x(t) - \Phi(v_2)_x(t)\|_H^2 \leq C \int_0^t \int_0^s \|v_1 - v_2\|_W^2 dr ds.$$

Now, integration over t yields

$$\int_0^t \|\Phi(v_1) - \Phi(v_2)\|_W^2 ds \leq C \int_0^t \int_0^s \|v_1 - v_2\|_W^2 dr ds,$$

where, as above $C = C(\delta, \zeta_0, \Delta\zeta_0, u_{0x})$.

This estimate shows that a high enough power of Φ is a contraction mapping on \mathcal{W} , so there exists a unique fixed point v for Φ . This v is then the unique solution to (52) - (55). However, by the uniqueness of the weak solution to (38)-(40), it follows that v is the solution to the weak abstract problem. Also, we note that the construction yields

$$cv_x \in L^\infty(0, T; H).$$

This proves the lemma.

Now, since $(c\zeta v_x)_x \in \mathcal{H}$, it follows that $c\zeta v_x \in L^2(0, T; H^1(0, L))$ and so $c\zeta v_x \in L^2(0, T; C([0, L]))$, therefore

$$v_x \in L^2(0, T; L^\infty(0, L)),$$

thus $u_x \in C([0, T]; L^\infty(0, L))$, hence,

$$u_x \in C([0, T]; L^\infty(0, L)).$$

This is the desired estimate on the strain which allows the elimination of the truncation function Q_M , proving the following local existence theorem.

Theorem 5. *Assume that the compatibility condition (47) holds, $u_{0x}(x) < M$ on $[0, L]$, where M is the truncation constant of Q_M (43), and $\zeta_0(x) \in (3\delta, 1]$. Then, there exists $T^* > 0$ such that, for $t \in [0, T^*)$, the unique solution (v, u, ζ) of (38) - (42) satisfies $\eta(\zeta(t)) = \zeta(t)$ and*

$$Q_M(u_x(t)) \mathcal{X}_{[l_1, l_2]}(x) = u_x^2(t) \mathcal{X}_{[l_1, l_2]}(x).$$

In addition, this solution has the following regularity,

$$c\zeta v_x \in L^2(0, T; H^1(0, L)), \quad v' \in \mathcal{H}, \\ \zeta \in C([0, T]; H^2(0, L)), \quad \zeta' \in L^2(0, T; H^2(0, L)).$$

6 Conclusions

Two models for the dynamic adhesive contact between two rods were presented. The first model assumes that the adhesive may be described as a rod made of an elastic-plastic material and then complete debonding occurs when the stress reaches the plasticity yield limit. In the second model the adhesive is also assumed to be a rod and the degradation of the adhesive is described by the introduction of material damage. Failure occurs when the material is completely damaged, or the damage reaches a critical floor value.

The analysis of the first model is routine. The second model was shown, in Section 4, to have a unique local (in time) weak solution. The proof was based on truncation of the strain energy and the damage function in the equation of motion. These allowed the use of standard tools to establish the existence of a weak global solution. Then, it was shown in Section 5 that with the appropriate initial conditions the weak solution is sufficiently regular so that the constraints (the truncations) are inactive on a time interval $[0, T^*)$, which means that the solution of the truncated problem is also the solution to the original problem.

Two quasistatic versions of the problem with material damage, with displacement or traction boundary condition at $x = 0$, were investigated in Section 3. The fact that the problems are one-dimensional allowed us to obtain a new condition for the damage source function, leading to the same and unusual parabolic nonlinear and nonlocal problem for the damage ζ , $P_{quas-\zeta}$ or $P_{\zeta q}$. The analysis of this problem will be done elsewhere.

In the limit when the thickness of the adhesive rod tends to zero a new adhesion source function was obtained, see the right-hand side of (26), which is unusual in that it contains ζ^{-1} which makes it non-Lipschitz, and different from the source functions used in [1, 2, 10, 20, 21, 22]. The problem was analyzed in Section 4.

Some future work, related issues, and unresolved questions follow. First, it may be of considerable interest to verify the model by comparing its predictions with experimental results. In this manner the model parameters may be estimated and then it may be used to predict the evolution of real systems. Because of the relative simplicity of the problem, it may be used as a bench-mark in applications, too.

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