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A PARABOLIC SHAPE OPTIMIZATION PROBLEM*

Dan Tiba † Masahiro Yamamoto ‡§

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Dedicated to Dr. Vasile Drăgan on the occasion of his 70^{th} anniversary

Abstract

In this article, we discuss some approximation methods for optimal design problems governed by evolution equations of parabolic type. The two investigated approaches are of fixed domain type. We also formulate supplementary questions and problems, related to this subject.

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1 Introduction

Optimal design problems usually take into account stationary processes, but there are studies related to evolutionary systems. For instance, the classical monographs by Pironneau [14], Sokolowski and Zolesio [15] devote sections to such subjects, in particular to hyperbolic systems and parabolic systems.

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†dan.tiba@imar.ro Institute of Mathematics, Romanian Academy and Academy of Romanian Scientists, Bucharest, Romania

‡myamac@ms.u-tokyo.ac.jp Graduate School of Mathematical Sciences, The University of Tokyo, Komaba, Meguro, Tokyo 153-8914, Japan,

§Honorary Member of Academy of Romanian Scientists and Peoples' Friendship University of Russia, Russian Federation

Here, we explore the application of methods of fixed domain type in the setting of parabolic shape optimization problems. See [10], [11], [12], [18] for some recent developments in this respect. We consider the minimization of a cost functional J defined on a family $\mathcal{O}$ of admissible domains in R^d; more precisely, for each $\Omega \in \mathcal{O}$, a functional J is defined and the value J depends on $x \in \Omega$, $t \in [0,T]$, $T > 0$, and the solution y_Ω of the following initial boundary value problem:

$$y'_\Omega - \Delta y_\Omega = f(x,t) \quad \text{in } \Omega \times]0,T[, \tag{1.1}$$

$$y_\Omega = 0 \quad \text{on } \partial\Omega \times [0,T], \tag{1.2}$$

$$y_\Omega(x,0) = y_0(x) \quad \text{in } \Omega, \tag{1.3}$$

where y' denotes the derivative with respect to $t \in [0,T]$. We assume $f \in L^2(\Omega \times]0,T[)$ and $y_0 \in L^2(\Omega)$. Moreover, in general, it is assumed that $\Omega \subset D$ for any $\Omega \in \mathcal{O}$, where D is some given Lipschitzian bounded domain. Regularity conditions on the admissible domains $\Omega \in \mathcal{O}$ and more hypotheses will be discussed later.

Examples of cost functionals J are:

$$\int_0^T \int_\Omega j(x,t,y_\Omega(x,t))dxdt, \tag{1.4}$$

$$\int_0^T \int_E j(x,t,y_\Omega(x,t))dxdt, \tag{1.5}$$

$$\int_\Omega j(x,y_\Omega(x,T))dx. \tag{1.6}$$

In (1.5), $E \subset D$ is another fixed domain satisfying $E \subset \Omega$ for any $\Omega \in \mathcal{O}$. Here j is some Carathéodory functional. It is also possible to consider boundary observation problems or the dependence of j on $\nabla y_\Omega(x,t)$, etc.

One physical interpretation of shape optimization problems involving the minimization of some functionals (1.4) - (1.6) subject to the state system (1.1) - (1.3), is to find the shape of an isolation wall in Ω protecting a certain subregion from heat or cooling sources situated in another subregion. This is similar to the hyperbolic case [14], [15], where the protection is considered with respect to some noise sources, for example.
In this paper, we approach the problem (1.1) - (1.6) via two methods: the Kawarada-Natori penalization idea (e.g., [6]) which we discuss in Section 2, and an approximation approach via controllability-type arguments in Section 3. The first one uses a perturbation of the state system and a

parametrization of the unknown geometry, while the second one does not perturb the parabolic differential operator, but introduces a supplementary control term on the right-hand side of the equation. In Section 2, the arguments known in the stationary case are adapted to the parabolic case, while Section 3 uses methods specific for the optimal control of parabolic problems. We underline that the well known level set method, due to Osher and Sethian [13] (see as well Allaire [2]), is essentially different from our approach. For instance, although we also apply level functions, no Hamilton-Jacobi equation is needed, but ordinary Hamiltonian systems are used instead (see [18], [8]). The final section includes a brief discussion of other problems and methods. The methodology used in this work enters the general setting of fixed domain approaches for shape optimization. We quote [8] and [9] for some recent applications in the stationary case.

2 Penalization of the state equation

We approximate the state system (1.1) - (1.3) by a penalized equation defined in D:

$$y'_\varepsilon - \Delta y_\varepsilon + \frac{1}{\varepsilon}\chi_{D\setminus\Omega} y_\varepsilon = f \quad \text{in } D\times]0,T[, \tag{2.1}$$

$$y_\varepsilon = 0 \quad \text{on } \partial D \times [0,T], \tag{2.2}$$

$$y_\varepsilon(x,0) = y_0(x) \quad \text{in } D, \tag{2.3}$$

where $\varepsilon > 0$. Here and henceforth $\chi_{D\setminus\Omega}$ denotes the characteristic function of the set $D \setminus \Omega$.

If Ω and D are open bounded sets, $f \in L^2(D\times]0,T[)$ and $y_0 \in L^2(D)$, then (2.1) - (2.3), respectively (1.1) - (1.3) possess unique weak solutions in $L^2(0,T;H_0^1(D)) \cap H^1(0,T;H^{-1}(D))$, respectively in $L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$. In both cases, the solutions are continuous on $[0,T]$ with values in $L^2(D)$, respectively in $L^2(\Omega)$. See Evans [3], Chapter 7.

Multiply (2.1) by y_ε and integrate by parts. Then, we have the estimate:

$$\frac{d}{dt}|y_\varepsilon|^2_{L^2(D)} + \int_D |\nabla y_\varepsilon|^2 dx + \frac{1}{\varepsilon}\int_D \chi_{D\setminus\Omega}|y_\varepsilon|^2 dx$$
$$\leq |f|_{L^2(D)}|y_\varepsilon|_{L^2(D)}, \quad \text{a.a. } t \in [0,T]. \tag{2.4}$$

After integration over $[0,t]$ in (2.4) and the application of the Gronwall inequality, we obtain that $\{y_\varepsilon\}$ is bounded in $C([0,T];L^2(D)) \cap L^2(0,T;H_0^1(D))$ and, moreover

$$|y_\varepsilon|_{L^2(0,T;L^2(D\setminus\Omega))} \longrightarrow 0 \tag{2.5}$$

as $\varepsilon \to 0$. On a subsequence, we may assume

$$y_\varepsilon \longrightarrow \widetilde{y} \quad \text{weakly in } L^2(0,T;H_0^1(D)). \tag{2.6}$$

As $y'_\varepsilon \in L^2(0,T;H^{-1}(D))$, we also have $y'_\varepsilon \in L^2(0,T;H^{-1}(\Omega))$. Multiply (2.1) by any $v \in L^2(0,T;H_0^1(\Omega)) \subset L^2(0,T;H_0^1(D))$ by using the zero extension. Then the last term disappears and by (2.6), we obtain that $\{y'_\varepsilon\}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$. Its weak limit on a subsequence (again denoted by y_ε) has to satisfy

$$y'_\varepsilon \longrightarrow \widetilde{y}' \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)) \tag{2.7}$$

by a distribution argument in $]0,T[$.

We now impose a very weak regularity property on $\partial\Omega$, the segment property (Adams [1], Tiba [17]) which is concerned with some continuity of the boundary $\partial\Omega$. Henceforth we say that Ω is of class C if Ω has the segment property.

Proposition 2.1. *If Ω is of class C, then $\widetilde{y}$ is the unique weak solution of (1.1) - (1.3).*

Proof. We have just to prove that $\widetilde{y}$ satisfies (1.2) and (1.3) as (2.6) and (2.7) already show that (1.1) is satisfied by $\widetilde{y}$ as a weak solution. We notice that by (2.5) and (2.6), we see that $\widetilde{y} \in L^2(0,T;H^1(D))$ and $\widetilde{y}(x,t) = 0$ a.e. in $(D \setminus \Omega) \times [0,T]$. Let $\mu(\cdot)$ and $\widetilde{\mu}(\cdot)$ denote the Lebesgue measures in $\mathbb{R}^{d+1}$ and $\mathbb{R}^d$, respectively.

We set $D_\Gamma = \{(x,t) \in (D \setminus \Omega) \times [0,T];\ \widetilde{y}(x,t) \neq 0\}$ and $D_t = \{x \in D \setminus \Omega;\ \widetilde{y}(x,t) \neq 0\}$. Then it is known that

$$0 = \mu(D_\Gamma) = \int_0^T \widetilde{\mu}(D_t)dt,$$

(see Vulikh [19], Chapter VIII.7). It yields that $\widetilde{\mu}(D_t) = 0$ for almost all $t \in [0,T]$.

Then, by the Hedberg-Keldys stability property (Theorem 2.3.9 in [10]), we see that $\widetilde{y} \in L^2(0,T;H_0^1(\Omega))$, that is, (1.2) is satisfied by $\widetilde{y}$. Moreover, $\widetilde{y}(x,0) = y_0(x)$ in Ω by $\widetilde{y} \in C([0,T];L^2(\Omega))$, due to (2.7) and $\widetilde{y} \in L^2(0,T;H_0^1(\Omega))$, and by applying the Mazur theorem (Yosida [20], Chapter 5.1).

Remark 2.2. Relation (2.5) can be sharpened to

$$|y_\varepsilon|_{L^2(0,T;L^2(D\setminus\Omega))} \le C\varepsilon^{\frac{1}{2}},$$

where the constant C is independent of $\varepsilon > 0$. This may be interpreted as an error estimate for the approximation.

Motivated by the above approximation property, we shall study the approximate optimization problem defined in D by

$$\min_{\Omega} \int_0^T \int_D \chi_\Omega j(x,t,y_\varepsilon(x,t))dxdt \tag{2.8}$$

subject to (2.1) - (2.3). Notice that the cost (2.8) is identified with (1.4). One may consider the cost (1.5) as well. The minimization parameter in (2.8) is the domain $\Omega \in \mathcal{O}$, provided that Ω is included in the given domain D.

We describe now the family $\mathcal{O}$ of admissible open subsets. We consider a family $\mathcal{F} \subset C(\overline{D})$ and to any $g \in \mathcal{F}$ we associate the open set

$$\Omega_g = \{x \in D;\, g(x) < 0\}. \tag{2.9}$$

Notice that the characteristic function of $D \setminus \Omega_g$ may be written $\chi_{D\setminus\Omega_g} = H(g)$, where H is the Heaviside function: $H(r) = 1$ if $r \geq 0$ and $H(r) = 0$ if $r < 0$. In the case where the cost (1.5) is considered and we impose a geometric constraint $E \subset \Omega_g$ for any $g \in \mathcal{F}$, this may be expressed by the simple inequality $g < 0$ in E for any $g \in \mathcal{F}$.

Henceforth we assume that $\mathcal{F}$ is a closed cone.

With this definition of $\mathcal{O}$, the approximate shape optimization problem (2.8), (2.1) - (2.3) becomes a control by the coefficients problem with respect to $g \in \mathcal{F} \subset C(\overline{D})$.

Let $H^\varepsilon \in C^\infty(\mathbb{R})$ be a regularization of the Heaviside function: for example, we can require that

$$H^\varepsilon(r) = \begin{cases} 1, & r > 0, \\ 0, & r < -\varepsilon. \end{cases}$$

See [11] for other choices.

Then, $H^\varepsilon(g)$ is a regularization of the characteristic function $\chi_{D\setminus\Omega_g}$, and we introduce the regularized state system:

$$\widehat{y}_\varepsilon' - \Delta\widehat{y}_\varepsilon + \frac{1}{\varepsilon}H^\varepsilon(g)\widehat{y}_\varepsilon = f \quad \text{in } D\times]0,T[, \tag{2.10}$$

$$\widehat{y}_\varepsilon = 0 \quad \text{on } \partial D \times [0,T], \tag{2.11}$$

$$\widehat{y}_\varepsilon(x,0) = y_0(x) \quad \text{in } D. \tag{2.12}$$

It turns out that Proposition 2.1 remains valid for (2.10) - (2.12) with a slight modification of the proof.

Corollary 2.3. *If Ω is of class C, then $\widehat{y}_\varepsilon \longrightarrow \widehat{y}$ weakly in $L^2(0,T;H_0^1(D))$ and $(\widehat{y}_\varepsilon)' \longrightarrow (\widehat{y})'$ weakly in $L^2(0,T;H^{-1}(\Omega))$ by taking subsequences. Moreover, $\widehat{y}$ restricted to Ω is the unique weak solution of (1.1) - (1.3).*
See Theorem 9 in [11] for the details. The system (2.10) - (2.12) enjoys good differentiability properties with respect to g.

Proposition 2.4. *The mapping $g \longrightarrow \widehat{y}_\varepsilon(g)$ from $\mathcal{F}$ to $L^2(0,T;H_0^1(D)) \cap H^1(0,T;H^{-1}(D))$ defined by (2.10) - (2.12) is Gâteaux differentiable. Moreover $z = \nabla\widehat{y}_\varepsilon(g)w \in L^2(0,T;H_0^1(D)) \cap H^1(0,T;H^{-1}(D))$ is a weak solution of the variational equation:*

$$z' - \Delta z + \frac{1}{\varepsilon}(H^\varepsilon)'(g)\widehat{y}_\varepsilon w + \frac{1}{\varepsilon}H^\varepsilon(g)z = 0 \quad in\ D\times]0,T[$$

$$z = 0 \quad on\ \partial D \times [0,T], \quad z(x,0) = 0 \quad in\ D$$

for any $w, g \in \mathcal{F}$.

Proof. We denote by $y_\lambda = \widehat{y}_\varepsilon(g + \lambda w)$ with small $|\lambda|$, for fixed $\varepsilon > 0$. Subtracting the corresponding equations and dividing by $\lambda \neq 0$, we obtain

$$\int_D \frac{y'_\lambda - \widehat{y}'_\varepsilon}{\lambda}(y_\lambda - \widehat{y}_\varepsilon)dx + \frac{1}{\lambda}\int_D |\nabla(y_\lambda - \widehat{y}_\varepsilon)|^2 dx + \frac{1}{\lambda\varepsilon}\int_D [H^\varepsilon(g+\lambda w) - H^\varepsilon(g)]\cdot$$

$$\cdot(y_\lambda - \widehat{y}_\varepsilon)\widehat{y}_\varepsilon dx + \frac{1}{\lambda\varepsilon}\int_D (H^\varepsilon(g+\lambda w))(y_\lambda - \widehat{y}_\varepsilon)^2 dx = 0. \tag{2.13}$$

Notice that $H^\varepsilon(\cdot)$ is C^1 and Lipschitzian with constant $\frac{1}{\varepsilon}$ (ε is fixed). Consequently

$$\frac{H^\varepsilon(g+\lambda w) - H^\varepsilon(g)}{\lambda} \longrightarrow (H^\varepsilon)'(g)w \tag{2.14}$$

uniformly on $\overline{D}$ as $\lambda \to 0$.

Relations (2.13) and (2.14), using Gronwall's inequality, show that $\{\frac{y_\lambda - \widehat{y}_\varepsilon}{\lambda}\}$ is bounded in $L^2(0,T;H_0^1(D)) \cap L^\infty(0,T;L^2(D))$. The weak formulation gives:

$$\int_D \frac{y'_\lambda - \widehat{y}'_\varepsilon}{\lambda} v dx + \int_D \nabla\left(\frac{y_\lambda - \widehat{y}_\varepsilon}{\lambda}\right)\nabla v dx + \frac{1}{\varepsilon}\int_D\int_D \frac{H^\varepsilon(g+\lambda w) - H^\varepsilon(g)}{\lambda}\widehat{y}_\varepsilon v dx$$

$$+\frac{1}{\varepsilon}\int_D [H^\varepsilon(g+\lambda w)]\frac{y_\lambda - \widehat{y}_\varepsilon}{\lambda} v dx = 0, \tag{2.15}$$

for any $v \in H_0^1(D)$ and almost all $t \in [0,T]$.

All the terms in (2.15), except for the first term, are known to be bounded with respect to λ. It yields that $\left\{\frac{y_\lambda - \widehat{y}_\varepsilon}{\lambda}\right\}$ is also bounded in

$H^1(0,T;H^{-1}(D))$. We denote by $z \in L^2(0,T;H_0^1(D)) \cap H^1(0,T;H^{-1}(D))$ the weak limit by choosing a subsequene of $\left\{\frac{y_\lambda - \widehat{y}_\varepsilon}{\lambda}\right\}$ as $\lambda \to 0$. It is possible to pass to the limit in (2.15) and we finish the proof. As the weak solution of the variational equation is unique and depends linearly on w, the limit is valid without taking subsequences and we prove the Gâteaux differentiability.

Remark 2.5. In the above proposition, it is implicitly assumed that $g + \lambda w \in \mathcal{F}$ for small $|\lambda|$. This depends on the specific definition of $\mathcal{F}$, in each application. See the examples in the end of this section.

We introduce now the adjoint system for the cost functionals (1.4) and (1.5), as follows. The case (1.6) will be considered in section 3.

$$-p' - \Delta p + \frac{1}{\varepsilon} H^\varepsilon(g) p = \chi_E j'_y(x,t,\widehat{y}_\varepsilon(x,t)) \quad \text{in } D, \tag{2.16}$$

$$p(x,T) = 0 \quad \text{in } D, \quad p = 0 \quad \text{on } \partial D \times [0,T] \tag{2.17}$$

for the functional (1.5). The system (2.16) - (2.17) has to be understood in the weak sense and we also assume that $j(x,t,\cdot)$ is differentiable.

For the cost functional (1.4), in the approximating the problem, one has to consider its regularization:

$$\int_0^T \int_D (1 - H^\varepsilon(g)) j(x,t,\widehat{y}_\varepsilon(x,t)) dxdt \tag{2.18}$$

and equation (2.16) is replaced by

$$-p' - \Delta p + \frac{1}{\varepsilon} H^\varepsilon(g) p = (1 - H^\varepsilon(g)) j'_y(x,t,\widehat{y}_\varepsilon(x,t)) \quad \text{in } D, \tag{2.19}$$

together with the conditions (2.17).

Proposition 2.6. *The directional derivative of the cost (1.5) is given by*

$$-\frac{1}{\varepsilon} \int_0^T \int_D (H^\varepsilon)'(g) \widehat{y}_\varepsilon w p \, dxdt. \tag{2.20}$$

Proof. We compute the corresponding limit:

$$\frac{1}{\lambda}\int_0^T\int_E [j(x,t,y_\lambda(x,t)) - j(x,t,\widehat{y}_\varepsilon(x,t))]dxdt$$
$$\longrightarrow \int_0^T\int_E j'_y(x,t,\widehat{y}_\varepsilon(x,t))zdxdt = \int_0^T\int_D \chi_E j'_y(x,t,\widehat{y}_\varepsilon(x,t))zdxdt$$
$$= \int_0^T\int_D [-p' - \Delta p + \frac{1}{\varepsilon}(H^\varepsilon(g))p]zdxdt$$
$$= \int_0^T\int_D [z' - \Delta z + \frac{1}{\varepsilon}(H^\varepsilon(g))z]pdxdt$$

by integration by parts, (2.16), and (2.17). Using Proposition 2.4, we can complete the proof.

For the case of the cost functional (1.4), approximating by (2.18) and using the adjoint equation (2.19) with (2.17), one can establish the following result in a similar manner:

Corollary 2.7. *The directional derivative of the cost (2.18) is given by*

$$-\int_0^T\int_D (H^\varepsilon)'(g)[j(x,t,\widehat{y}_\varepsilon(x,t)) + \frac{1}{\varepsilon}\widehat{y}_\varepsilon p]wdxdt \quad \textit{for any } w \in \mathcal{F} \tag{2.21}$$

with p satisfying (2.19) and (2.17).

Remark 2.8. From Proposition 2.6 and Corollary 2.7, one can obtain the corresponding cost gradients in the respective optimization problems, that is, the quantities multiplying w in (2.20) and (2.21). The corresponding descent directions are obtained from the gradients, just by changing the sign. Notice as well that $(H^\varepsilon)'(g)$ is positive, pointwisely defined and, consequently, other (simplified) descent directions can be achieved by removing this factor.

By the properties of $H^\varepsilon(\cdot)$ explained before (2.10), we see that $\text{supp}(H^\varepsilon)'(g) \subset \{x \in D; -\varepsilon < g(x) < 0\}$, which is in an interior neighborhood of $\partial\Omega_g$. This remains true for the gradients obtained in (2.21) and, consequently, the classical gradient methods are practically related to boundary variations. However, the simplified descent directions, explained above, remove this limitation and allow topological variations as well.

We recall a general descent algorithm with projection that may be used in this setting. The parameter $\varepsilon > 0$ is fixed and we do not mention it anymore.

Algorithm 2.9

Step 1. Start with $n = 0$, select some initial g_n and some tolerance parameter $\delta > 0$.

Step 2. Compute y_n, the corresponding solution of (2.10) - (2.12) with $g := g_n$.
Step 3. Compute the solution p_n of the adjoint system (2.16), (2.17) or (2.17), (2.19) with $\widehat{y}_\varepsilon := y_n$.
Step 4. Compute the gradient $\nabla J(g_n)$ of the cost functional (2.21) or (2.20).
Step 5. Generate the descent direction w_n starting from the gradient in Step 4, as explained in Remark 2.8.
Step 6. $\widetilde{g_n} = g_n + \lambda_n w_n$, where λ_n is determined via a line search.
Step 7. $g_{n+1} = \mathrm{Proj}_{\mathcal{F}}(\widetilde{g_n})$, where $\mathcal{F}$ is the admissible control set. Here $\mathrm{Proj}_{\mathcal{F}}$ denotes the projection onto $\mathcal{F}$
Step 8. $n := n + 1$. If $|g_n - g_{n+1}| < \delta$ or $|\nabla J(g_n)| < \delta$, then STOP. Here $|\cdot|$ denotes some appropriate norm. If not, GO TO Step 1.

Recall that $\mathcal{F} \subset C(\overline{D})$ is assumed to be a closed cone. We may include various restrictions on g in its definition, corresponding to geometric constraints on the admissible shapes, as explained in the example below (2.9).

If $\mathcal{F} \subset C^1(\overline{D})$ and the set $\{x \in D;\, g(x) = 0\}$ consists just of non-critical points for g, then its measure is null (see [8]) and (2.9) can be equivalently replaced by

$$\Omega_g = \mathrm{Int}\,\{x \in D;\, g(x) \leq 0\}, \tag{2.22}$$

which may be considered as an alternative definition. Then, for the geometric constraint $E \subset \Omega_g$, we may choose $\mathcal{F} = \{g \in C^1(\overline{D});\, g(x) \leq 0 \text{ for } x \in E\}$ and $\mathcal{F}$ is a closed cone in $C^1(\overline{D})$, not necessarily convex due to the above condition on the null level sets of $g \in \mathcal{F}$. Moreover, for $\lambda > 0$, the perturbation property from Remark 2.5 (and Proposition 2.4) is valid in this setting (see [8]).

We end this section with another example of geometric constraint, that may be easily included in shape optimization problems, by using level functions. If Γ is some given manifold of arbitrary codimensions (it may reduce even to one point), then the constraint $\Gamma \subset \partial\Omega_g$ for any $g \in \mathcal{F}$ can be expressed as $g(x) = 0$, $x \in \Gamma$ for any $g \in \mathcal{F}$ under definition (2.22) and if the set $\{x \in D;\, g(x) = 0\}$ consists just of non-critical points for g. The perturbation property from Remark 2.5 (and Proposition 2.4) is valid in this case.

3 The controllability approach

In this section, we concentrate on the cost functional (1.6) and we indicate some specific parabolic arguments based on approximation and controllability properties (e.g., [4], [5]). We underline that, in principle, the geometric controllability properties used in the elliptic case [10] can be extended to the present situation, but this requires more regularity and we shall not follow this path.

For any admissible $\Omega \in \mathcal{O}$, we define

$$y_\Omega \ \text{ in } \ H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$$

by (1.1) - (1.3) and we compute the cost (1.6). We also consider the control system

$$y_t - \Delta y = \chi_\Omega f + \chi_{D\setminus\Omega} u \quad \text{in } D\times]0,T[, \tag{3.1}$$

$$y = 0 \quad \text{on } \partial D \times [0,T], \quad y(x,0) = \chi_\Omega y_0(x) \quad \text{in } D, \tag{3.2}$$

where D bounded, smooth, $D \supset \Omega$ for any $\Omega \in \mathcal{O}$, $f \in L^2(D\times]0,T[)$, $y_0 \in L^2(D)$ and $u \in L^2(D\times]0,T[)$.

It is known ([5], [4]) that for any $T > 0$, $\Omega \subset D$ and any $\varepsilon > 0$, there exists $u_\varepsilon \in L^2(D\times]0,T[)$ depending on T, Ω such that the corresponding solution y_ε of (3.1) and (3.2) satisfies

$$|y_\varepsilon(x,T) - \widetilde{y}_\Omega(x,T)|_{L^2(D)} < \varepsilon, \tag{3.3}$$

where $\widetilde{y}_\Omega(\cdot,T)$ is the zero extension of $y_\Omega(\cdot,T)$ to D.

Due to (3.3), as $\varepsilon \to 0$, we have $y_\varepsilon(\cdot,T) \longrightarrow \widetilde{y}(\cdot,T)$ strongly in $L^2(D)$, but the limit $\widetilde{y}$ may not be attained for any $u \in L^2(D\times]0,T[)$ and any $\Omega \in \mathcal{O}$. We remark that the continuity with respect to the time ([3], Chapter 7) of the weak solutions for the linear parabolic equations discussed here, gives a clear sense to the initial conditions and the properties concerning the final values of these solutions.

We introduce a constrained optimal control problem in D:

$$\min_{\Omega,u} \int_D \chi_\Omega j(x, y(x,T))dx \tag{3.4}$$

subject to (3.1), (3.2) and the state constraint

$$y(x,T) = 0 \quad \text{a. e. in } D \setminus \Omega. \tag{3.5}$$

Notice that $\Omega \in \mathcal{O}$ is a domain and, in this section, no regularity is assumed on $\partial\Omega$: in particular, the trace does not exist. The constraint (3.5) may be just interpreted here as a weak form of the condition $y_\Omega(x,T) = 0$ on $\partial\Omega$ according to (1.2). The cost functional (3.4) is equivalent to (1.6). A control $u \in L^2(D\times]0,T[)$ is called *admissible* if the solution y of (3.1) and

(3.2) satisfies (3.5). The corresponding solution y or the pair $[y, u]$ are also called *admissible* if (3.5) is satisfied. These definitions depend on Ω as well.

Remark 3.1. For any $\Omega \in \mathcal{O}$, the set of admissible controls $u \in L^2(D\times]0,T[)$ is nonvoid due to the null controllability property for linear parabolic equations in $D\times]0,T[$ ([5]).

If the constraint (3.5) is replaced by a weaker one:

$$-\varepsilon \leq y(x,T) \leq \varepsilon \quad \text{a.e., in } D \setminus \Omega, \tag{3.6}$$

then the approximate controllability property shows that the corresponding set of admissible controls $u \in L^2(D)$ is large.

We now use the classical penalization technique in (3.1), (3.2), (3.4) and we define the penalized problem:

$$\min_{u\in L^2(D\times]0,T[),\Omega\in\mathcal{O}} \left(\int_D \chi_\Omega j(x, y(x,T))dx + \frac{1}{\delta} \int_D \chi_{D\setminus\Omega} |y(x,T)|^2 dx \right) \tag{3.7}$$

subject to (3.1) and (3.2), where $\delta > 0$ is the penalization parameter.

Notice that in (3.1), (3.7), the unknown geometry $\Omega \in \mathcal{O}$ is still present via the characteristic functions, although the state system is defined in the fixed domain $D\times]0,T[$. This can be removed as in the previous section by introducing the second family of controls $g \in \mathcal{F}$ and imposing the admissible domains to be defined via (2.9), but we shall not pursue this way here. For fixed Ω, we just discuss the approximating properties of (3.1), (3.2) and (3.7) when $\delta \to 0$.
In fact, the infimum in (3.7) is not necessarily attained with respect to $u \in L^2(D\times]0,T[)$ since we have no coercivity property in u (Ω is fixed now). By $[y_\delta, u_\delta]$ we denote a δ-optimal pair for the penalized problem (i.e., satisfying (3.1) and (3.2) and with cost at δ distance from the infimum in (3.7) with respect to $u \in L^2(D\times]0,T[)$).

For any admissible pair $[\overline{y}, \overline{u}]$ for the control problem (3.1), (3.2), (3.4) and (3.5), we have

$$J(y_\delta, u_\delta) \leq inf J(y,u) + \delta \leq \int_D \chi_\Omega j(x, \overline{y}(x,T))dx + \delta, \tag{3.8}$$

since the penalization integral is null on $\overline{y}$. Here $J(\cdot,\cdot)$ denotes the penalization cost (3.7):

$$J(y,u) = \int_D \chi_\Omega j(x, y(x,T))dx + \frac{1}{\delta} \int_D \chi_{D\setminus\Omega} |y(x,T)|^2 dx.$$

Proposition 3.2. *If we assume that $j(\cdot,\cdot)$ is bounded from below by a constant, then*

$$|y_\delta(\cdot,T)|_{L^2(D\setminus\Omega)} \longrightarrow 0 \tag{3.9}$$

as $\delta \to 0$.

Relation (3.9) is a clear consequence of (3.7) and (3.8). This gives a constructive approach to the controllability properties (3.3), (3.5) or (3.6), but the obtained information is weaker. In order to approximately extend the shape optimization problem (1.1)-(1.3) with (1.6), we have to use the penalized cost

$$\min_{u\in L^2(D\times]0,T[),\Omega\in\mathcal{O}} \left(\int_D \chi_\Omega j(x,y(x,T))dx + \frac{1}{\delta}\int_0^T\int_D \chi_{D\setminus\Omega}|y(x,t)|^2dxdt\right), \tag{3.10}$$

in (3.1) and (3.2), and apply similar arguments as in Proposition 3.2. The controllability property from Remark 3.1 is to be replaced by the following observation: one can assume that by a regularization of the extension by 0 of y_Ω, the penalization term in (3.10) remains bounded. See Remark 3.5 as well. If we denote again by $[y_\delta(x,t),u_\delta(x,t)]$ a δ-optimal pair for (3.10), (3.1) and (3.2), then we obtain $|y_\delta|_{L^2((D\setminus\Omega)\times[0,T])} \longrightarrow 0$ in addition to (3.9), which is again a weak approximation property.

Alternatively, let us now simply associate with the control system (3.1) and (3.2) the following cost

$$\min \frac{1}{2}\int_0^T\int_{D\setminus\Omega} y(x,t)^2dxdt, \tag{3.11}$$

which is just the penalization term in (3.10). Clearly, the infimum in (3.11), under conditions (3.1) and (3.2), is 0. See Remark 3.5. If some optimal pair $[y^*,u^*]\in L^2(0,T;H_0^1(D))\times L^2(D\times]0,T[)$ exists (which is not guaranteed in general), then $y^*(x,t)=0$ a.e. in $(D\setminus\Omega)\times[0,T]$. Assuming Ω to be of class C, the Hedberg-Keldysh stability property [10] shows (as in the previous section) that $y^*|_\Omega \in L^2(0,T;H_0^1(\Omega))$, that is, $y^*|_\Omega$ is the solution to (1.1) - (1.3) and u^* ensures an exact geometric controllability property (the exact extension of (1.1) - (1.3) to D).

Since this ideal situation is not valid in general, we introduce the Tikhonov regularization of (3.11), (3.1) and (3.2):

$$\min \frac{1}{2}\left(\int_0^T\left(\int_{D\setminus\Omega}(|y(x,t)|^2+\delta|u(x,t)|^2)dx\right)dt\right) \tag{3.12}$$

subject to (3.1) and (3.2). This approximating problem possesses a unique optimal pair $[\widehat{y}_\delta, \widehat{u}_\delta]$ in $[L^2(0,T:H_0^1(D))\cap H^1(0,T;H^{-1}(D))]\times L^2(D\times]0,T[)$.

Proposition 3.3. *The optimality conditions for the regularized problem (3.12) are given by (3.1), (3.2), the adjoint system*

$$-p_t - \Delta p = \chi_{D\setminus\Omega}\widehat{y}_\delta \quad \textit{in } D\times]0,T[$$

$$p|_{\partial D\times]0,T[} = 0, \quad p(x,T) = 0 \quad \textit{in } D$$

and the maximum principle

$$p + \delta\widehat{u}_\delta = 0 \quad \textit{in } (D\setminus\Omega)\times]0,T[.$$

Proof. The equation in variation is

$$z_t - \Delta z = \chi_{D\setminus\Omega} v \quad \text{in } D\times]0,T[, \tag{3.13}$$

$$z|_{\partial D\times]0,T[} = 0, \quad z(x,0) = 0 \quad \text{in } D. \tag{3.14}$$

One can write the adjoint system for (3.14) and (3.13):

$$-p_t - \Delta p = \chi_{D\setminus\Omega}\widehat{y}_\delta \quad \text{in } D\times]0,T[, \tag{3.15}$$

$$p|_{\partial D\times]0,T[} = 0, \quad p(x,T) = 0 \quad \text{in } D. \tag{3.16}$$

Take variations $[\widehat{y}_\delta, \widehat{u}_\delta] + \lambda[z,v]$ with $\lambda \in \mathbb{R}$ and $[z,v]$ satisfying (3.14) and (3.13). By using (3.13) - (3.16), the optimality of $[\widehat{y}_\delta, \widehat{u}_\delta]$ and integration by parts, simple calculations give

$$\begin{aligned}
0 =& \int_0^T \int_{D\setminus\Omega} \widehat{y}_\delta z dxdt + \delta \int_0^T \int_{D\setminus\Omega} \widehat{u}_\delta v dxdt \\
=&\delta \int_0^T \int_{D\setminus\Omega} \widehat{u}_\delta v dxdt - \int_0^T \int_D (p_t + \Delta p) z dxdt \\
=&\delta \int_0^T \int_{D\setminus\Omega} \widehat{u}_\delta v dxdt + \int_0^T \int_D (z_t - \Delta z) p dxdt \\
=& \int_0^T \int_{D\setminus\Omega} (p + \delta\widehat{u}_\delta) v dxdt.
\end{aligned}$$

As $v \in L^2((D\setminus\Omega)\times]0,T[)$ is arbitrary, the proof is finished.

Remark 3.4. We can eliminate $\widehat{u}_\delta$ in the optimality system given by Proposition 3.3 and obtain

$$y_t - \Delta y = \chi_\Omega f - \frac{1}{\delta}\chi_{D\setminus\Omega} p \quad \text{in } D\times]0,T[, \tag{3.17}$$

and

$$-p_t - \Delta p = \chi_{D\backslash\Omega} y \quad \text{in } D\times]0,T[\tag{3.18}$$

with initial and boundary conditions, which can be understood as an approximating extended system of (1.1)-(1.3) from Ω to D.

Take $u = 0$ in the regularized problem (3.1), (3.2) and (3.12) and by y_f denote the corresponding state. We have the inequality

$$\frac{1}{2}\int_0^T \int_{D\backslash\Omega} |\widehat{y}_\delta|^2 dxdt + \frac{\delta}{2}\int_0^T \int_{D\backslash\Omega} |\widehat{u}_\delta|^2 dxdt \le \frac{1}{2}\int_0^T \int_{D\backslash\Omega} |y_f|^2 dxdt.$$

Hence we see that $\{\widehat{y}_\delta\}$ and $\{\delta^{\frac{1}{2}}\widehat{u}_\delta\}$ are bounded in $L^2((D\setminus\Omega)\times]0,T[)$. We denote now the solution of (3.16) and (3.15) by $\{p_\delta\}$ (it depends in fact on δ). Consequently, $\{p_\delta\}$ is bounded in $L^2(0,T;H_0^1(D)) \cap H^1(0,T;H^{-1}(D))$. Moreover the maximum principle yields

$$|p_\delta|_{L^2((D\backslash\Omega)\times]0,T[)} \le C\delta^{\frac{1}{2}}.$$

We know that $\{\widehat{y}_\delta\}$ has some weak limit $\widehat{y}$ in $L^2((D\setminus\Omega)\times]0,T[)$ by taking a subsequence. By (3.15), (3.16) and the above convergence of $\{p_\delta\}$, we obtain $\widehat{y} = 0$ a.e. in $(D\setminus\Omega)\times]0,T[$. Consequently, the system (3.17) - (3.18) in D with the conditions (3.2) and (3.16), achieves an approximation of (1.1)-(1.3) in the above weak sense.

Remark 3.5. By taking above convex combinations of $\{\widehat{y}_\delta\}$ one may obtain convergence in the strong topology. In the system (3.17) - (3.18) with the conditions (3.2) and (3.16), it is not possible to introduce such convex combinations since they correspond to different values of δ. However, such a convex combination of $\delta^{-1}p_\delta$ satisfies as a distributed control the properties associated to the penalized cost functionals (3.10), respectively (3.7), associated to the state system (3.1), (3.2).

4 Conclusion

The range of shape optimization problems associated to evolution systems is very rich and the cases considered here or in [14], [15] give just an introduction to the subject. We underline that other state systems (hyperbolic, nonlinear, higher order) or cost functionals (for instance, defined on the boundary of $\Omega \in \mathcal{O}$) may be taken into account instead of (1.1) - (1.6). As questions of interest, beside the sensitivity analysis or the approximation methods already discussed in the existing scientific literature, a study of further discretization procedures is necessary, especially in the context of various significant applications in engineering problems. Moreover, a sharpening of the approximation results based on controllability/control arguments is needed in order to justify the relationship with shape optimization problems. It should be underlined that such applied problems in the evolution case are at least three dimensional (time and space dimension two), which is a clear hint of the high difficulties to be expected from the computational point of view.

In realistic problems, more constraints appear on the state (for instance, positivity or box constraints) or on the unknown geometry. In this respect, using the functional variations approach [11] from Section 2 (see (2.9), (2.22)), it is easy to express geometric conditions on the boundary of the unknown domains in an analytic way. For state constraints, classical penalization/regularization approaches are useful ([10]).

Another topic of fundamental interest is the existence of solutions for shape optimization problems in the parabolic case. The formalism from Section 2 is useful in this direction too. From [8] we recall the following property: if for any $g \in \mathcal{F} \subset C^1(\overline{D})$, on the set $\{x \in D;\, g(x) = 0\}$ we have $\nabla g(x) \neq 0$ and $g(x) > 0$, $x \in \partial D$, then the set $\{x \in D;\, g(x) = 0\}$ is a finite union of disjoint closed curves which are not self intersecting and not intersecting ∂D, in space dimension two. In particular, for any $g \in \mathcal{F}$, the corresponding Ω_g may have just a finite number of holes. The definitions (2.9) and (2.22) are equivalent under the above conditions. The presence of a hole in Ω_g is characterized by a change of sign of g at the border of the hole, since the normal derivative of g is not null in these points. These changes of signs are sharp since $\{x \in D;\, g(x) = 0\}$ is of measure zero. Moreover their number is finite for any $g \in \mathcal{F}$, as discussed above.

In this context, the famous Sverak compactness condition ([16], [10]) reads that there is some natural number m such that, for any $g \in \mathcal{F}$, the number of signs changes is bounded above by m. In the elliptic case, in dimension two, it is known that this property is sufficient for the passage to the limit (with respect to the domains) in the state system and the existence of optimal solutions. We conjecture that this property can be extended to parabolic shape optimization problems in spatial dimension two. Due to the non convex character of shape optimization problems (the dependence domain - solution is strongly nonlinear even for linear state systems), the uniqueness is not valid for shape optimization problems, with rare exceptions (e.g., [7], [10]).

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