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ON A NONLOCAL PROBLEM INVOLVING FRACTIONAL $p(x,\cdot)$-LAPLACIAN WITH NON-STANDARD GROWTH*

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Abstract

We are concerned in a nonlocal problem involving the fractional $p(x,\cdot)-$Laplacian operator and with a right-hand side that is a Carathéodory function satisfying only a non-standard growth condition. We show that our problem admits at least one weak solution. In order to do this, the main tool is the Berkovits degree theory for abstract Hammerstein type mappings.

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1 Introduction

The use of the functional framework provided by the classical Lebesgue and Sobolev spaces L^p and $W^{1,p}$ has shown to be not appropriate for studying various materials which present inhomogeneities. Indeed, for such materials the exponents involved in the constitutive law could be variable, which requires the use of the spaces $L^{p(x)}$ and $W^{1,p(x)}$. The use of these spaces

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is strongly motivated by their ability to model phenomena concerning electrorheological fluids [20, 22], thermorheological fluids [7], elastic materials [27] and image restoration [12]. The $p(x)$-Laplacian operator, which is an extension of the p-Laplacian, is involved in many of these problems and whose existence results are developed; see, for example, [13, 14] and references therein. Recently, some authors have further generalized the above mentioned operator to the fractional case (fractional operator $p(x,\cdot)$-Laplacian) and they have introduced a functional framework to study problems in which this fractional variable exponent operator is involved. See, for example, [5, 6, 11] and their references.

Let Ω be a smooth bounded open set in $\mathbb{R}^N$, $s \in (0,1)$ and let $p : \overline{\Omega} \times \overline{\Omega} \to (1, +\infty)$ be a continuous bounded function. We assume that

$$1 < p^- = \min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) \leq p(x,y) \leq p^+ = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) < +\infty, \quad (1)$$

and p is symmetric i.e.

$$p(x,y) = p(y,x), \quad \forall (x,y) \in \overline{\Omega} \times \overline{\Omega}. \quad (2)$$

Let us consider the fractional $p(x,\cdot)-$Laplacian operator given by

$$(-\Delta_{p(x,\cdot)})^s u(x) = p.v. \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}\, dy, \quad \forall x \in \Omega,$$

where $p.v.$ is a commonly used abbreviation in the principal value sense.

In this paper, we are concerned with the study of the following nonlinear elliptic problem,

$$\begin{cases} (-\Delta_{p(x,\cdot)})^s u(x) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad \text{(P)}$$

Note that $(-\Delta_{p(x,\cdot)})^s$ is the fractional version of well known $p(x)-$Laplacian operator $-\Delta_{p(x)}(u) = -div(|\nabla u|^{p(x)-2}\nabla u)$ for which Fan and Zhang in [16] and Iliaş in [17] present several sufficient conditions for the existence of solutions for a problem similar to (P), that is the Dirichlet problem of $p(x)$-Laplacian:

$$\begin{cases} -\Delta_{p(x)} u = f(x,u) & \text{in } \Omega, \\ u = 0 & x \in \partial\Omega. \end{cases}$$

The discussion is based on the theory of the spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ by using variational and topological methods.

Bendahmane and Wittbold in [10] have shown the existence and uniqueness of the renormalized solution for this problem where the right-hand side $f \in L^1(\Omega)$ and it not depends to u. We also refer to [23] for the existence and uniqueness of entropy solution. The same problem is studied by Messaho in [21] for $p \equiv cte$. Her approach is based to the truncation and epi-convergence method.

In [4], the autors study the problem (P). The main results are established by means of mountain pass theorem and Fountain theorem with Cerami condition.

Using another technical approach, that of the topological degree theory, and only with a growth condition, we prove in this paper the existence of at least one weak solution the problem (P). For more details about this theory and its applications, the reader can refer to [1, 2, 3, 8] and the references therein.

The paper is divided into three sections. In the second section we present some preliminary results on classes of operators related to the recent Berkovits degree and on Lebesgue and fractional Sobolev spaces with variable exponent. The third section is reserved for some technical lemmas and the main result concerning the existence weak solutions of the problem (P).

2 Some preliminary results

We first give the definitions of some classes of operators related to the Berkovits topological degree theory (see [8]).

Let X be a real separable reflexive Banach space with dual X^* and with continuous pairing $\langle . \, , \, . \rangle$ and let Ω be a nonempty subset of X. The symbol $\to$ ($\rightharpoonup$) stands for strong (weak) convergence and the sign $\circ$ denotes the composition of two operators.

Definition 1. *Let Y be a real Banach space. A mapping $F : \Omega \subset X \to Y$ is said to be*

1. *bounded,* if it takes any bounded set into a bounded set.
2. *demicontinuous,* if for any $(u_n) \subset \Omega$, $u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$.
3. *compact* if it is continuous and the image of any bounded set is relatively compact.

Definition 2. *A mapping $F : \Omega \subset X \to X^*$ is said to be*

1. *of class* (S_+), if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$ and $limsup\langle Fu_n, u_n - u\rangle \leq 0$, it follows that $u_n \to u$.

2. *quasimonotone* , if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, it follows that $limsup\langle Fu_n, u_n - u\rangle \geq 0$.

Definition 3. *For any operator* $F : \Omega \subset X \to X$ *and any bounded operator* $T : \Omega_1 \subset X \to X^*$ *such that* $\Omega \subset \Omega_1$, *we say that* F *satisfies condition* $(S_+)_T$, *if for any* $(u_n) \subset \Omega$ *with* $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ *and* $limsup\langle Fu_n, y_n - y\rangle \leq 0$, *we have* $u_n \to u$.

Let $\mathcal{O}$ be the collection of all bounded open set in X. For any $\Omega \subset X$, we consider the following classes of operators:

$$\begin{aligned}
\mathcal{F}_1(\Omega) &:= \{F : \Omega \to X^* \mid F \text{ is bounded, demicontinuous and satisfies condition } (S_+)\},\\
\mathcal{F}_{T,B}(\Omega) &:= \{F : \Omega \to X \mid F \text{ is bounded, demicontinuous and satisfies condition } (S_+)_T\},\\
\mathcal{F}_T(\Omega) &:= \{F : \Omega \to X \mid F \text{ is demicontinuous and satisfies condition } (S_+)_T\},\\
\mathcal{F}_B(X) &:= \{F \in \mathcal{F}_{T,B}(\bar{G}) \mid G \in \mathcal{O}, \mathrm{T} \in \mathcal{F}_1(\bar{\mathrm{G}})\}.
\end{aligned}$$

Let us now present a Berkovits lemma for abstract mappings of type Hammerstein and an impotent proposition deduced from this lemma and the properties of the Berkovits topological degree.

Lemma 1. *[8, Lemmas 2.2 and 2.4] Suppose that* $T \in \mathcal{F}_1(\bar{G})$ *is continuous and* $S : D_S \subset X^* \to X$ *is demicontinuous such that* $T(\bar{G}) \subset D_S$, *where* G *is a bounded open set in a real reflexive Banach space* X. *Then the following statements are true:*

(i) *If* S *is quasimonotone, then* $I + S \circ T \in \mathcal{F}_T(\bar{G})$, *where* I *denotes the identity operator.*

(ii) *If* S *is of class* (S_+), *then* $S \circ T \in \mathcal{F}_T(\bar{G})$

Proposition 1. *Let* $S : X \to X^*$ *and* $T : X^* \to X$ *be two operators bounded and continuous such that* S *is quasimonotone and* T *is an homeomorphism, strictly monotone and of class* (S_+). *If*

$$\Lambda := \{v \in X^* | v + tS \circ Tv = 0 \text{ for some } t \in [0, 1]\}$$

is bounded in X^*, *then the equation*

$$v + S \circ Tv = 0$$

admits at lest one solution in X^*.

Proof. Since Λ is bounded in X^*, there exists $R > 0$ such that

$$\|v\|_{X^*} < R \text{ for all } v \in \Lambda.$$

This says that

$$v + tS \circ Tv \neq 0 \text{ for all } v \in \partial B_R(0) \text{ and all } t \in [0,1]$$

where $B_R(0)$ is the ball of center 0 and radius R in X^*.
Thanks to the Minty-Browder Theorem [24, Theorem 26A], the inverse operator $L := T^{-1}$ is bounded, continuous and of type (S_+).
From Lemma 1 it follows that

$$I + S \circ T \in \mathcal{F}_T(\overline{B_R(0)}) \text{ and } I = L \circ T \in \mathcal{F}_T(\overline{B_R(0)}).$$

Since the operators I, S and T are bounded, $I + S \circ T$ is also bounded. We conclude that

$$I + S \circ T \in \mathcal{F}_{T,B}(\overline{B_R(0)}) \text{ and } I \in \mathcal{F}_{T,B}(\overline{B_R(0)}).$$

Consider a homotopy $H : [0,1] \times \overline{B_R(0)} \to X^*$ given by

$$H(t,v) := v + tS \circ Tv \text{ for } (t,v) \in [0,1] \times \overline{B_R(0)}.$$

Let us apply the homotopy invariance and normalization property of the Berkovits degree (which we note d) introduced in [8], we get

$$d(I + S \circ T, B_R(0), 0) = d(I, B_R(0), 0) = 1,$$

and hence there exists a point $v \in B_R(0)$ such that

$$v + S \circ Tv = 0.$$

□

To study the problem (P), we need also to introduce and clarify our functional framework. We first recall some useful properties of the variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$. For more details we refer the reader to [15, 18, 25] for more details.

Denote

$$C_+(\bar{\Omega}) = \{h \in C(\bar{\Omega}) | \inf_{x \in \bar{\Omega}} h(x) > 1\}.$$

For any $h \in C_+(\bar{\Omega})$, we define

$$h^+ := max\{h(x), x \in \bar{\Omega}\}, h^- := min\{h(x), x \in \bar{\Omega}\}.$$

For any $p \in C_+(\bar{\Omega})$ we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \{u;\ u : \Omega \to \mathbb{R} \text{ is measurable and } \int_{\Omega} |u(x)|^{p(x)}\, dx < +\infty\}.$$

Endowed with *Luxemburg norm*

$$\|u\|_{p(x)} = \inf\{\lambda > 0 / \rho_{p(\cdot)}(\frac{u}{\lambda}) \leq 1\}.$$

where

$$\rho_{p(\cdot)}(u) = \int_{\Omega} |u(x)|^{p(x)}\, dx, \quad \forall u \in L^{p(x)}(\Omega),$$

$(L^{p(x)}(\Omega), \|\cdot\|_{p(x)})$ is a Banach space, separable and reflexive. Its conjugate space is $L^{p'(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \Omega$.
We have also the following result

Proposition 2. *For any $u \in L^{p(x)}(\Omega)$ we have*

(i) $\|u\|_{p(x)} < 1 (= 1; > 1) \quad \Leftrightarrow \quad \rho_{p(\cdot)}(u) < 1 (= 1; > 1)$,

(ii) $\|u\|_{p(x)} \geq 1 \quad \Rightarrow \quad \|u\|_{p(x)}^{p^-} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(x)}^{p^+}$,

(iii) $\|u\|_{p(x)} \leq 1 \quad \Rightarrow \quad \|u\|_{p(x)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(x)}^{p^-}$.

(iv) $\lim_{n\to\infty} \|u_n - u\|_{p(x)} = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0$.

From this proposition, we can deduce the inequalities

$$\|u\|_{p(x)} \leq \rho_{p(\cdot)}(u) + 1, \tag{3}$$

$$\rho_{p(\cdot)}(u) \leq \|u\|_{p(x)}^{p^-} + \|u\|_{p(x)}^{p^+}. \tag{4}$$

If $p, q \in C_+(\overline{\Omega})$ such that $p(x) \leq q(x)$ for any $x \in \bar{\Omega}$, then there exists the continuous embedding $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$.

Next, we present the definition and some results on fractional Sobolev spaces with variable exponent that was introduced in [4, 9, 19]. Let s be a fixed real number such that $0 < s < 1$ and lets the assumptions (1) and (2) be satisfied, we define the fractional Sobolev space with variable exponent via the Gagliardo approach as follows:

$$\begin{aligned} W &= W^{s,p(x,y)}(\Omega) \\ &= \{u \in L^{\bar{p}(x)}(\Omega) : \int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{N+sp(x,y)}}\, dxdy < +\infty, \text{ for some } \lambda > 0\}, \end{aligned}$$

where $\bar{p}(x) = p(x,x)$. We equip the space W with the norm

$$\|u\|_W = \|u\|_{\bar{p}(x)} + [u]_{s,p(x,y)},$$

where $[\cdot]_{s,p(x,y)}$ is a Gagliardo seminorm with variable exponent, which is defned by

$$[u]_{s,p(x,y)} = \inf\{\lambda > 0 : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}\, dxdy \leq 1\}.$$

The space $(W, \|\cdot\|_W)$ is a Banach space (see [12]), separable and reflexive (see [9, Lemma 3.1]).

We also define W_0 as the subspace of W which is the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_W$. From [4, Theorem 2.1 and Remark 2.1],

$$\|\cdot\|_{W_0} := [\cdot]_{s,p(x,y)}$$

is a norm on W_0 which is equivalent to the norm $\|\cdot\|_W$, and we have the compact embedding $W_0 \hookrightarrow\hookrightarrow L^{\bar{p}(x)}(\Omega)$. So the space $(W_0, \|\cdot\|_{W_0})$ is a Banach space separable and reflexive.
We defne the modular $\rho_{p(\cdot,\cdot)} : W_0 \to \mathbb{R}$ by

$$\rho_{p(\cdot,\cdot)}(u) = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dxdy.$$

The modular $\rho_{p(\cdot,\cdot)}$ checks the following results, which is similar to Proposition 2 (see [26, Lemma 2.1])

Proposition 3. *For any $u \in W_0$ we have*

(i) $\|u\|_{W_0} \geq 1 \quad \Rightarrow \quad \|u\|_{W_0}^{p^-} \leq \rho_{p(\cdot,\cdot)}(u) \leq \|u\|_{W_0}^{p^+}$,

(ii) $\|u\|_{W_0} \leq 1 \quad \Rightarrow \quad \|u\|_{W_0}^{p^+} \leq \rho_{p(\cdot,\cdot)}(u) \leq \|u\|_{W_0}^{p^-}$.

3 Technical lemmas and main result

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a smooth bounded open set, $s \in (0,1)$ and we assume that (1) and (2) and holds. In this section, we present two technical lemmas that we will need to study our problem (P), then our main result.

Let denote $L : W_0 \to W_0^*$, the operator associated to the $(-\Delta_{p(x,\cdot)})^s$ defined by

$$\langle Lu, v\rangle = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp(x,y)}} |\nabla u|^{p(x)-2}\, dxdy,$$

for all $u, v \in W_0$, where W_0^* is the dual space of W_0.

Lemma 2. *[9]*

(i) *L is bounded and strictly monotone operator,*

(ii) *L is a mapping of type (S_+),*

(iii) *L is a homeomorphism.*

Now, we make the following assumptions on the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$:

(f_1) f satisfies the Carathéodory condition, that is, $f(.,\eta)$ is measurable on Ω for all $\eta \in \mathbb{R}$ and $f(x,\cdot)$ is continuous on $\mathbb{R}$ for a.e. $x \in \Omega$.

(f_2) f has the growth condition

$$|f(x,\eta)| \leq c(k(x) + |\eta|^{q(x)-1})$$

for a.e. $x \in \Omega$ and all $\eta \in \mathbb{R}$, where c is a positive constant, $k \in L^{\bar{p}'(x)}(\Omega)$ and $q \in C_+(\bar{\Omega})$ with $q^+ < \bar{p}^-$.

Lemma 3. *Under assumptions (f_1) and (f_2), the operator $S : W_0 \to W_0^*$ setting by*

$$\langle Su, v\rangle = -\int_\Omega f(x,u)v\,dx, \quad \forall u, v \in W_0$$

is compact.

Proof. Let $\phi : W_0 \to L^{\bar{p}'(x)}(\Omega)$ be an operator defined by

$$\phi u(x) := -f(x,u) \text{ for } u \in W_0 \text{ and } x \in \Omega.$$

We first show that ϕ is bounded and continuous.
For each $u \in W_0$, we have by the growth condition (f_2), the inequalities (3) and (4) that

$$\begin{aligned} \|\phi u\|_{\bar{p}'(x)} &\leq \rho_{\bar{p}'(x)}(\phi u) + 1 \\ &= \int_\Omega |f(x,u(x))|^{\bar{p}'(x)} + 1 \\ &\leq const(\rho_{\bar{p}'(x)}(k) + \rho_{r(x)}(u)) + 1 \\ &\leq const(\|k\|_{\bar{p}'(x)}^{\bar{p}'^+} + \|u\|_{r(x)}^{r^+} + \|u\|_{r(x)}^{r^-}) + 1, \end{aligned}$$

where $r(x) = (q(x)-1)\bar{p}'(x) \in C_+(\bar{\Omega})$ with $r(x) < \bar{p}(x)$. Then, by the continuous embedding $L^{\bar{p}(x)} \hookrightarrow L^{r(x)}$ and the compact embedding $W_0 \hookrightarrow\hookrightarrow L^{\bar{p}(x)}(\Omega)$, we have

$$\|\phi u\|_{\bar{p}'(x)} \leq const(\|k\|_{\bar{p}'(x)}^{\bar{p}'^+} + \|u\|_{W_0}^{r^+} + \|u\|_{W_0}^{r^-}) + 1.$$

This implies that ϕ is bounded on W_0.
To show that ϕ is continuous, let $u_n \to u$ in W_0. Then $u_n \to u$ in $L^{\bar{p}(x)}(\Omega)$. Hence there exist a subsequence (u_k) of (u_n) and measurable functions h in $L^{\bar{p}(x)}(\Omega)$ and g such that

$$u_k(x) \to u(x) \text{ and } |u_k(x)| \leq h(x)$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$. Since f satisfies the Carathéodory condition, we obtain that

$$f(x, u_k(x)) \to f(x, u(x)) \text{ a.e. } x \in \Omega.$$

it follows from (f_2) that

$$|f(x, u_k(x))| \leq c(k(x) + |h(x)|^{q(x)-1})$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.
Since

$$k + |h|^{q(x)-1} \in L^{\bar{p}'(x)}(\Omega),$$

and taking into account the equality

$$\rho_{\bar{p}'(x)}(\phi u_k - \phi u) = \int_\Omega |f(x, u_k(x)) - f(x, u(x))|^{\bar{p}'(x)} dx,$$

the dominated convergence theorem and the equivalence (iv) in Proposition 2 implies that

$$\phi u_k \to \phi u \text{ in } L^{\bar{p}'(x)}(\Omega).$$

Thus the entire sequence (ϕu_n) converges to ϕu in $L^{\bar{p}'(x)}(\Omega)$.
Since the embedding $I : W_0 \to L^{\bar{p}(x)}(\Omega)$ is compact, it is known that the adjoint operator $I^* : L^{\bar{p}'(x)}(\Omega) \to W_0^*$ is also compact. Therefore, the composition $S = I^* \circ \phi : W_0 \to W_0^*$ is compact. □

Definition 4. *We say that $u \in W_0$ is a weak solution of (P) if*

$$\langle Lu, v\rangle + \langle Su, v\rangle = 0, \ \ \forall v \in W_0.$$

Theorem 1. *Under assumptions* (1), (2) *(f_1) and (f_2), the problem (P) has a weak solution u in W_0.*

Proof. $u \in W_0$ is a weak solution of (P) if and only if

$$Lu = -Su. \tag{5}$$

Thanks to the properties of the operator L seen in Lemma 2 and in view of Minty-Browder Theorem [24, Theorem 26A], the inverse operator

$T := L^{-1} : W_0^* \to W_0$ is bounded, continuous and satisfies condition (S_+). Moreover, note by Lemma 3 that the operator S is bounded, continuous and quasimonotone.
Consequently, equation (5) is equivalent to

$$u = Tv \text{ and } v + S \circ Tv = 0. \tag{6}$$

To solve equation (6), we will apply the Proposition 1. To do this, we first claim that the set

$$\Lambda := \{v \in W_0^* | v + tS \circ Tv = 0 \text{ for some } t \in [0, 1]\}$$

is bounded. Indeed, let $v \in \Lambda$. Set $u := Tv$, then $\|Tv\|_{W_0} = \|u\|_{W_0}$.
If $\|u\|_{W_0} \leq 1$, then $\|Tv\|_{W_0}$ is bounded.
If $\|u\|_{W_0} > 1$, then we get by the implication (i) in Proposition 3 and the inequality (4) the estimate

$$\begin{aligned}
\|Tv\|_{W_0}^{p^-} &= \|u\|_{W_0}^{p-} \\
&\leq \rho_{p(\cdot,\cdot)}(u) \\
&= \langle Lu, u\rangle \\
&= \langle v, Tv\rangle \\
&= -t\langle S \circ Tv, Tv\rangle \\
&\leq t \int_\Omega (|u(x)|^{q(x)} + \lambda |u(x)|^{r(x)}) u \, dx \\
&\leq const(\|u\|_{q(x)}^{q^-} + \|u\|_{q(x)}^{q^+} + \|u\|_{r(x)}^{r^-} + \|u\|_{r(x)}^{r^+}).
\end{aligned}$$

From the continuous embedding $L^{q(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$ and the compact embedding $W_0 \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$, we can deduct the estimate

$$\|Tv\|_{W_0}^{p^-} \leq const(\|Tv\|_{W_0}^{q^+} + \|Tv\|_{W_0}^{r^+}).$$

It follows that $\{Tv | v \in \Lambda :\}$ is bounded.
Since the operator S is bounded, it is obvious from (6) that the set Λ is bounded in W_0^*. Hence, in virtu of Proposition 1, the equation $v + S \circ Tv$ have at lest one solution $\bar{v}$ in W_0^*. We conclude that $\bar{u} = T\bar{v}$ is a weak solution of (P). □

References

[1] M. Ait Hammou, E. Azroul. Construction of a topological degree theory in generalized Sobolev spaces. In *Recent Advances in Intuition-*

istic Fuzzy Logic Systems, Studies in Fuzziness and Soft Computing. Vol. 372, pp. 1-18, Springer Nature Switzerland AG, 2019.

[2] M. Ait Hammou, E. Azroul. Existence result for a nonlinear elliptic problem by topological degree in Sobolev spaces with variable exponent. *Moroccan J. of Pure and Appl. Anal.* 7 No. 1: 50-65, 2021.

[3] M. Ait Hammou, E. Azroul, B. Lahmi. Topological degree methods for partial differential operators in generalized Sobolev spaces. *Bol. Soc. Paran. Mat.* 39 No. 2: 39-61, 2021.

[4] E. Azroul, A. Benkirane, M. Shimi. On a nonlocal problem involving fractional $p(x, \cdot)$-Laplacian satisfyin Cerami condition. *Discrete Contin. Dyn. Syst. Ser. S* 2020.

[5] E. Azroul , A. Benkirane, A. Boumazourh, M. Shimi. Existence results for fractional $p(x, \cdot)$-Laplacian problem via the Nehari manifold approach. *Appl. Math. Optim.* 2020.

[6] E. Azroul, A. Benkirane, M. Shimi. Eigenvalue problems involving the fractional $p(x)-$Laplacian operator. *Adv. Oper. Theory* 4: 539-555, 2019.

[7] S.N. Antontsev, J.F. Rodrigues. On stationary thermo-rheological viscous flows. *Ann. Univ. Ferrara Sez. VII Sci. Mat.*52: 19-36, 2006.

[8] J. Berkovits. Extension of the Leray-Schauder degree for abstract Hammerstein type mappings. *J. Differential Equations* 234: 289-310, 2007.

[9] A. Bahrouni, V. Rădulescu. On a new fractional Sobolev space and applications to nonlocal variational problems with variable exponent. *Discrete Contin. Dyn. Syst.* 11:379-389, 2018.

[10] M. Bendahmane, P. Wittbold. Renormalized solutions for nonlinear elliptic equations with variable exponents and L^1-data. *Nonlinear Anal.* 70: 567-583, 2009.

[11] N.T. Chung. Eigenvalue Problems for Fractional $p(x, y)-$Laplacian Equations with Indefinite Weight. *Taiwanese J. Math.* 23: 1153-1173, 2019.

[12] Y. Chen, S. Levine, M. Rao. Variable exponent, linear growth functionals in image processing. *SIAM J. Appl. Math.* 66: 1383-1406, 2006.

[13] F. Cammaroto, L. Vilasi. Multiple solutions for a Kirchhoff-type problem involving the $p(x)-$Laplacian operator. *Nonlinear Anal.* 74: 1841-1852, 2011.

[14] X. L. Fan. On nonlocal $p(x)-$Laplacian Dirichlet problems. *Nonlinear Anal.* 72: 3314-3323, 2010.

[15] X.L. Fan, D. Zhao. On the Spaces $\mathrm{L}^{p(x)}(\Omega)$ and $\mathrm{W}^{m,p(x)}(\Omega)$. *J. Math. Anal. Appl.* 263: 424-446, 2001.

[16] X-L. Fan, Q-H. Zhang. Existence of solutions for $p(x)$-Laplacian Dirichlet problem. *Nonlinear Anal.* 52: 1843-1852, 2003.

[17] P. S. Iliaş. Dirichlet problem with $p(x)$-Laplacian. *Math. Reports* 10(60) No. 1: 43-56, 2008.

[18] O. Kováčik, J. Rákosník. On spaces $\mathrm{L}^{p(x)}$ and $\mathrm{W}^{1,p(x)}$. *Czechoslovak Math. J.* 41: 592-618, 1991.

[19] U. Kaufmann, J.D. Rossi, R. Vidal. Fractional Sobolev spaces with variable exponents and fractional $p(x)-$Laplacians. *Elect. J. Qual. Theory Differ. Equ.* 76: 1-10, 2017.

[20] Y. Liu, R. Davidson, P. Taylor. Investigation of the touch sensitivity of ER fluid based tactile display. In *Proceedings of SPIE, Smart Structures and Materials: Smart Structures and Integrated Systems 5764.* pp. 92-99, 2005.

[21] J. Messaho. Epiconvergence method to a nonlinear value boundary problem with L^1 Data. *Bol. Soc. Paran. Mat.* 34 No. 2: 35-41, 2016.

[22] M. Rŭžička. *Electrorheological fuids: modeling and mathematical theory.* Lecture Notes in Mathematics, Vol. 1748, Springer-Verlag, Berlin, 2000.

[23] M. Sanchón, J.M. Urbano. Entropy solutions for the $p(x)$-Laplace equation. *Trans. Amer. Math. Soc.* 361: 6387-6405, 2009.

[24] E. Zeidler. *Nonlinear Functional Analysis and its Applications.* II/B: Nonlinear Monotone Operators, Springer, New York, 1990.

[25] D. Zhao, W.J. Qiang, X.L. Fan. On generalized Orlicz spaces $\mathrm{L}^{p(x)}(\Omega)$, *J. Gansu Sci.* 9 No. 2: 1-7, 1996.

[26] C. Zhang, X. Zhang. Renormalized solutions for the fractional $p(x)-$Laplacian equation with L^1 data. arXiv:1708.04481v1

[27] V.V. Zhikov. Averaging of functionals in the calculus of variations and elasticity. *Math. USSR Izv.* 29: 33-66, 1987.