ISSN 2066-6594

Ann. Acad. Rom. Sci.
Ser. Math. Appl.
Vol. 14, No. 1-2/2022

ANISOTROPIC NONLINEAR ELLIPTIC SYSTEMS WITH VARIABLE EXPONENTS, DEGENERATE COERCIVITY AND $L^{q(\cdot)}$ DATA*

Naceri Mokhtar†

Abstract

The aim of this paper is to study the existence and maximal regularity for distributional solutions of degenerate anisotropic nonlinear elliptic systems with variable exponents where the right-hand side f is in $L^{q(\cdot)}$, $q(\cdot) : \overline{\Omega} \to (1, +\infty)$. The functional setting involves anisotropic Sobolev spaces with variable exponents as well as weak Lebesgue (Marcinkiewicz) spaces with variable exponents.

MSC: 35J60, 35J67, 35J70.

keywords: Degenerate system, elliptic, anisotropic, nonlinear, variable exponents, distributional solution

1 Introduction

Let Ω be a bounded open set in $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$, and let's consider the anisotropic nonlinear elliptic system

$$\begin{aligned} -\sum_{i=1}^{N} D_i\big(a_i(x,u)\sigma_i(x,D_iu)\big) + g(x,u) = f, &\quad \text{in } \Omega, \\ u = 0, &\quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

*Accepted for publication on February 6-th, 2022

†nasrimokhtar@gmail.com Address: ENS of Laghouat, BP 4033 Station post avenue of Martyrs, Laghouat, Algeria. And Laboratory EDPNL-HM, ENS-Kouba, Algiers, Algeria.

Our aim is to prove the existence and maximal regularity at least one distributional solution $u = (u_1, \dots, u_m)^\top$ $(m \geq 1)$ to (1), where the right-hand side f is in the anisotropic Lebesgue space $L^{q(x)}(\Omega; \mathbb{R}^m)$, $q(\cdot) : \overline{\Omega} \to (1, +\infty)$, $a_i : \Omega \times \mathbb{R}^m \to \mathbb{R}$, $i = 1, \dots, N$ are Carathéodory functions such that for a.e. $x \in \Omega$, for every $t \in \mathbb{R}^m$, we have

$$\frac{\alpha}{(1+|t|)^{\gamma(x)}} \leq a_i(x,t) \leq \beta, \quad i = 1, \dots, N, \tag{2}$$

where α, β are strictly positive real numbers and $\gamma \in C(\overline{\Omega})$, $\gamma(x) \geq 0$ for all $x \in \overline{\Omega}$, and the vector fields $\sigma_i : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$, $i = 1, \dots, N$, are Carathéodory functions and satisfying, a.e. $x \in \Omega$ and $\forall \xi, \xi' \in \mathbb{R}^m$ $(\xi \neq \xi')$, $a \in \mathbb{R}^m$, the following :

$$\sigma_i(x,\xi) \cdot \xi \geq c_1 |\xi|^{p_i(x)} - c_2, \tag{3}$$

$$|\sigma_i(x,\xi)| \leq c_3 \left(\sum_{j=1}^{N} |\xi|^{p_j(x)} + |h| \right)^{1-\frac{1}{p_i(x)}}, \quad h \in L^1(\Omega) \tag{4}$$

$$\left(\sigma_i(x,\xi) - \sigma_i(x,\xi')\right) \cdot \left(\xi - \xi'\right) \geq \Theta_i\left(x, \xi, \xi'\right), \tag{5}$$

$$\sigma_i(x,\xi) \cdot \left[\left(I - a \otimes a\right) \xi\right] \geq 0, \quad i = 1, \dots, N, \ |a| \leq 1. \tag{6}$$

Where

$$\Theta_i\left(x, \xi, \xi'\right) = \begin{cases} c_4 |\xi - \xi'|^{p_i(x)}, & \text{if } p_i(x) \geq 2 \\ c_5 \frac{|\xi - \xi'|^2}{(|\xi| + |\xi'|)^{2 - p_i(x)}}, & \text{if } 1 < p_i(x) < 2 \end{cases}$$

where c_l, $l = 1, \dots, 5$ are positive constants, $(I - a \otimes a)$ is the rank $m-1$ orthogonal projector onto the space orthogonal to the unit vector $a \in \mathbb{R}^m$, where $\otimes : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{M}^{m \times m}$ defined by

$$(a_1, \dots, a_m) \otimes (b_1, \dots, b_m) = \begin{pmatrix} a_1 b_1 & a_2 b_1 & \dots & a_m b_1 \\ a_1 b_2 & a_2 b_2 & \dots & a_m b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1 b_m & a_2 b_m & \dots & a_m b_m \end{pmatrix}.$$

Remark 1. *If $\sigma_{j,i}$, $j = 1, \dots, m$, denotes the components of the vector σ_i, then the angle condition can be stated more explicitly as*

$$\sum_{j,l=1}^{m} \sigma_{j,i}(x,\xi) \xi_l (\delta_{j,l} - a_j a_l) \geq 0.$$

Clearly, condition (6) *is void in the scalar case.*

The variable exponents $p_i : \overline{\Omega} \to (1, \infty)$ $i = 1, \dots, N$ are continuous functions. We assume that the nonlinearity $g : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is a Caratéodory function and satisfies for a.e. $x \in \Omega$ the following conditions:

$$g(x,r) \cdot (r - r') \geq 0, \quad \forall r, r' \in \mathbb{R}^m, \ |r| \leq |r'|, \tag{7}$$

$$\sup_{|r| \leq \tau} |g(x,r)| \in L^1(\Omega; \mathbb{R}^m) \quad \forall r \in \mathbb{R}^m \text{ and } \forall \tau \in \mathbb{R}, \tag{8}$$

$$g(x,r) \cdot r \geq |r|^{q(x)s(x)+1}, \quad \forall r \in \mathbb{R}^m, \tag{9}$$

$$g(x,r) \cdot r \geq |r|^{s(x)+1}, \quad \forall r \in \mathbb{R}^m, \text{if } f \in \left(\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega; \mathbb{R}^m)\right)^*, \tag{10}$$

where $s(\cdot) > 0$ is continuous function on $\overline{\Omega}$.
As prototype example, we consider the following model:

$$\begin{aligned} -\sum_{i=1}^{N} D_i \left(\frac{|D_i u|^{p_i(x)-2} D_i u}{(1+|u|)^{\gamma(x)}} \right) + |u|^{s(x)-1} u = f, \quad &\text{in } \Omega, \\ u = 0, \quad &\text{on } \partial\Omega, \end{aligned} \tag{11}$$

where $f \in L^{q(\cdot)}(\Omega; \mathbb{R}^m)$, $s(\cdot)$ and $p_i(\cdot)$ are restricted as in Theorem 1 or Theorem 2.

In this paper we study the existence of solutions in the sense of distributions for the anisotropic nonlinear elliptic system with variable exponents and degenerate coercivity where the right-hand side f of the system is in $L^{q(\cdot)}(\Omega; \mathbb{R}^m)$, $q(\cdot) : \overline{\Omega} \to (1, +\infty)$, with p_i is assumed to be merely a continuous function, and we treat the regularity of distributional solution u depending simultaneously on $s(\cdot)$ and $q(\cdot)$.
We note that, the existence and regularity results for weak solutions in the framework of anisotropic Sobolev spaces for a nonlinear anisotropic elliptic equations and systems with variable exponents are proved in [1, 2, 3, 6, 7, 8], the scalar case ($p_i(\cdot) = p_i$ constant) processed in [4], and the isotropic case ($p_i(\cdot) = p$ constant) studie in [17].
We mention also that the existence results for distributional solutions of nonlinear elliptic systems with variable exponents and measure data have been obtained in [5], and anisotropic scalar case ($p_i(\cdot) = p_i$ constant) has been studied in [4, 13].
Here we will prove the existence of a solution to (1), under the conditions $1 < q(x) < \frac{N\overline{p}(x)}{1+(N+1)(\overline{p}(x)-1)}$, $\overline{p}(x) < N$ in $\overline{\Omega}$.
So, in the case when $s(x) \geq \frac{1+\gamma(x)}{q(x)-1}$ (see Theorem 1), then the case when $\frac{1+\gamma(x)}{q(x)-1} > s(x) > \max\left(\frac{1+\gamma(x)}{p_i(x)q(x)-1}; (1+\gamma(x))(p_i(x)-1)\right)$ (see Theorem 2).

The proof requires a priori estimates for a sequence of suitable approximate solutions (u_n), which in turn is proving its existence, and then to pass to the limit .

We prove the a.e. convergence of the partial derivatives $D_i u_n$, which can be turned into strong L^1 convergence. Equipped with this convergence we pass to the limit in the strong L^1 sense in the nonlinear vector fields $\sigma_i(x, D_i u_n)$ then $a_i(x, T_n(u_n))\sigma_i(x, D_i u_n)$, and the nonlinearity function $g(x, u_n)$, and finally conclude that the approximate solutions u_n converge to a solution of (1).

2 Mathematical preliminaries

In this section we first recall some facts on variable exponent spaces $L^{p(\cdot)}(\Omega)$. We refer to [11, 12, 14] and references therein for further properties of variable exponent Lebesgue-Sobolev spaces.

Let $p : \overline{\Omega} \to [1, \infty)$ be a continuous function. We denote by $L^{p(\cdot)}(\Omega)$ the space of measurable function $f(x)$ on Ω such that

$$\rho_{p(\cdot)}(f) := \int_\Omega |f(x)|^{p(x)} dx < +\infty.$$

The space $L^{p(\cdot)}(\Omega)$ equipped with the norm

$$\|f\|_{p(\cdot)} := \|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 \mid \rho_{p(\cdot)}(f/\lambda) \le 1\right\}$$

becomes a Banach space. Moreover, if $p^- = \min_{x\in\overline{\Omega}} p(x) > 1$, then $L^{p(\cdot)}(\Omega)$ is reflexive and the dual of $L^{p(\cdot)}(\Omega)$ can be identified with $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the Hölder type inequality:

$$\left|\int_\Omega uv\, dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)} \le 2\|u\|_{p(\cdot)} \|v\|_{p'(\cdot)},$$

holds true.

We define also the Banach space $W_0^{1,p(\cdot)}(\Omega)$ by

$$W_0^{1,p(\cdot)}(\Omega) := \left\{f \in L^{p(\cdot)}(\Omega) : |Df| \in L^{p(\cdot)}(\Omega) \text{ and } f = 0 \text{ on } \partial\Omega\right\}$$

endowed with the norm $\|f\|_{W_0^{1,p(\cdot)}(\Omega)} = \|Df\|_{p(\cdot)}$. The space $W_0^{1,p(\cdot)}(\Omega)$ is separable and reflexive provided that with $1 < p^- \le p^+ = \max_{x\in\overline{\Omega}} p(x) < \infty$.

The smooth functions are in general not dense in $W_0^{1,p(\cdot)}(\Omega)$, but if the exponent variable $p(x) > 1$ is logarithmic Hölder continuous (12), that is

$$\exists M > 0 : |p(x) - p(y)| \leq -\frac{M}{ln(|x-y|)} \quad \forall x \neq y \text{ such that } |x-y| \leq \frac{1}{2}, \tag{12}$$

then the smooth functions are dense in $W_0^{1,p(\cdot)}(\Omega)$.
For $u \in W_0^{1,p(\cdot)}(\Omega)$ with $p \in C(\overline{\Omega}, [1, +\infty))$, the Poincaré inequality holds (see [12])

$$\|u\|_{p(\cdot)} \leq C\|Du\|_{p(\cdot)}, \tag{13}$$

for some constant C which depends on Ω and the function $p(x)$.
The following Lemma will be used later.

Lemma 1 ([11, 12]). *If* (u_n), $u \in L^{p(\cdot)}(\Omega)$, *then the following relations hold*

(i) $\|u\|_{p(\cdot)} < 1$ (*respectively* $= 1, > 1$) $\Longleftrightarrow \rho_{p(\cdot)}(u) < 1$ (*respectively* $= 1, > 1$),

(ii) $\min\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right) \leq \|u\|_{p(\cdot)} \leq \max\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right)$,

(iii) $\min\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right) \leq \rho_{p(\cdot)}(u) \leq \max\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right)$,

(iv) $\|u\|_{p(\cdot)} \leq \rho_{p(\cdot)}(u) + 1$,

(v) $\|u_n - u\|_{p(\cdot)} \to 0 \Longleftrightarrow \rho_{p(\cdot)}(u_n - u) \to 0$.

Remark 2. *Note that the following inequality*

$$\int_\Omega |f|^{p(x)}\, dx \leq C \int_\Omega |Df|^{p(x)}\, dx,$$

in general does not hold (see [14]). But by Lemma 1 and (13) *we have*

$$\int_\Omega |f|^{p(x)} dx \leq C \max\{\|Df\|_{L^{p(\cdot)}(\Omega)}^{p^+}; \|Df\|_{L^{p(\cdot)}(\Omega)}^{p^-}\}. \tag{14}$$

Let $p_i : \overline{\Omega} \to [1, \infty)$ be a continuous functions.
We introduce the anisotropic variable exponent Sobolev space

$$\begin{aligned} W^{1,p_i(\cdot)}(\Omega) &:= \left\{u \in L^{p_i(\cdot)}(\Omega) : D_i u \in L^{p_i(\cdot)}(\Omega)\right\}, \\ W_0^{1,p_i(\cdot)}(\Omega) &:= \left\{u \in W_0^{1,1}(\Omega) : D_i u \in L^{p_i(\cdot)}(\Omega)\right\}, \end{aligned}$$

which are Banach spaces under the norm

$$\|u\|_i := \|u\|_{L^{p_i(\cdot)}(\Omega)} + \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}, \quad i = 1, \dots, N.$$

we present the anisotropic Sobolev space with variable exponent which is used for the study of problems (1). First of all, let $p_i(\cdot) : \overline{\Omega} \to [1, +\infty)$, $i = 1, \dots, N$ be a continuous functions, we set $\overrightarrow{p}(\cdot) = (p_1(\cdot), \dots, p_N(\cdot))$ and $p_+(x) = \max_{1 \le i \le N} p_i(x)$, $p_-(x) = \min_{1 \le i \le N} p_i(x)$, $\forall x \in \overline{\Omega}$.

Definition 1. *The anisotropic variable exponent Sobolev space $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is defined as follow*

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) = \left\{ u \in L^{p_+(\cdot)}(\Omega), D_i u \in L^{p_i(\cdot)}(\Omega),\ i = 1, \dots, N \right\},$$

which is Banach space with respect to the norm

$$\|u\|_{W^{1,\overrightarrow{p}(\cdot)}(\Omega)} = \|u\|_{p_+(\cdot)} + \sum_{i=1}^{N} \|D_i u\|_{p_i(\cdot)} .$$

Definition 2. *We define the spaces $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$, and $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ as the intersection of $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $W_0^{1,1}(\Omega)$, and we write*

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,\overrightarrow{p}(\cdot)}(\Omega)}, \ \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) = W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega).$$

Remark 3 ([10])**.** *If Ω is a bounded open set with Lipschitz boundary $\partial\Omega$, then*

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) = \left\{ u \in W^{1,\overrightarrow{p}(\cdot)}(\Omega),\ u_{|\partial\Omega} = 0 \right\},$$

where, $u_{|\partial\Omega}$ denotes the trace on $\partial\Omega$ of u in $W^{1,1}(\Omega)$.

Remark 4 ([10])**.** *It is well-known that in the constant exponent case, that is, when $\overrightarrow{p}(\cdot) = \overrightarrow{p} \in [1, +\infty)^N$, $W_0^{1,\overrightarrow{p}}(\Omega) = \mathring{W}^{1,\overrightarrow{p}}(\Omega)$. However in the variable exponent case, in general $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \subset \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and the smooth functions are in general not dense in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, but if for each $i = 1, \dots, N$,*
p_i is log-Hölder continuous, then $C_0^\infty(\Omega)$ is dense in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, thus $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Remark 5. *We use standard notation for the vector and matrix-valued versions of the space/ norm introduced above. For example, the $\mathbb{R}^m$-valued version of $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is denoted by $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega; \mathbb{R}^m)$.*

We set

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}),\ p(x) > 1,\ \text{for any } x \text{ in } \overline{\Omega}\}. \tag{15}$$

For any $p \in C_+(\overline{\Omega})$, we denote

$$p^+ = \max_{x\in\overline{\Omega}} p(x), \text{ and } \quad p^- = \min_{x\in\overline{\Omega}} p(x).$$

And we set for all $x \in \overline{\Omega}$

$$\overline{p}(x) = \frac{N}{\sum\limits_{i=1}^{N} \frac{1}{p_i(x)}}, \quad p_+(x) = \max_{1\le i\le N} p_i(x), \quad p_+^+ = \max_{x\in\overline{\Omega}} p_+(x),$$
$$p_-(x) = \min_{1\le i\le N} p_i(x), \quad p_-^- = \min_{x\in\overline{\Omega}} p_-(x),$$

and we define

$$\overline{p}^\star(x) = \begin{cases} \frac{N\overline{p}(x)}{N-\overline{p}(x)}, & \text{for } \overline{p}(x) < N, \\ +\infty, & \text{for } \overline{p}(x) \ge N. \end{cases}$$

We have the following embedding results.

Lemma 2 ([14])**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overrightarrow{p}(\cdot) \in (C_+(\overline{\Omega}))^N$. If $\varrho \in C_+(\overline{\Omega})$ and $\forall x \in \overline{\Omega}$, $\varrho(x) < \max(p_+(x),\ \overline{p}^\star(x))$. Then the embedding*

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{\varrho(\cdot)}(\Omega),$$

is compact.

Lemma 3 ([14])**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overrightarrow{p}(\cdot) \in (C_+(\overline{\Omega}))^N$. Suppose that*

$$\forall x \in \overline{\Omega},\ p_+(x) < \overline{p}^\star(x). \tag{16}$$

Then the following Poincaré-type inequality holds

$$\|u\|_{L^{p_+(\cdot)}(\Omega)} \le C \sum_{i=1}^{N} \|D_i u\|_{L^{p_i(\cdot)}(\Omega)},\ \forall u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega), \tag{17}$$

where C is a positive constant independent of u. Thus $\sum_{i=1}^N \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}$ is an equivalent norm on $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Proposition 1. *If $u : \Omega \to \mathbb{R}$ is a measurable function such that $T_t(u) \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ for all $t > 0$, then there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that*

$$\nabla T_t(u) = v 1_{\{|u|\leq t\}} \text{ a.e. in } \Omega, \quad T_t(r) = \max\{-t, \min\{t, r\}\}. \tag{18}$$

Moreover, if $u \in W_0^{1,1}(\Omega)$ then v coincides with the standard distributional gradient of u.

A function u such that $T_t(u) \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ for any $t > 0$, does not necessarily belong to $W_0^{1,1}(\Omega)$. However, according to the above proposition, it is possible to define its weak gradient, still denoted by ∇u, as the unique function v which satisfies (18).

The following embedding results for the anisotropic constant exponent Sobolev space are well-known [15].

Lemma 4. *Let $\alpha_i \geq 1,\ i = 1, \ldots, N$, we pose $\overrightarrow{\alpha} = (\alpha_1, \ldots, \alpha_N)$. Suppose $u \in W_0^{1,\overrightarrow{\alpha}}(\Omega)$, and set*

$$\frac{1}{\overline{\alpha}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\alpha_i}, \qquad r = \begin{cases} \overline{\alpha}^\star = \frac{N\overline{\alpha}}{N-\overline{\alpha}} & \text{if } \overline{\alpha} < N, \\ \text{any number from } [1, +\infty) & \text{if } \overline{\alpha} \geq N. \end{cases}$$

Then there exists a constant C depending on $N, p_1, \ldots, p_N$ if $\overline{\alpha} < N$ and also on r and $|\Omega|$ if $\overline{\alpha} \geq N$, such that

$$\|u\|_{L^r(\Omega)} \leq C \prod_{i=1}^{N} \|D_i u\|_{L^{\alpha_i}(\Omega)}^{\frac{1}{N}}. \tag{19}$$

In this paper we will use the weak Lebesgue (Marcinkiewicz) spaces with variable exponents $\mathcal{M}^{h(\cdot)}(\Omega)$ where $h(\cdot)$ a measurable function such that $h^- > 0$ (see [9]). They contain the measurable functions $u : \Omega \to \mathbb{R}$ for which there exists a positive constant M such that

$$\int_{\{|u|>t\}} t^{h(x)}\, dx \leq M, \quad \text{for all } t > 0.$$

Moreover, it is clear that $u \in \mathcal{M}^{h(\cdot)}(\Omega)$ if $|u|^{h(\cdot)} \in L^1(\Omega)$. Indeed,

$$\int_{\{|u|>t\}} t^{h(x)}\, dx \leq \int_\Omega |u|^{h(x)}\, dx$$

In particular, $L^{h(\cdot)}(\Omega) \subset \mathcal{M}^{h(\cdot)}$, for all $h(\cdot) \geq 1$.
Similarly to the anisotropic Sobolev spaces with variable exponents, we use standard notation for the vector/matrix-valued versions of the weak Lebesgue spaces with variable exponents.

Remark 6. *We remark that for $h(\cdot) = h$ constant this definition coincides with the classical definition of the Marcinkiewicz space $\mathcal{M}^h(\Omega)$, they contain the measurable functions $f : \Omega \to \mathbb{R}$ for which the distribution function*

$$\lambda_f(t) := \big| \{x \in \Omega \, : \, |f(x)| > t\} \big|, \quad t \geq 0,$$

satisfies an estimate of the form

$$\lambda_f(t) \leq C t^{-h}, \quad \textit{for some finite constant } C.$$

For any $t > 0$, the standard scalar truncation function T_t on $[0, \infty)$ (at height t) is defined as

$$T_t(r) := \begin{cases} r, & \text{if } r \leq t, \\ t, & \text{if } r > t. \end{cases}$$

Now we need the following Lemma, thanks to Proposition 2.5 in [9], we have

Lemma 5. *Let $\rho(\cdot), r(\cdot)$ in $C(\overline{\Omega})$ such that $r^- > 0, (\rho - r)^- > 0$.
If $u \in \mathcal{M}^{\rho(\cdot)}(\Omega)$, then $|u|^{r(\cdot)} \in L^1(\Omega)$. In particular, $\mathcal{M}^{\rho(\cdot)}(\Omega) \subset L^{r(\cdot)}(\Omega)$ for all $\rho(\cdot), r(\cdot) \geq 1$ such that $(\rho - r)^- > 0$.*

For any $t > 0$, define the spherial (radially symmetric) truncation function $T_t : \mathbb{R}^m \to \mathbb{R}^m$ by

$$T_t(r) := \begin{cases} r, & \text{if } |r| \leq t, \\ \frac{r}{|r|} t, & \text{if } |r| > t. \end{cases} \tag{20}$$

This function will be used repeatedly to derive a priori estimates for our approximate solutions. We also need its derivative (see [1])

$$DT_t(r) = \begin{cases} I, & \text{if } |r| < t, \\ \frac{t}{|r|}\big(I - \frac{r \otimes r}{|r|^2}\big), & \text{if } |r| > t. \end{cases} \tag{21}$$

And if $m = 1$

$$DT_t(r) = \begin{cases} 1, & |r| < t, \\ 0, & |r| > t. \end{cases} \tag{22}$$

In particular, (6) implies for all $\xi, r \in \mathbb{R}^m$ the crucial property

$$\sigma_l(x, \xi) \cdot DT_t(r)\xi \geq \sigma_l(x, \xi) \cdot \xi \, \chi_{|r| < t}, \quad l = 1, \ldots, N. \tag{23}$$

We refer to [16] for a discussion of T_t and other test functions for elliptic systems, which indeed is a delicate issue.

Definition 3. *A distributional solution of* (1) *is a vector-valued function* $u : \Omega \to \mathbb{R}^m$ *satisfying* $u \in W_0^{1,1}(\Omega; \mathbb{R}^m)$, $g(., u)$ *and* $\sigma_i(., D_i u) \in L^1(\Omega; \mathbb{R}^m)$, $i = 1, \ldots, N$, *and for all* $\varphi \in C_c^\infty(\Omega; \mathbb{R}^m)$,

$$\int_\Omega \sum_{i=1}^N a_i(x, u)\sigma_i\big(x, D_i u\big) \cdot D_i\varphi \, dx + \int_\Omega g(x, u) \cdot \varphi \, dx = \int_\Omega f(x).\varphi \, dx.$$

3 Main results

Our main results are the following.

Theorem 1. *Let* $q(\cdot) : \overline{\Omega} \to (1, +\infty)$, $p_i : \overline{\Omega} \to (1, +\infty)$, *and* $s : \overline{\Omega} \to (0, +\infty)$ *be continuous functions such that* (16) *holds and for all* $x \in \overline{\Omega}$

$$1 < q(x) < \frac{N\overline{p}(x)}{1 + (N+1)(\overline{p}(x) - 1)}, \ \overline{p}(x) < N, \quad \nabla q \in L^\infty(\Omega), \tag{24}$$

and

$$s(x) \geq \frac{1 + \gamma(x)}{q(x) - 1}, \quad \nabla s \in L^\infty(\Omega), \nabla\gamma \in L^\infty. \tag{25}$$

Let $f \in L^{q(\cdot)}(\Omega; \mathbb{R}^m)$ *and let* a_i *are satisfying* (2), σ *satisfying* (3)-(6) *and* g *satisfy* (7)- (9). *Then, the problem* (1) *has at least one distributional solution* $u \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega; \mathbb{R}^m) \cap L^{s(\cdot)q(\cdot)}(\Omega; \mathbb{R}^m)$.

Theorem 2. *Let* $f \in L^{q(\cdot)}(\Omega)$ *with* q *as in* (24), $p_i : \overline{\Omega} \to (1, +\infty)$, *and* $s : \overline{\Omega} \to (0, +\infty)$ *be continuous functions. Assume* (16), *and for all* $x \in \overline{\Omega}$,

$$\frac{1 + \gamma(x)}{q(x) - 1} > s(x) > \max\Big(\frac{1 + \gamma(x)}{p_i(x)q(x) - 1}; (1 + \gamma(x))(p_i(x) - 1)\Big), \tag{26}$$

and $\nabla s \in L^\infty(\Omega)$, $\overline{p}(x) < N$.
Let a_i *are satisfying* (2), σ *satisfying* (3)-(6) *and* g *satisfy* (7)- (9). *Then, the problem* (1) *has at least one distributional solution* u *such that* $|u|^{q(x)s(x)} \in L^1(\Omega)$ *and* $u \in \mathring{W}^{1,\vec{r}(\cdot)}(\Omega; \mathbb{R}^m)$ *where* $r_i(\cdot)$ *are continuous functions on* $\overline{\Omega}$ *satisfying*

$$1 < r_i(x) < \frac{p_i(x)q(x)s(x)}{s(x) + 1 + \gamma(x)}, \quad \forall x \in \overline{\Omega}, \ i = 1, \ldots, N. \tag{27}$$

Remark 7. *Observe that the conditions* (24), (25), *and* (16) *guarantee that*

$$s(x) > (1 + \gamma(x))(p_i(x) - 1), \quad \forall x \in \overline{\Omega}, \ i = 1, \ldots, N.$$

Remark 8. *In Theorem 2, the conditions* (24) *and* (16) *imply that the assumption* (26) *is not empty since we have*

$$\frac{1}{q(x)-1} > p_i(x) - 1, \quad \forall x \in \overline{\Omega},\ i = 1, \dots, N. \tag{28}$$

3.1 Approximate solutions

• We must first prove the following Lemma

Lemma 6. *Let f in $\big(\mathring{W}^{1,\vec{p}(\cdot)}(\Omega;\mathbb{R}^m)\big)^*$. Assume that $p_i(\cdot), i = 1, \dots, N$, $s(\cdot) \geq p_i(\cdot)$ are continuous functions on Ω such that* (2), (3)-(6), (7),(8), (10) *and* (16) *holds. Then the system*

$$\begin{gathered} -\sum_{i=1}^{N} D_i\Big(a_i(x, T_n(u_n))\sigma_i(x, D_i u_n)\Big) + g(x, u_n) = f \quad \text{in } \Omega, \\ u_n = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{29}$$

has at least one solution in the sense that

$$\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n))\sigma_i(x, D_i u_n).D_i\varphi\, dx + \int_\Omega g(x, u_n).\varphi\, dx = \langle f, \varphi\rangle, \tag{30}$$

for every $\varphi \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega;\mathbb{R}^m) \cap L^{s(\cdot)}(\Omega;\mathbb{R}^m)$.

Proof. Let f in $\big(\mathring{W}^{1,\vec{p}(\cdot)}(\Omega;\mathbb{R}^m)\big)^*$, and consider the approximate system :

$$\begin{gathered} -\sum_{i=1}^{N} D_i\big(a_i(x, T_n(u_{n_k}))\sigma_i\big(x, D_i u_{n_k}\big)\big) + g_k(x, u_{n_k}) = f, \quad \text{in } \Omega, \\ u_{n_k} = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{31}$$

where, $g_k(x,\xi) = \frac{g(x,\xi)}{1+|g(x,\xi)|/k}, \quad \forall k \in \mathbb{N}^*$.
Note that $\big|g_k(x,\xi)\big| \leq \mid g(x,\xi) \mid$, and $\big|g_k(x,\xi)\big| \leq k$.
Let's prove that the system (31) has at least one solution u_{n_k} in $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega;\mathbb{R}^m)$ in the following sense, $\forall \varphi \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega;\mathbb{R}^m)$

$$\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_{n_k}))\sigma_i(x, D_i u_{n_k}).D_i\varphi\, dx + \int_\Omega g_k(x, u_{n_k}).\varphi\, dx = \langle f, \varphi\rangle \tag{32}$$

For $u_{n_k}, v \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega;\mathbb{R}^m)$, we denote by $\mathbf{A_k}$ the operator

$$\mathbf{A_k} : u_{n_k} \mapsto \Big(v \mapsto \int_\Omega \sum_{i=1}^{N} a_i(x, T_n(u_{n_k}))\sigma_i\big(x, D_i u_{n_k}\big) \cdot D_i v + \int_\Omega g_k(x, u_{n_k}) \cdot v\Big).$$

We consider

$$b_k(u_{n_k},v)=\int_\Omega\sum_{i=1}^N a_i(x,T_n(u_{n_k}))\sigma_i\big(x,D_iu_{n_k}\big)\cdot D_iv,$$

$$c_k(u_{n_k},v)=\int_\Omega g_k(x,u_{n_k})\cdot v\,dx,$$

and we seek $u_{n_k}\in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^m)$ such that

$$b_k(u_{n_k},v)+c_k(u_{n_k},v)=\langle f,v\rangle,\ \forall v\in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^m). \tag{33}$$

The generalized problem (33) corresponding to (31) is equivalent to

$$\mathbf{A_k}(u_{n_k})(v)=\langle f,v\rangle,\ \forall v\in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^m),$$

where, $\mathbf{A_k}:=\mathbf{B_k}+\mathbf{C_k}$, with
$\mathbf{B_k},\mathbf{C_k}:\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^m)\longrightarrow\big(\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^m)\big)^*$ characterized by
$\langle\mathbf{B_k}(u_{n_k}),v\rangle=b_k(u_{n_k},v),\quad\langle\mathbf{C_k}(u_{n_k}),v\rangle=c_k(u_{n_k},v)$.
Here $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^m)$ and $(\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^m))^*$. We put in the following
$\|.\|_{\overrightarrow{p}(\cdot)}=\|.\|_{\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^m)},\quad\|.\|_{p(\cdot)}=\|.\|_{L^{p(\cdot)}(\Omega)}$
$*$ From (2), (4), and Hölder inequality we have

$$\begin{aligned}
\big|\langle\mathbf{B_k}(u_{n_k}),v\rangle\big|&\le\sum_{i=1}^N\int_\Omega|a_i(x,T_n(u_{n_k}))||\sigma_i\big(x,D_iu_{n_k}\big)|\big|D_iv\big|\,dx\\
&\le c\sum_{i=1}^N\int_\Omega\left(\sum_{j=1}^N|D_iu_{n_k}|^{p_j(x)}+|h(x)|\right)^{1-\frac{1}{p_i(x)}}\big|D_iv\big|\,dx\\
&\le 2c\sum_{i=1}^N\left\|\left(\sum_{j=1}^N|D_iu_{n_k}|^{p_j(x)}+|h(x)|\right)^{1-\frac{1}{p_i(x)}}\right\|_{p_i'(\cdot)}\big\|\big|D_iv\big|\big\|_{p_i(\cdot)}\\
&\le 2c\sum_{i=1}^N\left(1+\int_\Omega\left(\sum_{j=1}^N|D_iu_{n_k}|^{p_j(x)}+|h(x)|\right)dx\right)^{1-\frac{1}{p_-^-}}\sum_{i=1}^N\big\|\big|D_iv\big|\big\|_{p_i(\cdot)}\\
&\le 2c\sum_{i=1}^N\left(1+\int_\Omega\left(N\sum_{j=1}^N|D_ju_{n_k}|^{p_j(x)}+|h(x)|\right)dx\right)^{1-\frac{1}{p_-^-}}\big\|v\big\|_{\overrightarrow{p}(\cdot)}\\
&\le 2cN\left(N\big\|u_{n_k}\big\|_{\overrightarrow{p}(\cdot)}^{p_+^+}+C\right)^{1-\frac{1}{p_-^-}}\big\|v\big\|_{\overrightarrow{p}(\cdot)}.
\end{aligned}$$

Which implies the boundedness of $\mathbf{B_k}$.
On the other hand, thanks to the Hölder inequality, we have for all $u_{n_k}, v \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^m)$

$$\begin{aligned}\big|\langle \mathbf{C_k}(u_{n_k}), v\rangle\big| &= \Big|\int_\Omega g_k(x,u_{n_k})\cdot v\,dx\Big| \\ &\leq \big(\frac{1}{p_+^-}+\frac{1}{(p'_+)^-}\big)\big\||g_k(x,u_{n_k})|\big\|_{L^{p'_+(\cdot)}(\Omega)}\big\||v|\big\|_{L^{p_+(\cdot)}(\Omega)} \\ &\leq \big(\frac{1}{p_+^-}+\frac{1}{(p'_+)^-}\big)\big(1+\int_\Omega k^{p'_+(x)}\big)^{1/(p'_+)^-}\|v\|_{\overrightarrow{p}(\cdot)} \\ &\leq \big(\frac{1}{p_+^-}+\frac{1}{(p'_+)^-}\big)\big(1+k^{(p'_+)^+}|\Omega|\big)^{1/(p'_+)^-}\|v\|_{\overrightarrow{p}(\cdot)}.\end{aligned}$$

Which implies the boundedness of $\mathbf{C_n}$.
$*$ Through (2) and (3), observe that

$$\begin{aligned}\frac{\langle \mathbf{A_k}(u_{n_k}), u_{n_k}\rangle}{\|u_{n_k}\|_{\overrightarrow{p}(\cdot)}} &\geq \frac{\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_{n_k}))\sigma_i(x,D_iu_{n_k})\cdot D_iu_{n_k}\,dx}{\|u_{n_k}\|_{\overrightarrow{p}(\cdot)}} \\ &\geq \frac{c\big(\sum_{i=1}^N\int_\Omega|D_iu_{n_k}|^{p_i(x)}\big)-c'}{\|u_{n_k}\|_{\overrightarrow{p}(\cdot)}} \\ &\geq \frac{c\big(\sum_{i=1}^N\min\{\|D_iu_{n_k}\|_{p_i(\cdot)}^{p_i^-},\|D_iu_{n_k}\|_{p_i(\cdot)}^{p_i^+}\}\big)-c'}{\|u_n\|_{\overrightarrow{p}(\cdot)}} \\ &\geq \frac{c\big(\sum_{i=1}^N\|D_iu_{n_k}\|_{p_i(\cdot)}^{p_-^-}-N\big)-c'}{\|u_{n_k}\|_{\overrightarrow{p}(\cdot)}} \\ &\geq \frac{c\big(\frac{1}{N}\sum_{i=1}^N\|D_iu_{n_k}\|_{p_i(\cdot)}\big)^{p_-^-}-(cN+c')}{\|u_{n_k}\|_{\overrightarrow{p}(\cdot)}} \\ &= \frac{c}{N^{p_-^-}}\|u_{n_k}\|_{\overrightarrow{p}(\cdot)}^{p_-^--1}-\frac{Nc+c'}{\|u_{n_k}\|_{\overrightarrow{p}(\cdot)}}.\end{aligned}$$

This implies that $\mathbf{A_k}$ is coercive.
$*$ Let us prove the pseudo-monotonicity of the operator $\mathbf{A_k}$,
Let $(u_{n_k})_k$ be a sequence in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^m)$ such that

$$\begin{cases} u_{n_k} \rightharpoonup u_n \text{ in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^m), \\ \limsup_{k\to\infty}\langle \mathbf{A_n}(u_{n_k}), u_{n_k}-u_n\rangle \leq 0. \end{cases} \tag{34}$$

We have to prove that,

$$\liminf_{k\to\infty}\langle \mathbf{A_k}(u_{n_k}), u_{n_k}-v\rangle \geq \langle \mathbf{A_k}(u_n), u_n - v\rangle, \quad \forall\, v\in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^m). \quad (35)$$

We remark that

$$\langle \mathbf{A_k}(u_{n_k}), u_{n_k}-v\rangle = \int_\Omega \sum_{i=1}^N a_i(x,T_n(u_{n_k}))\sigma_i\big(x, D_i u_{n_k}\big)\cdot (D_i u_{n_k} - D_i v)\,dx + \int_\Omega g_k(x,u_{n_k})\cdot(u_{n_k}-v)\,dx. \quad (36)$$

We will separately study the two terms of the right-hand side.
First, let's prove that

$$\lim_{k\to+\infty}\int_\Omega \sum_{i=1}^N J_i(x)\,dx = 0, \quad (37)$$

where,

$$J_i(x) = a_i(x,T_n(u_{n_k}))\left(\sigma_i\big(x,D_i u_{n_k}\big) - \sigma_i\big(x, D_i u_n\big)\right)\cdot(D_i u_{n_k} - D_i u_n).$$

Note that,

$$\langle \mathbf{A_k}(u_{n_k}), u_{n_k}-u_n\rangle = \int_\Omega \sum_{i=1}^N a_i(x,T_n(u_{n_k}))\sigma_i\big(x, D_i u_{n_k}\big)\cdot (D_i u_{n_k} - D_i u_n) + \int_\Omega g_k(x,u_{n_k})\cdot(u_{n_k}-u_n).$$

We observe that $\int_\Omega g_k(x,u_{n_k}).(u_{n_k}-u_n) \longrightarrow 0$, since $u_{n_k}\longrightarrow u_n$ in $L^{\varrho(\cdot)}(\Omega;\mathbb{R}^m)$ where $\varrho(\cdot)$ defined in Lemma 2, and the sequence $(g_k(x,u_{n_k}))$ is bounded in $L^{p_i'(\cdot)}(\Omega;\mathbb{R}^m)$ due to the hypotheses on g_k. By (34) and that $D_i u_{n_k}\rightharpoonup D_i u_n$ in $L^{p_i(\cdot)}$ this implies that

$$\limsup_{k\to+\infty}\int_\Omega \sum_{i=1}^N J_i(x)\,dx \leq 0.$$

Through this and using hypothesis (2) and (5), we get the desired result (37).
Right Now, let's prove the following, for all $i=1,\dots,N$,

$$\lim_{k\to+\infty}\int_\Omega |D_i u_{n_k} - D_i u_n|^{p_i(x)} = 0. \quad (38)$$

For that we put

$$\Omega_i^1 = \{x \in \Omega, p_i(x) \geq 2\}, \text{ and } \Omega_i^2 = \{x \in \Omega, 1 < p_i(x) < 2\},$$

then, by hypothesis (2) and (5) we have

$$\int_\Omega J_i(x)\, dx \geq c_4 \int_{\Omega_i^1} |D_i u_{n_k} - D_i u_n|^{p_i(x)}. \tag{39}$$

On the other hand, we have

$$\begin{aligned}
&\int_{\Omega_i^2} |D_i u_{n_k} - D_i u_n|^{p_i(x)}\, dx \\
&\leq \int_{\Omega_i^2} \frac{|D_i u_{n_k} - D_i u_n|^{p_i(x)}}{(|D_i u_{n_k}| + |D_i u_n|)^{\frac{p_i(x)(2-p_i(x))}{2}}} (|D_i u_{n_k}| + |D_i u_n|)^{\frac{p_i(x)(2-p_i(x))}{2}}\, dx \\
&\leq 2 \left\| \frac{|D_i u_{n_k} - D_i u_n|^{p_i(x)}}{(|D_i u_{n_k}| + |D_i u_n|)^{\frac{p_i(x)(2-p_i(x))}{2}}} \right\|_{L^{\frac{2}{p_i(\cdot)}}(\Omega_i^2)} \\
&\quad \times \left\| (|D_i u_{n_k}| + |D_i u_n|)^{\frac{p_i(x)(2-p_i(x))}{2}} \right\|_{L^{\frac{2}{2-p_i(\cdot)}}(\Omega_i^2)} \\
&\leq 2 \max \Big\{ \Big(\int_{\Omega_i^2} \frac{|D_i u_{n_k} - D_i u_n|^2}{\big(|D_i u_{n_k}| + |D_i u_n|\big)^{2-p_i(x)}}\, dx \Big)^{\frac{p_i^-}{2}}, \\
&\qquad\qquad \Big(\int_{\Omega_i^2} \frac{|D_i u_{n_k} - D_i u_n|^2}{\big(|D_i u_{n_k}| + |D_i u_n|\big)^{2-p_i(x)}}\, dx \Big)^{\frac{p_i^+}{2}} \Big\} \\
&\quad \times \max \Big\{ \Big(\int_\Omega \big(|D_i u_{n_k}| + |D_i u_n|\big)^{p_i(x)}\, dx \Big)^{\frac{2-p_i^+}{2}}, \\
&\qquad\qquad \Big(\int_\Omega \big(|D_i u_{n_k}| + |D_i u_n|\big)^{p_i(x)}\, dx \Big)^{\frac{2-p_i^-}{2}} \Big\}.
\end{aligned} \tag{40}$$

By hypothesis (5), boundedness of $(u_{n_k})_k$ in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega; \mathbb{R}^m)$,
$u_n \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega; \mathbb{R}^m)$, and (37), after letting $k \longrightarrow +\infty$ in (39) and in (40), we find (38),
which implies, for all $i = 1, \ldots, N$,

$$D_i u_{n_k} \longrightarrow D_i u_n \text{ strongly in } L^{p_i(\cdot)}(\Omega; \mathbb{R}^m) \text{ and a.e. in } \Omega. \tag{41}$$

By (41) and (34), we have

$$a_i(x, T_n(u_{n_k}))\sigma_i\big(x, D_i u_{n_k}\big) \rightharpoonup a_i(x, T_n(u_n))\sigma_i\big(x, D_i u_n\big) \text{ in } L^{p_i'(\cdot)}(\Omega; \mathbb{R}^m),$$

then we have, for every $v \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^m)$, $i = 1,\dots,N$

$$\int_\Omega a_i(x,T_n(u_{n_k}))\sigma_i\big(x,D_iu_{n_k}\big)\cdot D_iv \longrightarrow \int_\Omega a_i(x,T_n(u_n))\sigma_i\big(x,D_iu_n\big)\cdot D_iv \tag{42}$$

On the other hand,
as $a_i(x,T_n(u_{n_k}))\sigma_i\big(x,D_iu_{n_k}\big) \longrightarrow a_i(x,T_n(u_n))\sigma_i\big(x,D_iu_n\big)$
and $D_iu_{n_k} \longrightarrow D_iu_n$ a.e. in Ω, Fatou's Lemma implies that

$$\begin{aligned}\liminf_{k\to+\infty}\int_\Omega\sum_{i=1}^N a_i(x,T_n(u_{n_k}))\sigma_i\big(x,D_iu_{n_k}\big)\cdot D_iu_{n_k}\\ \geq \int_\Omega\sum_{i=1}^N a_i(x,T_n(u_n))\sigma_i\big(x,D_iu_n\big)\cdot D_iu_n.\end{aligned} \tag{43}$$

From (42) and (43), we deduce that

$$\begin{aligned}\liminf_{k\to+\infty}\int_\Omega\sum_{i=1}^N a_i(x,T_n(u_{n_k}))\sigma_i\big(x,D_iu_{n_k}\big)\cdot(D_iu_{n_k}-D_iv)\\ \geq \int_\Omega\sum_{i=1}^N a_i(x,T_n(u_n))\sigma_i\big(x,D_iu_n\big)\cdot(D_iu_n-D_iv).\end{aligned}$$

Since $u_{n_k} \longrightarrow u_n$ in $L^{\varrho(\cdot)}(\Omega;\mathbb{R}^m)$ where $\varrho(\cdot)$ defined in Lemma (2), then $g_k(x,u_{n_k}) \rightharpoonup g_k(x,u_n)$ weakly in $L^{p_i'(\cdot)}(\Omega;\mathbb{R}^m)$, and this implies that

$$\liminf_{k\to+\infty}\int_\Omega g_k\big(x,u_{n_k}\big)\cdot(u_{n_k}-v) \geq \int_\Omega g_k(x,u_n)\cdot(u_n-v).$$

We thus have obtained (35).
Therefore $\mathbf{A_k}$ is pseudo-monotone.
$*$ Moreover, the operator $\mathbf{A_k}$ is bounded, pseudomonotone , and coercive then, the main Theorem on pseudo-monotone operators applies and ensures existence of least one weak solution $u_{n_k} \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^m)$ to (31) in the sense that, $\forall\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^m)$

$$\int_\Omega\sum_{i=1}^N a_i(x,T_n(u_{n_k}))\sigma_i\big(x,D_iu_{n_k}\big)\cdot D_i\varphi\,dx + \int_\Omega g_k(x,u_{n_k})\cdot\varphi\,dx = \langle f,\varphi\rangle. \tag{44}$$

Passage to the limit: Put $X = \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^m)$.
Choosing $\varphi = u_{n_k}$ in (44), by (2), (3), and (9) , we have

$$\alpha(1+n)^{-\gamma^+}\left(c_1\int_\Omega\sum_{i=1}^N |\ D_iu_{n_k}\ |^{p_i(x)}\ dx - c'\right) \leq \|f\|_{X^*}\|u_{n_k}\|_X,$$

where, $c' = N\big|\Omega\big|c_2$.
Using Young's inequality, (iii) of Lemma 1, and the fact that, for all $i = 1, \dots, N$

$$| D_i u_{n_k} |^{p_i(x)} + 1 \geq | D_i u_{n_k} |^{p_-^-},$$

we get, for all $\varepsilon > 0$

$$\frac{\alpha c_1}{N^{p_-^-}} \big\|u_{n_k}\big\|_X^{p_-^-} \leq (1+n)^{\gamma^+} \left(C(\varepsilon)\big\|f\big\|_{X^*}^{(p_-^-)'} + \varepsilon \big\|u_{n_k}\big\|_X^{p_-^-} \right) + \alpha(N\big|\Omega\big|c_1 + c').$$

After taking $\varepsilon = \frac{\alpha c_1 (1+n)^{-\gamma^+}}{2N^{p_-^-}}$ it follows that the sequence $(u_{n_k})_k$ is bounded in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^m)$.
So, there exists a function $u_n \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^m)$ and a subsequence (still denoted by $(u_{n_k})_k$), such that

$$u_{n_k} \rightharpoonup u_n \quad \text{weakly in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^m) \text{ and a.e in } \Omega. \tag{45}$$

Now, choosing $\varphi = u_{n_k} - u_n$ in (44) as a test function, we get

$$\begin{aligned} &\int_\Omega \sum_{i=1}^N \big(a_i(x, T_n u_{n_k})\sigma_i(x, D_i u_{n_k}) - a_i(x, T_n(u_n))\sigma_i(x, D_i u_n)\big) \\ &\quad \cdot (D_i u_{n_k - D_i u_n}) dx \\ &\quad + \int_\Omega \sum_{i=1}^N a_i(x, T_n(u_n))\sigma_i(x, D_i(u_n)) \cdot (D_i u_{n_k} - D_i u_n) dx \\ &\quad + \int_\Omega g_k(x, u_{n_k}) \cdot (u_{n_k - u_n}) dx \\ &= \langle f, u_{n_k} - u_n \rangle. \end{aligned}$$

We observe that $\int_\Omega g_k(x, u_{n_k}).(u_{n_k} - u_n) \longrightarrow 0$, since $u_{n_k} \longrightarrow u_n$ in $L^{\varrho(\cdot)}(\Omega;\mathbb{R}^m)$ where $\varrho(\cdot)$ defined in Lemma 2, and the hypotheses on g_k give us that $(g_k(x, u_{n_k}))_k$ is bounded in $L^{p_i'(\cdot)}(\Omega;\mathbb{R}^m)$, and that $D_i u_{n_k} \rightharpoonup D_i u_n$ in $L^{p_i(\cdot)}$, this implies that

$$\lim_{k \to +\infty} \int_\Omega \sum_{i=1}^N I_i(x)\, dx \longrightarrow 0.$$

where,
$I_i(x) = \big(a_i(x, T_n(u_{n_k}))\sigma_i\big(x, D_i u_{n_k}\big) - a_i(x, T_n(u_n))\sigma_i\big(x, D_i u_n\big)\big) \cdot (D_i u_{n_k} - D_i u_n)$ In the same way as proving (41) we can get, for all $i = 1, \dots, N$,

$$D_i u_{n_k} \longrightarrow D_i u_n \text{ strongly in } L^{p_i(\cdot)}(\Omega;\mathbb{R}^m) \text{ and a.e. in } \Omega. \tag{46}$$

Now, by (4), (2), and (45), we have

$$\begin{aligned}\int_\Omega |a_i(x,T_n(u_{n_k}))\sigma_i(x,D_iu_{n_k})|^{p_i'(\cdot)}\,dx \leq & C\int_\Omega \left(\sum_{j=1}^N |D_iu_{n_k}|^{p_j(x)}+|h|\right)\,dx\\ \leq & C\int_\Omega \left(N\sum_{j=1}^N |D_ju_{n_k}|^{p_j(x)}+|h|\right)\,dx\\ \leq & CN\|u_{n_k}\|^{p_+^+}_{\vec{p}(\cdot)}+C'\leq C''.\end{aligned}$$

And therefore

$$a_i(x,T_n(u_{n_k}))\sigma_i(x,D_iu_{n_k}) \text{ is bounded in } L^{p_i'(\cdot)}(\Omega;\mathbb{R}^m). \tag{47}$$

From (46) and (47), we have

$$a_i(x,T_n(u_{n_k}))\sigma_i\big(x,D_iu_{n_k}\big) \rightharpoonup a_i(x,T_n(u_n))\sigma_i\big(x,D_iu_n\big) \text{ in } L^{p_i'(\cdot)}(\Omega;\mathbb{R}^m).$$

From this, we obtain, for every $\varphi\in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega;\mathbb{R}^m)$, $i=1,\ldots,N$

$$a_i(x,T_n(u_{n_k}))\sigma_i\big(x,D_iu_{n_k}\big)\cdot D_i\varphi \longrightarrow a_i(x,T_n(u_n))\sigma_i\big(x,D_iu_n\big)\cdot D_i\varphi. \tag{48}$$

Now, we have to prove that

$$g_k(x,u_{n_k}) \longrightarrow g(x,u_n) \quad \text{strongly in } L^1(\Omega;\mathbb{R}^m). \tag{49}$$

From (45), we have

$$g_k(x,u_{n_k}) \longrightarrow g(x,u_n) \quad \text{a.e. in } \Omega. \tag{50}$$

Let $E\subset\Omega$ be any measurable set, we write for all $t>0$

$$\int_E |g_k(x,u_{n_k})|\,dx = \int_{E_1} |g_k(x,u_{n_k})|\,dx + \int_{E_2} |g_k(x,u_{n_k})|\,dx,$$

where, $E_1 = E\cap\{|u_{n_k}|\leq t\}$, $E_2 = E\cap\{|u_{n_k}|>t$
Let $0<M<t$, and observe that

$$\big|T_t(u_{n_k})\big| \leq \big|T_t(u_{n_k})\big|\mathbf{1}_{\{|u_n|\leq M\}} + \big|T_t(u_{n_k})\big|\mathbf{1}_{\{|u_{n_k}|>M\}} \leq M + t\mathbf{1}_{\{|u_{n_k}|>M\}}.$$

Using this decomposition in (44) after taking $\varphi=T_t(u_{n_k})$, and by (8), we conclude the equi-integrability of $g_k(x,u_{n_k})$ in $L^1(\Omega;\mathbb{R}^m)$, and since (50), Vitalis theorem implies (49).

Therefore, we can obtain (30) by passing to the limit in (44). Thus the proof of Lemma 6 was concluded.

• Let (f_n) be a sequence in $\big(\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^m)\big)^* \cap L^{q(\cdot)}(\Omega;\mathbb{R}^m)$ of bounded functions which converges to f in $L^{q(\cdot)}(\Omega;\mathbb{R}^m)$, and which verifies the inequality

$$\|f_n\|_{q(\cdot)} \leq \|f\|_{q(\cdot)}, \quad \forall n \geq 1.$$

Thanks to Lemma 6, there exists at least one solution in the sense of distributions for the approximate problem

$$-\sum_{i=1}^{N} D_i\big(a_i(x,T_n(u_n))\sigma_i(x,D_iu_n)\big) + g(x,u_n) = f_n, \quad \text{in } \Omega,$$
$$u_n = 0 \quad \text{in } \partial\Omega,$$

satisfies the weak formulation

$$\int_\Omega \sum_{i=1}^{N} a_i(x,T_n(u_n))\sigma_i\big(x,Du_n\big)\cdot D_i\varphi\, dx + \int_\Omega g(x,u_n)\cdot\varphi\, dx = \int_\Omega f_n.\varphi\, dx, \tag{51}$$

for every $\varphi \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega;\mathbb{R}^m) \cap L^{s(\cdot)}(\Omega;\mathbb{R}^m)$. □

3.2 A priori estimates

In this section, we state and prove an uniform estimates for the approximate solutions u_n of the problem (30). Throughout the paper, we will denote by C_n (or C) the positive constants depending only on the data of the problem, but not on n.

Lemma 7. *There exists a constant $C > 0$ such that*

$$\|u_n\|_{\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^m)} \leq C. \tag{52}$$

Proof. Put $X = \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^m)$.
Choosing $\varphi = u_n$ in (51), by (2), (3), and (9) , we have

$$\alpha(1+n)^{-\gamma^+}\left(c_1\int_\Omega \sum_{i=1}^{N} |\ D_iu_n\ |^{p_i(x)}\ dx - c'\right) \leq \big\|f_n\big\|_{X^*}\big\|u_n\big\|_X,$$

where, $c' = N\big|\Omega\big|c_2$.
Using Young's inequality, (iii) of Lemma 1, and the fact that,
for all $i = 1,\dots,N$

$$|\ D_iu_n\ |^{p_i(x)} + 1 \geq |\ D_iu_n\ |^{p_-^-},$$

we get, for all $\varepsilon > 0$

$$\frac{\alpha c_1}{N^{p_-^-}}\|u_n\|_X^{p_-^-} \leq (1+n)^{\gamma^+}\left(C(\varepsilon)\|f_n\|_{X^*}^{(p_-^-)'} + \varepsilon\|u_n\|_X^{p_-^-}\right) + \alpha(N|\Omega|c_1 + c').$$

After taking $\varepsilon = \frac{\alpha c_1(1+n)^{-\gamma^+}}{2N^{p_-^-}}$ it follows that the sequence $(u_n)_n$ is bounded in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^m)$. □

Lemma 8. *Let q, s, p_i and γ be restricted as in Theorem 2. Then, there exists a constant $C > 0$ such that*

$$\sum_{i=1}^{N}\int_{\{|u_n|\leq t\}} \frac{|D_i u_n|^{p_i(x)}}{(1+|u_n|)^{\gamma(x)-(q(x)-1)s(x)}}\,dx \leq C(1+t). \tag{53}$$

$$\int_\Omega |g(x,u_n)|\,dx \leq C. \tag{54}$$

$$\int_\Omega |u_n|^{s(x)q(x)}\,dx \leq C. \tag{55}$$

$$\sum_{i=1}^{N}\int_{\{|u_n|\leq t\}} t^{(q(x)-1)s(x)-1-\gamma(x)}|D_i u_n|^{p_i(x)}\,dx \leq C, \quad \forall t \geq 1. \tag{56}$$

Proof. Taking $\psi(x,u_n) = \left((1+|u_n|)^{(q(x)-1)s(x)}\right)T_t(u_n)$ in (51) as a test function, by the fact that for a.e. $x \in \Omega$ and for all $i = 1,\dots,N$

$$\begin{aligned} D_i\psi(x,u_n) = {} & D_i\left((q(x)-1)s(x)\right)\frac{T_t(u_n)\ln(1+|u_n|)}{(1+|u_n|)^{(1-q(x))s(x)}} \\ & + (q(x)-1)s(x)\frac{T_t(u_n)D_i|u_n|}{(1+|u_n|)^{(1-q(x))s(x)+1}} \\ & + \frac{D_i u_n DT_t(u_n)}{(1+|u_n|)^{(1-q(x))s(x)}}, \end{aligned}$$

we have,

$$\begin{aligned} & \int_\Omega \sum_{i=1}^{N}\sigma_i\big(x, D_i u_n\big)\cdot D_i\psi(x,u_n)\,dx \\ & \quad + \int_\Omega g(x,u_n)\cdot\psi(x,u_n)\,dx \\ & \quad = \int_\Omega f_n\cdot\psi(x,u_n)\,dx. \end{aligned} \tag{57}$$

And this is equivalent to what follows after compensation

$$\begin{aligned}
&\int_\Omega \sum_{i=1}^{N} \frac{a_i(x, T_n(u_n)\sigma_i\big(x, D_i u_n\big) \cdot DT_t(u_n) D_i u_n}{(1+|u_n|)^{(1-q(x))s(x)}} dx \\
&+ \int_\Omega \frac{g(x, u_n) \cdot T_t(u_n)}{(1+|u_n|)^{(1-q(x))s(x)}}\, dx \\
&+ \int_\Omega \sum_{i=1}^{N} (q(x)-1)s(x) \frac{a_i(x, T_n(u_n)\sigma_i\big(x, D_i u_n\big) \cdot T_t(u_n) D_i |u_n|}{(1+|u_n|)^{(1-q(x))s(x)+1}} dx \\
&= \int_\Omega \frac{f_n \cdot T_t(u_n)}{(1+|u_n|)^{(1-q(x))s(x)}} dx \\
&- \int_\Omega \sum_{i=1}^{N} D_i((q(x)-1)s(x)) \frac{a_i(x, T_n(u_n)\sigma_i\big(x, D_i u_n\big) \cdot T_t(u_n) \ln(1+|u_n|)}{(1+|u_n|)^{(1-q(x))s(x)}}\, dx.
\end{aligned} \tag{58}$$

Using (23), (2) and (3), we obtain

$$\begin{aligned}
&\int_\Omega \sum_{i=1}^{N} \frac{a_i(x, T_n(u_n)\sigma_i\big(x, D_i u_n\big) \cdot DT_t(u_n) D_i u_n}{(1+|u_n|)^{(1-q(x))s(x)}} \\
&\qquad \geq \int_\Omega \sum_{i=1}^{N} \frac{a_i(x, T_n(u_n)\sigma_i(x, D_i u_n) \cdot D_i u_n \, \chi_{|u_n|<t}}{(1+|u_n|)^{(1-q(x))s(x)}} \\
&\qquad \geq c \sum_{i=1}^{N} \int_{\{|u_n| \leq t\}} (1+|u_n|)^{(q(x)-1)s(x)-\gamma(x)} \Big(c_1 |D_i u_n|^{p_i(x)} - c_2 \Big)\, dx.
\end{aligned} \tag{59}$$

By the observation that $(q(x)-1)s(x) - 1 - \gamma(x) < 0$, we get

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\{|u_n| \leq t\}} (1+|u_n|)^{(q(x)-1)s(x)-\gamma(x)} dx \\
&\qquad \leq \sum_{i=1}^{N} \int_{\{|u_n| \leq t\}} (1+|u_n|)(1+|u_n|)^{(q(x)-1)s(x)-1-\gamma(x)} dx \\
&\qquad \leq (1+t) \sum_{i=1}^{N} \int_{\{|u_n| \leq t\}} dx \\
&\qquad \leq N|\Omega|(1+t).
\end{aligned}$$

We find that, (59) give us

$$\begin{aligned} &\int_\Omega \sum_{i=1}^{N} \frac{a_i(x, T_n(u_n)\sigma_i\big(x, D_i u_n\big) \cdot DT_t(u_n) D_i u_n}{(1+|u_n|)^{(1-q(x))s(x)}} dx \\ &\qquad \geq c' \sum_{i=1}^{N} \int_{\{|u_n| \leq t\}} \frac{|D_i u_n|^{p_i(x)}}{(1+|u_n|)^{\gamma(x)+(1-q(x))s(x)}} dx - c'''(1+t) \end{aligned} \tag{60}$$

And after using (7), we have

$$\int_\Omega \frac{g(x,u_n)\cdot T_t(u_n)}{(1+|u_n|)^{(1-q(x))s(x)}}\, dx \geq t \int_{\{|u_n|>t\}} (1+|u_n|)^{(q(x)-1)s(x)} \frac{u_n}{|u_n|}.g(x,u_n)\, dx. \tag{61}$$

Using the fact that

$$(1+|u_n|)^{(q(x)-1)s(x)-1-\gamma(x)} \frac{u_n}{|u_n|}.g(x,u_n) \geq |g(x,u_n)|, \quad |u_n|>0, \tag{62}$$

it is produced through the following:
By (7) with the observation that $(q(x)-1)s(x)-1-\gamma(x)<0$ since (26), we get

$$\frac{1}{\big|u_n\big|} g(x,u_n). \left((1+|u_n|)^{(q(x)-1)s(x)-1-\gamma(x)} u_n - \big|u_n\big| \frac{g(x,u_n)}{\big|g(x,u_n)\big|} \right) \geq 0,$$

we get

$$\int_\Omega \frac{g(x,u_n)\cdot T_t(u_n)}{(1+|u_n|)^{(1-q(x))s(x)}}\, dx \geq t(1+t)^{1+\gamma^+} \int_{\{|u_n|>t\}} |g(x,u_n)|\, dx. \tag{63}$$

On the other hand by (2) we have it as well

$$\begin{aligned} &\int_\Omega \sum_{i=1}^{N} (q(x)-1)s(x) \frac{a_i(x, T_n(u_n)\sigma_i\big(x, D_i u_n\big) \cdot T_t(u_n) D_i |u_n|}{(1+|u_n|)^{(1-q(x))s(x)+1}}\, dx \\ &\qquad \geq \alpha(q^- - 1)s^- \int_\Omega \sum_{i=1}^{N} \frac{\sigma_i\big(x, D_i u_n\big)\cdot T_t(u_n) D_i |u_n|}{(1+|u_n|)^{(1-q(x))s(x)+1+\gamma(x)}}\, dx. \end{aligned} \tag{64}$$

Through all this we find that (58) give us

$$
\begin{aligned}
&c' \sum_{i=1}^{N} \int_{\{|u_n| \le t\}} \frac{|D_i u_n|^{p_i(x)}}{(1+|u_n|)^{\gamma(x)+(1-q(x))s(x)}}\, dx \\
&+ t(1+t)^{1+\gamma^+} \int_{\{|u_n|>t\}} |g(x,u_n)|\, dx \\
&\le \int_\Omega \frac{f_n \cdot T_t(u_n)}{(1+|u_n|)^{(1-q(x))s(x)}}\, dx \\
&- \alpha(q^- - 1)s^- \int_\Omega \sum_{i=1}^{N} \frac{\sigma_i\big(x, D_i u_n\big) \cdot T_t(u_n) D_i |u_n|}{(1+|u_n|)^{(1-q(x))s(x)+1+\gamma(x)}}\, dx \\
&- \int_\Omega \sum_{i=1}^{N} D_i((q(x)-1)s(x)) \frac{a_i(x, T_n(u_n)\sigma_i\big(x, D_i u_n\big) \cdot T_t(u_n) \ln(1+|u_n|)}{(1+|u_n|)^{(1-q(x))s(x)}}\, dx \\
&+ c'''(1+t).
\end{aligned}
\tag{65}
$$

So, from (65) with using (2), we get

$$
\begin{aligned}
&c' \sum_{i=1}^{N} \int_{\{|u_n| \le t\}} \frac{|D_i u_n|^{p_i(x)}}{(1+|u_n|)^{\gamma(x)+(1-q(x))s(x)}}\, dx \\
&+ t(1+t)^{1+\gamma^+} \int_{\{|u_n|>t\}} |g(x,u_n)|\, dx \\
&\le t \int_\Omega |f_n| (1+|u_n|)^{(q(x)-1)s(x)}\, dx \\
&+ \alpha(q^- - 1)s^- t \int_\Omega \sum_{i=1}^{N} \frac{|\sigma_i\big(x, D_i u_n\big)||D_i |u_n||}{(1+|u_n|)^{(1-q(x))s(x)+1+\gamma(x)}}\, dx \\
&+ c'' t \int_\Omega \sum_{i=1}^{N} (1+|u_n|)^{(q(x)-1)s(x)} |\sigma_i\big(x, D_i u_n\big)| \ln(1+|u_n|)\, dx \\
&+ c'''(1+t).
\end{aligned}
\tag{66}
$$

Now, from Hölder inequality, (4), (52), and the fact that

$$|D_i|u_n|| \le |D_i u_n|,\ |u_n| > 0$$

(Which is a direct result of equality $D_i|u_n| = \frac{u_n \cdot D_i u_n}{|u_n|}$ for the Euclidean norm on $\mathbb{R}^m$ for example), we have

$$\begin{aligned}
&\int_\Omega \sum_{i=1}^{N} \frac{|\sigma_i(x, D_i u_n)||D_i|u_n||}{(1+|u_n|)^{(1-q(x))s(x)+1+\gamma(x)}}\,dx \\
&\quad\le \sum_{i=1}^{N}\int_\Omega |\sigma_i(x, D_i u_n)|\big|D_i u_n\big|\,dx \\
&\quad\le c_3 \sum_{i=1}^{N}\int_\Omega \left(\sum_{j=1}^{N}|D_i u_n|^{p_j(x)}+|h|\right)^{1-\frac{1}{p_i(x)}} \big|D_i u_n\big|\,dx \\
&\quad\le c\sum_{i=1}^{N}\left\|\left(\sum_{j=1}^{N}|D_i u_n|^{p_j(x)}+|h|\right)^{1-\frac{1}{p_i(x)}}\right\|_{p_i'(\cdot)} \big\|\big|D_i u_n\big|\big\|_{p_i(\cdot)} \\
&\quad\le c\sum_{i=1}^{N}\left(1+\int_\Omega\left(\sum_{j=1}^{N}|D_i u_n|^{p_j(x)}+|h|\right)dx\right)^{1-\frac{1}{p_-^-}} \sum_{i=1}^{N}\big\|\big|D_i u_n\big|\big\|_{p_i(\cdot)} \\
&\quad\le c\sum_{i=1}^{N}\left(1+\int_\Omega\left(N\sum_{j=1}^{N}|D_j u_n|^{p_j(x)}+|h|\right)dx\right)^{1-\frac{1}{p_-^-}} \big\|u_n\big\|_{\overrightarrow{p}(\cdot)} \\
&\quad\le Nc\left(N\big\|u_n\big\|_{\overrightarrow{p}(\cdot)}^{p_+^+}+C\right)^{1-\frac{1}{p_-^-}} \big\|u_n\big\|_{\overrightarrow{p}(\cdot)} \le C'.
\end{aligned} \tag{67}$$

By Young enequality, we have $\forall \varepsilon > 0$

$$\int_\Omega \left((1+|u_n|)^{(q(x)-1)s(x)}\right)|f_n|\,dx \le \frac{1}{\varepsilon}\int_\Omega |f|^{q(x)}\,dx + \varepsilon\int_\Omega (1+|u_n|)^{q(x)s(x)}\,dx.$$

Using that $1+|u_n|^{q(x)s(x)} \ge \min\{1; 2^{1-q^+s^+}\}(1+|u_n|)^{q(x)s(x)}$, (8), and (9), we have

$$\int_\Omega \left((1+|u_n|)^{(q(x)-1)s(x)}\right)|f_n|\,dx \le \frac{1}{\varepsilon}\rho_{q(\cdot)}(|f|) + c\varepsilon(c' + \int_{|u_n|>t}|g(x,u_n)|\,dx).$$

In order to choose $\varepsilon = 1/(2c)$, we get

$$\int_\Omega \left((1+|u_n|)^{(q(x)-1)s(x)}\right)|f_n|\,dx \le \frac{1}{2}\int_{|u_n|>t}|g(x,u_n)|\,dx + C. \tag{68}$$

Now, by Young's inequality, and remark that $(q(x)-1)s(x)-\gamma(x)-1 \leq 0$, we have

$$\begin{aligned}
&\int_\Omega (1+|u_n|)^{(q(x)-1)s(x)}|\sigma_i(x,D_iu_n)|\ln(1+|u_n|)dx \\
&= \int_\Omega \Big[(1+|u_n|)^{\frac{(q(x)-1)s(x)+(p_i(x)-1)(\gamma(x)+1)}{p_i(x)}}(\ln(1+|u_n|))\Big] \\
&\quad \times |\sigma_i(x,D_iu_n)|(1+|u_n|)^{\frac{(q(x)-1)s(x)-\gamma(x)-1}{p_i'(x)}}dx \\
&\leq \varepsilon \int_\Omega (1+|u_n|)^{(q(x)-1)s(x)+(p_i(x)-1)(\gamma(x)+1)}(\ln(1+|u_n|))^{p_i(x)}dx \\
&\quad + c(\varepsilon)\int_\Omega |\sigma_i(x,D_iu_n)|^{p_i'(x)}(1+|u_n|)^{(q(x)-1)s(x)-\gamma(x)-1}dx \\
&\leq \varepsilon \int_\Omega (1+|u_n|)^{(q(x)-1)s(x)+(p_i(x)-1)(\gamma(x)+1)}(\ln(1+|u_n|))^{p_i(x)}dx \\
&\quad + c(\varepsilon)\int_\Omega |\sigma_i(x,D_iu_n)|^{p_i'(x)}\,dx \qquad (69)
\end{aligned}$$

Thanks to Remark 7 we have $s(x) > (p_i(x)-1)(\gamma(x)+1)$, so

$$(p_i(x)-1)(\gamma(x)+1)-s(x) < 0,$$

and therefore $(1+|t|)^{(p_i(x)-1)(\gamma(x)+1)-s(x)}\ln(1+|t|)^{p_i(x)}$ is bounded by $C>0$ for all $x\in\overline{\Omega}$ and $t\in\mathbb{R}$.
So, through this, (4), (8), (9), (52), and after choosing $\varepsilon = \frac{1}{2cc''C}$ we have

$$\begin{aligned}
&\int_\Omega (1+|u_n|)^{(q(x)-1)s(x)}|\sigma_i\big(x,D_iu_n\big)|\ln(1+|u_n|)\,dx \\
&\qquad\qquad \leq \frac{1}{2c''}\int_{|u_n|>t}|g(x,u_n)|\,dx + C''. \qquad (70)
\end{aligned}$$

Now, by using (67), (68), (70) in (66) we get

$$\begin{aligned}
&c'\sum_{i=1}^N \int_{\{|u_n|<t\}} \frac{|D_iu_n|^{p_i(x)}}{(1+|u_n|)^{\gamma(x)+(1-q(x))s(x)}}dx \\
&\quad + t((1+t)^{1+\gamma^+}-1)\int_{|u_n|>t}|g(x,u_n)|dx \\
&\leq c(1+t). \qquad (71)
\end{aligned}$$

After dropping the nonnegative term in (71) we get (53), and from this we get, for any choice of $t>0$ in (71), we have

$$\int_{\{|u_n|>t\}} |g(x,u_n)|\,dx \leq C. \qquad (72)$$

Then, by (8) and (72), we derive (54). Finally we combine (9) and (54) to obtain (55).

Now let's prove (56), by (53), (26), and the observation that

$$(q(x)-1)s(x)-1-\gamma(x) \leq 0 \text{ in } \overline{\Omega},$$

we have for all $t \geq 1$

$$\begin{aligned}
&\int_{\{|u_n|\leq t\}} t^{(q(x)-1)s(x)-1-\gamma(x)} |D_i u_n|^{p_i(x)}\, dx \\
&\qquad \leq \int_{\{|u_n|\leq t\}} (\frac{1+t}{2})^{(q(x)-1)s(x)-1-\gamma(x)} |D_i u_n|^{p_i(x)}\, dx \\
&\qquad \leq c(1+t)^{-1} \int_{\{|u_n|\leq t\}} (1+|u_n|)^{(q(x)-1)s(x)-\gamma(x)} |D_i u_n|^{p_i(x)}\, dx \\
&\qquad \leq C.
\end{aligned}$$

□

Lemma 9. *Let q, s, and p_i be restricted as in Theorem 2. Then, $(D_i u_n)$ is bounded in $L^{r_i}(\Omega;\mathbb{R}^m)$ for all $(r_1(\cdot),\ldots,r_N(\cdot)) \in (C(\overline{\Omega}))^N$ such that*

$$1 < r_i(x) < \frac{p_i(x)q(x)s(x)}{s(x)+1+\gamma(x)}, \quad \forall x \in \overline{\Omega},\ i = 1,\ldots,N.$$

Proof. For all $i = 1,\ldots,N$ setting $\alpha_i(\cdot) = \frac{p_i(\cdot)}{s(\cdot)+1+\gamma(\cdot)}$, then we have
∗ For $0 < t < 1$, we have trivially that

$$\int_{\{|D_i u_n|^{\alpha_i(x)} > t\}} t^{s(x)q(x)}\, dx \leq |\Omega|.$$

∗ For $t \geq 1$, using (55), and (26), we have

$$\begin{aligned}
&\int_{\{|D_i u_n|^{\alpha_i(x)} > t\}} t^{q(x)s(x)}\, dx \\
&\qquad \leq \int_{\{|D_i u_n|^{\alpha_i(x)} > t\}\cap\{|u_n|\leq t\}} t^{q(x)s(x)}\, dx + \int_{\{|u_n|>t\}} t^{q(x)s(x)}\, dx \\
&\qquad \leq \int_{\{|u_n|\leq t\}} t^{q(x)s(x)} \left(\frac{|D_i u_n|^{\alpha_i(x)}}{t}\right)^{\frac{p_i(x)}{\alpha_i(x)}} dx + \int_{\Omega} |u_n|^{q(x)s(x)}\, dx \\
&\qquad \leq \int_{\{|u_n|\leq t\}} t^{(q(x)-1)s(x)-1-\gamma(x)} |D_i u_n|^{p_i(x)}\, dx + c.
\end{aligned}$$

With (56), we have

$$\int_{\{|D_i u_n|^{\alpha_i(x)}>t\}} t^{q(x)s(x)}\, dx \le C'.$$

This shows that, for all $i = 1, \ldots, N$, $|D_i u_n|^{\alpha_i(\cdot)}$ is bounded in $\mathcal{M}^{q(\cdot)s(\cdot)}(\Omega)$, and hence we conclude from Lemma 5 that $|D_i u_n|$ is bounded in $L^{r_i(\cdot)}(\Omega)$ for all $r_i(\cdot)$ in $C(\overline{\Omega})$ satisfying

$$1 < r_i(x) < \frac{p_i(x)q(x)s(x)}{s(x)+1+\gamma(x)} \quad in \quad \overline{\Omega}.$$

This completes the proof of Lemma 9. □

Lemma 10. *Let m, s, p_i, and γ be restricted as in Theorem 1. Then, there exists a constant $C > 0$ such that*

$$\int_\Omega |u_n|^{s(x)q(x)}\, dx \le C. \tag{73}$$

$$\int_\Omega |g(x,u_n)|\, dx \le C. \tag{74}$$

Proof. After choosing $\varphi = u_n$ in (30), by (2), (3), and (9) , we have

$$c_1 \int_\Omega \sum_{i=1}^N |\ D_i u_n\ |^{p_i(x)}\ dx + \int_\Omega |u_n|^{s(x)q(x)+1}\, dx \le \|f_n\|_{X^*}\|u_n\|_X + c'.$$

Then, by (52) we get

$$\int_\Omega |u_n|^{s(x)q(x)+1}\, dx \le C. \tag{75}$$

By the fact, $1 + |u_n|^{s(x)q(x)+1} \ge |u_n|^{s(x)q(x)}$, we conclud (73).
Now, choosing $\varphi = T_t(u_n),\ t > 0$ in (30), by (2), (3), and (9) , we have

$$\begin{aligned}\int_\Omega \sum_{i=1}^N & a_i(x, T_n(u_n))\sigma_i\big(x, D_i u_n\big) \cdot DT_t(u_n) D_i u_n\, dx \\ & + t \int_{\{|u_n|>t\}} \frac{u_n}{|u_n|} \cdot g(x,u_n)\, dx \\ \le & t \int_\Omega |f_n|\, dx.\end{aligned} \tag{76}$$

Using (23), (2), and (3), we obtain

$$\begin{aligned}\int_\Omega \sum_{i=1}^{N} a_i(x,T_n(u_n))\sigma_i\big(x,D_iu_n\big)\cdot DT_t(u_n)D_iu_n \\ \geq \int_\Omega \sum_{i=1}^{N} a_i(x,T_n(u_n))\sigma_i(x,D_iu_n)\cdot D_iu_n\chi_{|u_n|<t} \\ \geq \frac{c_1}{(1+n)^{\gamma^+}}\Big(\sum_{i=1}^{N}\int_{\{|u_n|\leq t\}}|D_iu_n|^{p_i(x)}\,dx - c'\Big).\end{aligned}$$

By this equation, the fact $\frac{u_n}{|u_n|}.g(x,u_n) \geq |g(x,u_n)|$, $|u_n| > 0$ (Which results from (7)), and after dropping the nonnegative term, we find that (76) give us

$$t\int_{\{|u_n|>t\}}|g(x,u_n)|\,dx \leq ct\big(1+\int_\Omega |f|^{q(\cdot)}\,dx\big)+c'. \tag{77}$$

Then, for any choice to $t>0$, we have

$$\int_{\{|u_n|>t\}}|g(x,u_n)|\,dx \leq C. \tag{78}$$

Then, by (8) and (78), we derive (74). □

3.3 Proofs for Theorem 1 and Theorem 2:

The Proof of Theorem 1: by (52), the sequence (u_n) is bounded in $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega;\mathbb{R}^m)$. This implies that we can extract a subsequence (denote again by (u_n)), such that

$$\begin{aligned} u_n \rightharpoonup u \quad \text{weakly in } \mathring{W}^{1,\vec{p}(\cdot)}(\Omega;\mathbb{R}^m), \\ u_n \to u \quad \text{strongly in } L^{p_0}(\Omega;\mathbb{R}^m), \quad p_0=\min_{1\leq i\leq N}\min_{x\in\overline{\Omega}} p_i(x), \\ u_n \to u \quad \text{a.e. in } \Omega. \end{aligned} \tag{79}$$

This implies that

$$g(x,u_n)\to g(x,u) \quad \text{a.e. in } \Omega \tag{80}$$

Let $E\subset\Omega$ be any measurable set, we write

$$\int_E |g(x,u_n)|\,dx = \int_{E\cap\{|u_n|\leq t\}}|g(x,u_n)|\,dx + \int_{E\cap\{|u_n|>t\}}|g(x,u_n)|\,dx.$$

Let $0 < M < t$, and observe that

$$\big|T_t(u_n)\big| \leq \big|T_t(u_n)\big|\mathbf{1}_{\{|u_n|\leq M\}} + \big|T_t(u_n)\big|\mathbf{1}_{\{|u_n|>M\}} \leq M + t\mathbf{1}_{\{|u_n|>M\}},$$

Using this decomposition in (51) after taking $\varphi = T_t(u_n)$, yields

$$t\int_{\{|u_n|>t\}} \big|g(x,u_n)\big|\,dx \leq M\int_\Omega \big|f_n\big|\,dx + t\int_{\{|u_n|>M\}} \big|f_n\big|\,dx. \tag{81}$$

From (81) and (8) , we conclude the equi-integrability of $g(x,u_n)$ in $L^1(\Omega;\mathbb{R}^m)$, and since (80), Vitali's theorem implies that

$$g(x,u_n) \longrightarrow g(x,u) \quad \text{strongly in } L^1(\Omega;\mathbb{R}^m). \tag{82}$$

Now, choosing $\varphi = u_n - u$ in (51) as a test function, we get

$$\begin{aligned}\int_\Omega \sum_{i=1}^N & a_i(x,T_n(u_n))\big(\sigma_i\big(x,D_iu_n\big) - \sigma_i\big(x,D_iu\big)\big)\cdot(D_iu_n - D_iu)\,dx+\\ &\int_\Omega \sum_{i=1}^N a_i(x,T_n(u_n))\sigma_i\big(x,D_iu\big)\cdot(D_iu_n - D_iu)\,dx\\ &+\int_\Omega g(x,u_n)\cdot(u_n-u)\,dx\\ &=\int_\Omega f_n\cdot(u_n-u)\,dx.\end{aligned}$$

We observe that $\int_\Omega g(x,u_n)\cdot(u_n-u) \longrightarrow 0$, since $u_n \longrightarrow u$ in $L^{\varrho(\cdot)}(\Omega;\mathbb{R}^m)$ where $\varrho(\cdot)$ is defined in lemma 2, and $(g(x,u_n))$ is bounded in $L^{p_i'(\cdot)}(\Omega;\mathbb{R}^m)$, and that $D_iu_n \rightharpoonup D_iu$ in $L^{p_i(\cdot)}$, this implies that

$$\lim_{n\to+\infty}\int_\Omega \sum_{i=1}^N a_i(x,T_n(u_n))\big(\sigma_i\big(x,D_iu_n\big)-\sigma_i\big(x,D_iu\big)\big)\cdot(D_iu_n-D_iu)\,dx \longrightarrow 0,$$

and this convergence gives us, for all $i = 1,\ldots,N$

$$a_i(x,T_n(u_n))\big(\sigma_i\big(x,D_iu_n\big) - \sigma_i\big(x,D_iu\big)\big)\cdot(D_iu_n - D_iu) \longrightarrow 0 \quad \text{a.e. in } \Omega. \tag{83}$$

From (83) and (2), we have

$$\Big|\big(\sigma_i\big(x,D_iu_n\big) - \sigma_i\big(x,D_iu\big)\big)\cdot(D_iu_n - D_iu)\Big| \leq C(x),$$

for some function $C(x)$. Up to a Lebesgue measure zero set Z, the above inequality holds pointwise. Let us prove that there exists a function c such that

$$\big|D_i u_n(x)\big| \le c(x). \tag{84}$$

By hypothese (5), we have

$$C(x) \ge \begin{cases} c_4 \left((|D_i u_n| - |D_i u|)^{p_-^-} - 1 \right), & \text{if } p_i(x) \ge 2 \\ c_5 \left(\frac{|D_i u_n| - |D_i u|}{1 + |D_i u_n| + |D_i u|} \right)^2, & \text{if } 1 < p_i(x) < 2 \end{cases}$$

and this implies (84).
We are going to prove that

$$D_i u_n(x) \longrightarrow D_i u(x) \text{ in } \Omega \backslash Z. \tag{85}$$

Assume by contradiction that there exists $x_0 \in \Omega \backslash Z$ such that $D_i u_n(x_0)$ does not converge to $D_i u(x_0)$. The Bolzano Weierstrass theorem implies that $D_i u_n(x_0) \longrightarrow b$, for some $b \in \mathbb{R}^N$, up to a subsequence. Passing to the limit in

$$\big(\sigma_i\big(x_0, D_i u_n(x_0)\big) - \sigma_i\big(x_0, D_i u(x_0)\big)\big) \cdot (D_i u_n(x_0) - D_i u(x_0)),$$

we get

$$\big(\sigma_i\big(x_0, b\big) - \sigma_i\big(x_0, D_i u(x_0)\big)\big) \cdot (b - D_i u(x_0)) = 0,$$

which yields $b = D_i u(x_0)$ by hypothesis (5).
Through this we get, for all $i = 1, \dots, N$,

$$D_i u_n \longrightarrow D_i u \quad \text{a.e. in}\, \Omega. \tag{86}$$

From (86),(52), Vitali's theorem gives , for all $i = 1, \dots, N$

$$D_i u_n \longrightarrow D_i u \text{ in } L^1(\Omega; \mathbb{R}^m) \text{ and a.e. in } \Omega. \tag{87}$$

So we have

$$\sigma_i\big(x, D_i u_n\big) \longrightarrow \sigma_i\big(x, D_i u\big) \quad \text{a.e. in } \Omega. \tag{88}$$

Now, we prove that,

$$\big(\sigma_i\big(x, D_i u_n\big)\big)_n \text{ is uniformly bounded in } L^{p_i(\cdot)}(\Omega; \mathbb{R}^m).$$

Using the assumption (4), we get for all $i = 1, \dots, N$,

$$|\sigma_i(x, D_i u_n)|^{p_i(x)} \leq C\left(\sum_{j=1}^{N} |D_i u_n|^{p_j(x)} + |h|\right)^{p_i(x)-1} \leq C'\left(\sum_{i=1}^{N} |D_i u_n|^{p_i(x)} + |h|\right)^{p_i(x)-1}. \quad (89)$$

Then, by (52), we conclude that, for all $i = 1, \dots, N$,

$$\big(\sigma_i\big(x, D_i u_n\big)\big)_n \quad \text{is uniformly bounded in } L^{p_i(\cdot)}(\Omega; \mathbb{R}^m),$$

So, by (88) and Vitali's theorem, we derive, , for all $i = 1, \dots, N$,

$$\sigma_i\big(x, D_i u_n\big) \longrightarrow \sigma_i\big(x, D_i u\big) \text{ strongly in } L^1(\Omega; \mathbb{R}^m). \quad (90)$$

So, we can easily pass to the limit in (51) for all $\varphi \in C_c^\infty(\Omega; \mathbb{R}^m)$.
This proves Theorem (1).
The Proof of Theorem 2: By Lemma 9 the sequence (u_n) is bounded in $\mathring{W}^{1,\vec{r}(\cdot)}(\Omega)$ where $r_i(\cdot)$ is defined as (27). Without loss of generality, we can therefore assume that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{weakly in } \mathring{W}^{1,\vec{r}(\cdot)}(\Omega), \\ u_n \to u \quad \text{strongly in } L^{r_0}(\Omega), \quad r_0 = \min_{1\leq i\leq N} \min_{x\in\overline{\Omega}} r_i(x), \\ u_n \to u \quad \text{a.e. in} \quad \Omega. \end{gathered} \quad (91)$$

This implies that

$$g(x, u_n) \to g(x, u) \quad \text{a.e. in } \Omega. \quad (92)$$

By proceeding as in Theorem 1, we have

$$g(x, u_n) \to g(x, u) \quad \text{in } L^1(\Omega; \mathbb{R}^m), \quad (93)$$

and

$$D_i u_n \longrightarrow D_i u \quad \text{a.e. in } \Omega. \quad (94)$$

By (94) we have

$$\sigma_i\big(x, D_i u_n\big) \longrightarrow \sigma_i\big(x, D_i u\big) \text{ a.e. in } \Omega. \quad (95)$$

Now, we prove that,

$$\big(\sigma_i\big(x, D_i u_n\big)\big)_n \quad \text{uniformly bounded in } L^{\frac{r_i(\cdot)}{p_i(\cdot)-1}}(\Omega; \mathbb{R}^m),$$

where $r_i(\cdot)$ is a continuous function on $\overline{\Omega}$ satisfying (27) in $\overline{\Omega}$.
Then, we have, for all $x \in \overline{\Omega}$

$$1 < \frac{r_i(x)}{p_i(x)-1} < \frac{p_i(x)q(x)s(x)}{(s(x)+1+\gamma(x))(p_i(x)-1)}, \quad i = 1, \dots, N. \tag{96}$$

The choice of $\frac{r_i(x)}{p_i(x)-1} > 1$ is possible since we have (26).
Now, let $\psi(\cdot) : \overline{\Omega} \to (0,1)$ be a continuous function such that, for all $x \in \overline{\Omega}$

$$\frac{r_i(x)}{p_i(x)} < \psi(x) < \frac{q(x)s(x)}{s(x)+1+\gamma(x)} < 1, \quad i = 1, \dots, N. \tag{97}$$

Therefore, we derive for all $x \in \overline{\Omega}$

$$\begin{aligned} &0 < \psi(x)p_i(x) < \frac{p_i(x)q(x)s(x)}{s(x)+1+\gamma(x)}, \\ &\text{and } \frac{r_i(x)}{p_i(x)\psi(x)} < 1, \quad i = 1, \dots, N. \end{aligned} \tag{98}$$

Using the assumption (4), we get for all $i = 1, \dots, N$,

$$|\sigma_i(x, D_i u_n)|^{\frac{r_i(x)}{p_i(x)-1}} \leq C \left(\sum_{i=1}^{N} |D_i u_n|^{\psi(x)p_j(x)} + |h|^{\psi(x)}(x) \right)^{\frac{r_i(x)}{p_i(x)\psi(x)}} \tag{99}$$

then, by Lemma 9, and (98), we conclude that, for all $i = 1, \dots, N$,

$$\big(\sigma_i\big(x, D_i u_n\big)\big)_n \quad \text{uniformly bounded in } L^{\frac{r_i(\cdot)}{p_i(\cdot)-1}}(\Omega; \mathbb{R}^m), \tag{100}$$

where $r_i(\cdot)$ is a continuous function on $\overline{\Omega}$ satisfying (27).
So, by (95) and Vitali's theorem, we derive, , for all $i = 1, \dots, N$,

$$\sigma_i\big(x, D_i u_n\big) \longrightarrow \sigma_i\big(x, D_i u\big) \ \text{ strongly in } L^1(\Omega; \mathbb{R}^m) \tag{101}$$

From (2), (91) and (101) , we derive

$$a_i(x, T_n(u_n))\sigma_i\big(x, D_i u_n\big) \longrightarrow a_i(x, u)\sigma_i\big(x, D_i u\big) \ \text{ strongly in } L^1(\Omega; \mathbb{R}^m). \tag{102}$$

So, we can easily pass to the limit in (30) for all $\varphi \in C_c^\infty(\Omega; \mathbb{R}^m)$.
Now, from (55), and since $|u_n|^{s(x)q(x)} \geq 0$, by using Fatous Lemma, we deduce that

$$0 \leq \int_\Omega |u|^{s(x)q(x)}\, dx \leq C.$$

Then $|u|^{s(x)q(x)} \in L^1(\Omega)$, which completes the proof of Theorem 2.

References

[1] M. Naceri, F. Mokhtari. Anisotropic nonlinear elliptic systems with variable exponents and degenerate coercivity. *Appl. Anal.* 100(11), 2021, 2347-2367.

[2] M. Naceri, M.B. Benboubker, Distributional solutions of anisotropic nonlinear elliptic systems with variable exponents: existence and regularity, *Advances in Operator Theory.* 7(2), 1–34 (2022).

[3] M. Naceri, Singular Anisotropic Elliptic Problems with Variable Exponents, *Memoirs on Differential Equations and Mathematical Physics.* 85(2022), 119–132 (2022).

[4] M. Bendahmane, K. Karlsen. Anisotropic nonlinear elliptic systems with measure data and anisotropic harmonic maps into spheres. *Journal of Differential Equations*, 1-29, 2006.

[5] M. Bendahmane, M. Fares. Nonlinear elliptic systems with variable exponents and measure data. *Moroccan. J. Pure Appl. Anal.*1, 108-125 (2015).

[6] M. Fares. Nonlinear anisotropic elliptic equations in $\mathbb{R}^N$ with variable exponents and locally integrable data. *Math. Meth. Appl. Sci.* 40, 2265-2276 (2017)

[7] M. Fares. Regularity of the Solution to Nonlinear Anisotropic Elliptic Equations with Variable Exponents and Irregular Data. *Mediterr. J. Math.* 14:141 (2017)

[8] M. Bendahmane, K. H. Karlsen, M. Saad. Nonlinear anisotropic elliptic and parabolic equations with variable exponents and L^1 data. *Commun. Pure Appl.Anal.*12, 1201-1220 (2013)

[9] M. Sanchón, M. Urbano. Entropy solutions for the $p(x)$-Laplace equation. *Trans. Amer. Math.* Soc. 361 (2009) 6387-6405.

[10] X. Fan. Anisotropic variable exponent Sobolev spaces and -Laplacian equations. *Complex Variables and Elliptic Equations.* Soc. 55, 1-20 (2010).

[11] L. Diening, P. Harjulehto, P. Hästö, M. Ruzicka. Lebesgue and Sobolev Spaces with Variable Exponents, Lecture Notes in Mathematics. *Springer.* vol. 2017, New York (2011)

[12] D. Cruz-Uribe, A. Fiorenza, M. Ruzhansky, and J. Wirth. Variable Lebesgue Spaces and Hyperbolic Systems. *Advanced Courses in Mathematics - CRM Barcelona. Birkhäuser,* Basel, 2014.

[13] G. Dolzmann, N. Hungerbühler, and S. Müller. Non-linear elliptic systems with measure-valued right hand side. *Math. Z.* 226(4):545-574, 1997.

[14] X. Fan, D. Zhao. On the spaces $L^{p(x)}(\Omega)$ and $W^{1;p(x)}(\Omega)$, *J. Math. Anal. Appl.* 263 (2001), 424-446.

[15] M. Troisi. Teoremi di inclusione per spazi di sobolev non isotropi. *Ricerche. Mat.* 18:3–24, 1969.

[16] R. Landes. Testfunctions for elliptic systems and maximum principles. *Forum Math.* 12(1):23–52, 2000.

[17] S. Zhou. A note on nonlinear elliptic systems involving measures. *Electron. J. Differential Equations*, pages No. 08, 6 pp. (electronic), 2000.