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ON SOME MOTIONS OF SECOND GRADE FLUIDS INDUCED BY A SPHERE THAT APPLIES OSCILLATING SHEAR STRESSES TO THE FLUID

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Abstract. *Exact solutions for the laminar basic flow of second grade fluids due to a sphere that applies oscillating shear stresses to the fluid are presented as a sum of permanent and transient solutions. The corresponding solutions for Newtonian fluids, as expected, are obtained as limiting cases of general solutions and the required time to reach the steady-state is graphically determined. This time is very small for both type of oscillating motions. Consequently, the steady-state or permanent solutions corresponding to such motions are most important.*

Keywords: *Exact solutions; Oscillating shear stresses on a sphere; Second grade fluids*.

1. Introduction

Flows near spinning bodies are of interest both for theory and practice. Their direct applications to centrifugal pumps and lubrication problems justify the growing interest for motions near rotating bodies [1]. Such flows also appear on a large scale in meteorology and astrophysics and more elementary forms of motion around a sphere or between two concentric spheres have been investigated. The first approximate solutions for the motion of a non-Newtonian fluid due to an oscillating sphere seem to be those of Frater [2]. Later, Zierep and Kulman [3] studied the motion of Newtonian fluids inside a rotating sphere and established a closed-form solution for its primary component. Other exact solutions for motions of Newtonian or second grade fluids in a sphere have been obtained by Zierep [4],

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respectively Fetecau and Zierep [5]. However, all these solutions correspond to motions induced by a moving sphere when the velocity is given on the boundary.

In some physical situations, contrary to what is usually assumed, the force with which the sphere is moved can be prescribed. To reiterate, in Newtonian mechanics force is the cause and kinematics is the effect [6]. Prescribing the shear stress on the sphere surface is tantamount to prescribe the (shear) force applied to move it and Renardy [7] showed how to formulate a well-posed shear stress boundary-value problem. Unfortunately, exact solutions corresponding to such motions in spherical domains are lack in the existing literature.

The main purpose of this work is to provide some exact solutions for motions of second grade fluids due to a sphere that applies oscillatory shear stresses to the fluid. These solutions, presented in simple forms in terms of some standard or modified Bessel functions, can be easy reduced to the similar solutions for Newtonian fluids. They correspond to laminar basic flows which are symmetric with respect to the equator plane.

2. Constitutive and governing equations

The Cauchy stress tensor for an incompressible second grade fluid is [5, 8]

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1}$$

where $-p\mathbf{I}$ denotes the indeterminate spherical stress, μ is the coefficient of viscosity, α_1 and α_2 are the normal stress moduli while $\mathbf{A}_1$ and $\mathbf{A}_2$ are kinematic tensors defined by

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T, \quad \mathbf{A}_2 = \frac{d}{dt}\mathbf{A}_1 + \mathbf{A}_1\mathbf{L} + \mathbf{L}^T\mathbf{A}_1, \tag{2}$$

Here $\mathbf{L}$ is the velocity gradient, d/dt denotes the material time derivative and the superscript 'T' indicates the transpose operation.

In the following we consider a unsteady unidirectional motion whose velocity field $\mathbf{v}$ in the spherical coordinate system r, θ and φ has the form [8, Eq. (2.11)]

$$\mathbf{v} = \mathrm{v}(r,t)\sin\theta\,\mathbf{e}_\varphi, \tag{3}$$

where $\mathbf{e}_\varphi$ is the unit vector in the φ-direction. For such motions, the constraint of incompressibility is identically satisfied while [9, Anhang B]

$$\mathbf{L} = \begin{pmatrix} 0 & 0 & -\frac{1}{r}\mathrm{v}(r,t)\sin\theta \\ 0 & 0 & -\frac{1}{r}\mathrm{v}(r,t)\cos\theta \\ \frac{\partial \mathrm{v}(r,t)}{\partial r}\sin\theta & \frac{1}{r}\mathrm{v}(r,t)\cos\theta & 0 \end{pmatrix}. \tag{4}$$

Introducing Eq. (4) into (2) in order to obtain $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_1^2$ and using them in Eq. (1), it results that the non-trivial shear stress $\tau_{r\varphi}(r,\theta,t)$ is given by

$$\tau_{r\varphi}(r,\theta,t) = \tau(r,t)\sin\theta, \tag{5}$$

where

$$\tau(r,t) = \left(\mu + \alpha_1 \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\mathrm{v}(r,t). \tag{6}$$

As the shear stress $\tau_{\theta\varphi} = 0$, the balance of linear momentum yields

$$\rho \frac{\partial \mathrm{v}(r,t)}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{3}{r}\right)\tau(r,t), \tag{7}$$

where ρ is the constant density of the fluid. Into above equation $\partial p/\partial\varphi$ is zero due to the rotational symmetry and the body force has been neglected.

In order to solve a problem with shear stress on the boundary, we eliminate the velocity $\mathrm{v}(r,t)$ between Eqs. (6) and (7) and find that

$$\frac{\partial \tau(r,t)}{\partial t} = \left(\nu + \alpha \frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{6}{r^2}\right)\tau(r,t), \tag{8}$$

where $\nu = \mu/\rho$ is the kinematic viscosity and $\alpha = \alpha_1/\rho$. Making the change of unknown function

$$\tau(r,t) = r^{-1/2}\tau^*(r,t) \tag{9}$$

and dropping out the star notation, we attain to the more suitable partial differential equation

$$\frac{\partial \tau(r,t)}{\partial t} = \left(\nu + \alpha \frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{25}{4r^2}\right)\tau(r,t). \tag{10}$$

3. Starting solutions for oscillating motions inside a sphere

Let us consider an incompressible second grade fluid at rest into a sphere of radius R. At time $t = 0^+$ the sphere begins to oscillate about the axis $\theta = 0$ due to an oscillating torque $2\pi Rf\sin(\omega t)\sin(\theta)$ or $2\pi Rf\cos(\omega t)\sin(\theta)$. Owing to the shear the fluid is moved and its velocity has the form (3). More exactly, the fluid is deflected away in a thin radial jet at the equator and sucked back near the poles, producing a meridional flow component [1]. Because of the rotation, the fluid moves in spirals from poles to the equator and for latitudes smaller than 40° no turbulent flow could be detected. Furthermore, beyond 30° the flow was always laminar for all Reynolds numbers that have been tested [10].

The governing equation for the shear stress $\tau(r,t)$ is given by Eq. (10) and the corresponding initial and boundary conditions are

$$\tau(r,0)=0; \quad r\in[0,R), \tag{11}$$

$$\tau(R,t)=f\sqrt{R}H(t)\sin(\omega t) \quad \text{or} \quad \tau(R,t)=f\sqrt{R}H(t)\cos(\omega t), \tag{12}$$

where $H(t)$ is the Heaviside unit step function and f and ω are the amplitude, respectively, the frequency of the oscillations. The natural condition

$$|\tau(0,t)|<\infty, \tag{13}$$

has to be also satisfied.

In order to solve this problem we shall use the finite Hankel transform

$$g_J(r_n)=\int_0^R rg(r)J_{5/2}(r\,r_n)\mathrm{d}r, \tag{14}$$

of $g(r)$ and the identity [11, Sec. 14, Eq. (59)]

$$\int_0^R rJ_{5/2}(r\,r_n)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{25}{4r^2}\right)g(r)\mathrm{d}r=-Rr_n g(R)\mathrm{J}'_{5/2}(R\,r_n)-r_n^2 g_J(r_n). \tag{15}$$

If $g(r)$ satisfies Dirichlet's conditions in the interval $(0,R)$, then at any point of $(0,R)$ where this function is continuous

$$g(r)=\frac{2}{R^2}\sum_{n=1}^{\infty} g_J(r_n)\frac{\mathrm{J}_{5/2}(r\,r_n)}{[\mathrm{J}'_{5/2}(R\,r_n)]^2}, \tag{16}$$

where the sum is taken over all positive roots of the transcendental equation

$$\mathrm{J}_{5/2}(R\,r)=0. \tag{17}$$

Consequently, multiplying Eq. (10) by $r\mathrm{J}_{5/2}(r\,r_n)$, integrating the result with respect to r from 0 to R and bearing in mind the identity (15) and the boundary conditions (12), we find that

$$\begin{aligned}&\frac{\mathrm{d}\tau_{Js}(r_n,t)}{\mathrm{d}t}+\frac{\nu r_n^2}{1+\alpha r_n^2}\tau_{Js}(r_n,t)+fR^{3/2}\frac{r_n\mathrm{J}'_{5/2}(R\,r_n)}{1+\alpha r_n^2}\big[\nu H(t)\sin(\omega t)\\&+\alpha\omega H(t)\cos(\omega t)+\alpha\delta(t)\sin(\omega t)\big]=0; \quad n=1,2,3,...,\end{aligned} \tag{18}$$

$$\begin{aligned}&\frac{\mathrm{d}\tau_{Jc}(r_n,t)}{\mathrm{d}t}+\frac{\nu r_n^2}{1+\alpha r_n^2}\tau_{Jc}(r_n,t)+fR^{3/2}\frac{r_n\mathrm{J}'_{5/2}(R\,r_n)}{1+\alpha r_n^2}\big[\nu H(t)\cos(\omega t)\\&-\alpha\omega H(t)\sin(\omega t)+\alpha\delta(t)\cos(\omega t)\big]=0; \quad n=1,2,3,...,\end{aligned} \tag{19}$$

where $\delta(.)$ is the Dirac delta function and the finite Hankel transforms $\tau_{Js}(r_n,t)$ and $\tau_{Jc}(r_n,t)$ of the problem solutions satisfy the initial conditions

$$\tau_{Js}(r_n,0)=0, \quad \tau_{Jc}(r_n,0)=0 \quad \text{for} \quad n=1,2,3,... \tag{20}$$

Solving the linear ordinary differential equations (18) and (19) with the boundary conditions (20) and using the inversion formula (16), we find that

$$\tau_s(r,t)=\frac{2f}{\sqrt{R}}H(t)\sum_{n=1}^{\infty}\frac{1}{\nu^2r_n^4+\omega^2(1+\alpha r_n^2)^2}\frac{r_n\mathrm{J}_{5/2}(r r_n)}{\mathrm{J}_{3/2}(R r_n)}$$
$$\times\left\{\nu\omega\cos(\omega t)-\left[\nu^2r_n^2+\alpha\omega^2(1+\alpha r_n^2)\right]\sin(\omega t)-\nu\omega\exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right)\right\}, \tag{21}$$

$$\tau_c(r,t)=\frac{2f}{\sqrt{R}}H(t)\sum_{n=1}^{\infty}\frac{1}{\nu^2r_n^4+\omega^2(1+\alpha r_n^2)^2}\frac{r_n\mathrm{J}_{5/2}(r r_n)}{\mathrm{J}_{3/2}(R r_n)}$$
$$\times\left\{-\left[\nu^2r_n^2+\alpha\omega^2(1+\alpha r_n^2)\right]\cos(\omega t)-\nu\omega\sin(\omega t)+\frac{\nu^2r_n^2}{1+\alpha r_n^2}\exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right)\right\}. \tag{22}$$

In order to obtain the starting solutions (21) and (22), we also used the known results

$$\mathrm{J}'_{5/2}(R r_n)=\mathrm{J}_{3/2}(R r_n)\quad\text{if}\quad \mathrm{J}_{5/2}(R r_n)=0\quad\text{and}$$
$$(\delta * g)(t)=\int_0^t\delta(t-s)\mathrm{g}(s)\mathrm{d}s=\int_0^t\delta(s)\mathrm{g}(t-s)\mathrm{d}s=g(t). \tag{23}$$

The starting solutions (21) and (22) describe the motion of the fluid some time after its initiation. After that time, when the transients disappear, the motion of fluid is described by the steady-state (permanent) solutions

$$\tau_{sp}(r,t)=-\frac{2f}{\sqrt{R}}\sin(\omega t)\sum_{n=1}^{\infty}\frac{\nu^2r_n^2+\alpha\omega^2(1+\alpha r_n^2)}{\nu^2r_n^4+\omega^2(1+\alpha r_n^2)^2}\frac{r_n\mathrm{J}_{5/2}(r r_n)}{\mathrm{J}_{3/2}(R r_n)}$$
$$+\frac{2f}{\sqrt{R}}\nu\omega\cos(\omega t)\sum_{n=1}^{\infty}\frac{1}{\nu^2r_n^4+\omega^2(1+\alpha r_n^2)^2}\frac{r_n\mathrm{J}_{5/2}(r r_n)}{\mathrm{J}_{3/2}(R r_n)}, \tag{24}$$

$$\tau_{cp}(r,t)=-\frac{2f}{\sqrt{R}}\cos(\omega t)\sum_{n=1}^{\infty}\frac{\nu^2r_n^2+\alpha\omega^2(1+\alpha r_n^2)}{\nu^2r_n^4+\omega^2(1+\alpha r_n^2)^2}\frac{r_n\mathrm{J}_{5/2}(r r_n)}{\mathrm{J}_{3/2}(R r_n)}$$
$$-\frac{2f}{\sqrt{R}}\nu\omega\sin(\omega t)\sum_{n=1}^{\infty}\frac{1}{\nu^2r_n^4+\omega^2(1+\alpha r_n^2)^2}\frac{r_n\mathrm{J}_{5/2}(r r_n)}{\mathrm{J}_{3/2}(R r_n)}, \tag{25}$$

which are periodic in time and independent of the initial conditions but satisfy the governing equation and the boundary conditions. However, in this form, the solutions (24) and (25) seem to not satisfy the boundary conditions. In order to take away this drawback we use the result (see [11] the entry 2 of Table X)

$$r^{5/2}=-2R^{3/2}\sum_{n=1}^{\infty}\frac{\mathrm{J}_{5/2}(r r_n)}{r_n\mathrm{J}_{3/2}(R r_n)}\quad if\quad \mathrm{J}_{5/2}(R r_n)=0 \tag{26}$$

and write them in the equivalent forms

$$\tau_{sp}(r,t)=f\frac{r^{5/2}}{R^2}\sin(\omega t)$$
$$+\frac{2f}{\sqrt{R}}\left(\frac{\omega}{\nu}\right)^2\sin(\omega t)\sum_{n=1}^{\infty}\frac{1+\alpha r_n^2}{r_n[r_n^4+(\omega/\nu)^2(1+\alpha r_n^2)^2]}\frac{J_{5/2}(r r_n)}{J_{3/2}(R r_n)} \tag{27}$$
$$+\frac{2f}{\sqrt{R}}\frac{\omega}{\nu}\cos(\omega t)\sum_{n=1}^{\infty}\frac{r_n}{r_n^4+(\omega/\nu)^2(1+\alpha r_n^2)^2}\frac{J_{5/2}(r r_n)}{J_{3/2}(R r_n)},$$

$$\tau_{cp}(r,t)=f\frac{r^{5/2}}{R^2}\cos(\omega t)$$
$$+\frac{2f}{\sqrt{R}}\left(\frac{\omega}{\nu}\right)^2\cos(\omega t)\sum_{n=1}^{\infty}\frac{1+\alpha r_n^2}{r_n[r_n^4+(\omega/\nu)^2(1+\alpha r_n^2)^2]}\frac{J_{5/2}(r r_n)}{J_{3/2}(R r_n)} \tag{28}$$
$$-\frac{2f}{\sqrt{R}}\frac{\omega}{\nu}\sin(\omega t)\sum_{n=1}^{\infty}\frac{r_n}{r_n^4+(\omega/\nu)^2(1+\alpha r_n^2)^2}\frac{J_{5/2}(r r_n)}{J_{3/2}(R r_n)}.$$

The corresponding velocity fields can be immediately obtained by introducing Eqs. (21) and (22) into Eq. (7), integrating with respect to the temporal variable from 0 to t and using the initial condition

$$\mathrm{v}(R,0)=0;\quad r\in[0,R). \tag{29}$$

It is worth pointing out that making $\alpha=0$ into Eqs. (21), (22), (27) and (28), the solutions corresponding to Newtonian fluids performing the same motions are obtained. The starting solutions (21) and (22), for instance, take the simple forms

$$\tau_{sN}(r,t)=\frac{2fH(t)}{\sqrt{R}}\sum_{n=1}^{\infty}\frac{1}{r_n^4+(\omega/\nu)^2}\left\{\frac{\omega}{\nu}\cos(\omega t)-r_n^2\sin(\omega t)-\frac{\omega}{\nu}e^{-\nu r_n^2 t}\right\}$$
$$\times\frac{r_n J_{5/2}(r r_n)}{J_{3/2}(R r_n)}, \tag{30}$$

$$\tau_{cN}(r,t)=\frac{2fH(t)}{\sqrt{R}}\sum_{n=1}^{\infty}\frac{1}{r_n^4+(\omega/\nu)^2}\left\{-\frac{\omega}{\nu}\sin(\omega t)-r_n^2\cos(\omega t)+r_n^2 e^{-\nu r_n^2 t}\right\}$$
$$\times\frac{r_n J_{5/2}(r r_n)}{J_{3/2}(R r_n)}. \tag{31}$$

Furthermore, by now letting $\omega=0$ into Eqs. (22), (25),(28) and (31), the solutions corresponding to the motion induced by a sphere that applies a torque $\tau_{r\varphi}(R,\theta,\varphi)=2\pi Rf\sin\theta$ (that is constant in time) to the fluid are obtained. Eq. (22), for instance, becomes (see again Eq. (26))

$$\tau(r,t)=\frac{f}{\sqrt{R}}H(t)\left[\frac{r^{5/2}}{R^{3/2}}+2\sum_{n-1}^{\infty}\frac{1}{1+\alpha r_n^2}\frac{J_{5/2}(r r_n)}{r_n J_{3/2}(R r_n)}\exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right)\right]. \tag{32}$$

Generally, the starting solutions for unsteady motions of fluids are important for those who want to eliminate the transients from their experiments. Consequently, an important problem regarding the technical relevance of these solutions is to find the required time to reach the steady-state. More exactly, to determine the time after which the fluid moves according to the permanent solutions. This time, as it results from Figs. 1 and 2, is very small for both motions induced by sine or cosine oscillations of the shear stress on the boundary. Indeed, with an error of 10^{-6} order, the profiles of starting shear stresses (21) and (22) are almost identical to those corresponding to the permanent solutions (24) and (25). Consequently, in such motions, the permanent solutions are very important.

4. Permanent solutions for motions around a sphere

Let us now assume that a solid sphere of radius R, immersed in an incompressible second grade fluid, applies the same oscillatory torques to the fluid. Due to the shear the fluid around the sphere is gradually moved and the meridional component of velocity has again the same form (3). The governing equation for the shear stress $\tau(r,t)$ is given by Eq. (10) and the corresponding boundary conditions for permanent motions are

$$\tau_p(R,t)=f\sqrt{R}\sin(\omega t) \quad \text{or} \quad \tau_p(R,t)=f\sqrt{R}\cos(\omega t). \tag{33}$$

Denoting by $\tau_{sp}(r,t)$ and $\tau_{cp}(r,t)$ the steady-state solutions corresponding to the two distinct motions and by

$$T_p(r,t)=\tau_{cp}(r,t)+i\,\tau_{sp}(r,t), \tag{34}$$

the complex shear stress, it results that

$$\frac{\partial T_p(r,t)}{\partial t}=\left(\nu+\alpha\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{25}{4r^2}\right)T_p(r,t). \tag{35}$$

The corresponding boundary condition is

$$T_p(r,t)=f\sqrt{R}e^{i\omega t}. \tag{36}$$

Furthermore, we assume that the fluid is quiescent at infinity. Consequently, there is no shear in the free stream and

$$\tau_p(r,t)\to 0 \quad \text{as} \quad r\to\infty. \tag{37}$$

In order to solve this problem, we seek a separable solution

$$T_p(r,t)=G(r)e^{i\omega t}. \tag{38}$$

By substituting Eq. (38) into (35), we find that

$$G''(r)+\frac{1}{r}G'(r)-\left(\frac{25}{4r^2}+\frac{i\omega}{\nu+i\alpha\omega}\right)G(r)=0, \tag{39}$$

where the prime denotes differentiation. Introducing a new variable

$$s=r\sqrt{\gamma};\quad \gamma=i\omega/(\nu+i\alpha\omega), \tag{40}$$

Eq. (39) takes the form of a Bessel equation, namely

$$s^2\frac{\mathrm{d}^2G(s)}{\mathrm{d}s^2}+s\frac{\mathrm{d}G(s)}{\mathrm{d}s}-\left(\frac{25}{4}+s^2\right)G(s)=0. \tag{41}$$

The general solution of Eq. (41) is

$$G(s)=C_1\mathrm{I}_{5/2}(s)+C_2\mathrm{K}_{5/2}(s), \tag{42}$$

where C_1 and C_2 are arbitrary constants and $\mathrm{I}_{5/2}(\cdot)$, $\mathrm{K}_{5/2}(\cdot)$ are modified functions of the first and second kind of order 5/2. Bearing in mind Eqs. (34), (38), the boundary condition (36) as well as the natural condition (37), it results that

$$\begin{aligned}\tau_{cp}(r,t)&=f\sqrt{R}\,\mathrm{Re}\left\{\frac{\mathrm{K}_{5/2}(r\sqrt{\gamma})}{\mathrm{K}_{5/2}(R\sqrt{\gamma})}\mathrm{e}^{i\omega t}\right\},\\ \tau_{sp}(r,t)&=f\sqrt{R}\,\mathrm{Im}\left\{\frac{\mathrm{K}_{5/2}(r\sqrt{\gamma})}{\mathrm{K}_{5/2}(R\sqrt{\gamma})}\mathrm{e}^{i\omega t}\right\},\end{aligned} \tag{43}$$

where Re and Im denote the real part, respectively the imaginary part of the complex number that follows.

The velocity fields corresponding to these motions, namely,

$$\begin{aligned}\mathrm{v}_{cp}(r,t)&=\frac{f}{\rho\omega}\sqrt{\frac{R}{r}}\,\mathrm{Re}\left\{\frac{\sqrt{\gamma}\mathrm{K}_{3/2}(r\sqrt{\gamma})}{\mathrm{K}_{5/2}(R\sqrt{\gamma})}\mathrm{e}^{i(\omega t-\pi/2)}\right\},\\ \mathrm{v}_{sp}(r,t)&=\frac{f}{\rho\omega}\sqrt{\frac{R}{r}}\,\mathrm{Im}\left\{\frac{\sqrt{\gamma}\mathrm{K}_{3/2}(r\sqrt{\gamma})}{\mathrm{K}_{5/2}(R\sqrt{\gamma})}\mathrm{e}^{i(\omega t-\pi/2)}\right\},\end{aligned} \tag{44}$$

are obtained by introducing Eqs. (43) into (9) and using Eq. (7) and the identity [12, chapter VIII, Eq. (7.7)]

$$x\mathrm{K}'_{5/2}(x)+\frac{5}{2}\mathrm{K}_{5/2}(x)=x\mathrm{K}_{3/2}(x). \tag{45}$$

5. Conclusions

The motion of a second grade fluid inside a sphere that applies oscillatory shear stresses to the fluid is studied by means of the finite Hankel transform. More exactly, exact expressions are determined for the non-trivial shear stress corresponding to the meridional flow component of velocity. The solutions that have been obtained, presented as a sum of steady-state and transient components, describe the motion of the fluid at least for latitudes smaller than 40° where turbulent flows couldn't be detected [10]. They satisfy all imposed initial and boundary conditions and can be immediately reduced to the similar solutions for Newtonian fluids. The corresponding velocity of the fluid can be easy obtained by means of a simple integration of the motion equation. Further, the shear stress corresponding to the motion induced by a sphere that applies a constant shear to the fluid for each θ is obtained as a limiting case of one of our solutions.

The starting solutions for unsteady motions of fluids are useful for those who want to eliminate the transients from their rheological experiments. They describe the motion of the fluid some time after its initiation. After that time when the transients disappear, the fluid moves according to the steady-state solutions that are periodic in time and independent of the initial conditions. Consequently, an important problem regarding the technical relevance of starting solutions is to determine the required time to reach the steady state. This time, as it clearly results from Figs. 1 and 2, is very small for both types of oscillating motions.

In conclusion, the steady-state or permanent solutions corresponding to oscillating motions due to a sphere that applies oscillating shear stresses to the fluid are of great importance. This is the reason that, for oscillating motions around a sphere, only the permanent solutions have been determined. Finally, for completion, we mention that the permanent solutions corresponding to oscillating motions through a sphere can also be written under the simple forms (43) and (44) with $\mathrm{I}_{5/2}(\cdot)$ and $\mathrm{I}_{3/2}(\cdot)$ instead of $\mathrm{K}_{5/2}(\cdot)$, respectively $\mathrm{K}_{3/2}(\cdot)$.

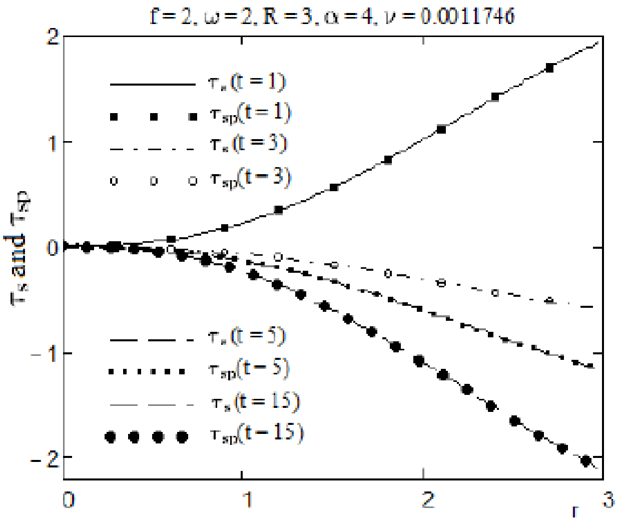

Fig. 1. Profiles of starting and permanent shear stresses

(τ_s and τ_{sp}) given by Eqs. (21) and (24).

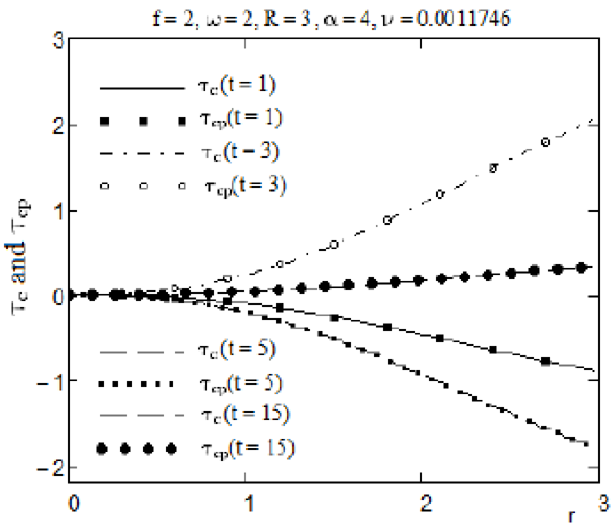

Fig. 2. Profiles of starting and permanent shear stresses (τ_c and τ_{cp}) given by Eqs. (22) and (25).

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