NON-LINEAR BEHAVIORS IN THE DYNAMICS OF COMPLEX SYSTEMS WITH POTENTIAL ECONOMY APPLICATION. QUALITATIVE ANALYSIS FROM MULTI-FRACTAL PERSPECTIVE

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Abstract: *In a Schrödinger-type and Madelung-type scenarios for the description of complex economics system dynamics, SL(2R) symmetries are highlighted. The emergence of such symmetries has several consequences: the existence of analogic-type behavior as a gauge invariance of Riccati type as well as the existence of digital-type behavior through the spontaneous symmetry breaking of the same gauge invariance.*

When said symmetries are discussed in the context of economics dynamics, the individual reaction to market signals can be associated to period doubling and modulated dynamics (i.e. to the digital signals) while, the behaviors of large investors and of the State, through banking or monetary policies, can associated to the "complex economics system background" (i.e. analogical signals).

Moreover, the markets have a fractal/multi-fractal structure on the long term, being characterized by a "self-memory". The economic structures emphasize fluctuations but, they never reach the chaos state. Thus, a holographic approach on complex economics system dynamics (and, on economics complex economics systems) provides a valid and more natural perspective, compared to the standard approaches. Our research provides a qualitative insight of economics complex system dynamics, remaining a more rigorous study which reveals a quantitative analysis of financial fractal bubbles to be done in further research.

Keywords: Scale Relativity Theory, multifractal, Schrödinger type scenario, Madelung type scenario, Riccati type gauge

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1. Introduction

The usual physical models used in describing the dynamics of complex economics systems are based on the hypothesis of the differentiability of the econo-physical quantities used to describe their evolution. As a consequence, the validity of these models must be understood gradually, in areas where differentiability and integrability are still functional [1-5]. However, when discussing nonlinearity and chaoticity in the dynamics of complex economics systems, differentiable and integrable mathematical procedures are of little use. Therefore, in order to properly describe the dynamics of complex economics systems, it is necessary to introduce the scale resolution both into the expressions of the physical variables as well as into the expressions of the fundamental equations governing these dynamics [6-9].

Accepting the above affirmation, any physical variable (used in the description of complex economics system dynamics), will depend on both the usual mathematical procedures on spatial and time coordinates as well as on a scale resolution. Specifically, instead of working with a single physical variable (a strictly non-differentiable mathematical function), it is possible to operate only with approximations of this mathematical function, resulting by averaging it at different scale resolutions. Thus, any physical variable used to describe the dynamics of complex economics systems will operate as the limit of a family of mathematical functions, the function being non-differentiable for zero scale resolution and differentiable for non-zero scale resolution [6-9].

This way of describing the dynamics of complex economics systems, obviously implies the development of both new geometric structures and physical theories, consistent with these geometric structures, for which the laws of motion, invariant to time coordinate transformations are also invariant to transformations with respect to scale resolution. Such a geometric structure is the one based on the concept of the fractal/multifractal and the corresponding physical model described in the Scale Relativity Theory (SRT) [7-9]. This article aims to present a novel approach to understanding the dynamics of complex economic systems through a holographic implementation. By examining the behavior of the structural units within these intricate systems, we explore the use of continuous, non-differentiable curves, such as fractal or multifractal curves, to provide a comprehensive and explicit depiction of their dynamics. Through this perspective, we shed light on the interconnections and complexities inherent in economic systems, offering new insights into their functioning and behavior.

Let it be shown that the information above becomes functional for economic systems. Dynamic and historical through its very nature, the economic activity was carried out based on time- and place-dependent conditions, changing its organization as a function of the nature of the economic system. Andre Marchal [10] defined the economic system as a coherent ensemble of social and institutional, economic and technical, psychological or mental structures. The complexity of a system derives from its functionality, from its number of participants or deciders, from the nature or intensity of specific relations or from the magnitude of its imbalances. The theory of complex economics systems assumes that individual entities interact in a non-linear manner and the effect of said interactions is a consequence or a result of the collective actions. By fulfilling the criteria of a complex economics system, economics has become a fertile domain for the application of specific mathematical or physical theories. In such a context, the study of complex economics systems through the fractal theory perspective has become, in the last decades, an important endeavor for researchers. The pioneering research of Pareto [11], in order to identify the economic "optimum", or the analyses of Gini [12] paved the way towards the holographic analysis of economic phenomena.

The work of Mandelbrot [6] introduced the scientific world to the concept of fractality. He found empirically that a chart of market price changes of cotton price looks similar to another chart with different time resolution. The holographic analysis thus comes off as a necessity in simulations, being able to model and predict the generally statistical nature of a system, without predicting its behavior in a certain moment. The classic and neoclassic theories, based on the hypothesis of rationality, supported the self-regulating capacity of the market. The premise was that supply and demand automatically adjust through the price mechanism, thus inducing an equilibrium trend. Empirical analyses have proven that self-regulating, or the tendency towards reaching an equilibrium, are not consistent with reality. Demand is dominated by the imperfect behavior of consumers and supply is adjusted through the signals transmitted by prices.

Initially applied in the consumer goods market, the fractal/multi-fractal theory has encompassed other economics domain such as: price changes in open market, the distribution of income of companies and the scaling relation of company's size fluctuations [13,14]

The financial markets, through their inherently speculative character, can constitute a trans- and interdisciplinary domain. The globalization of financial markets and the application of new informational technologies emphasize the complexity degree of these markets.

The 2008 economic crisis has challenged the mainstream economic theory. In such a context, the speculative bubbles, an phenomenon increasingly encountered on these markets, is the result of individual reaction to market signals through the price at that particular moment (an aspect which can be assimilated to the digital signals), in the context of an optimistic state induced by the behaviors of large investors and of the State, through banking or monetary policies (aspect which can be assimilated to analogical signals) [15,16]. The results of contemporary empirical studies prove that the market dynamics and the evolution of economic indicators are not a random phenomenon. Besides this our proposed model, there is another approach to explain the nonlinear behavior of complex economics system through fractional derivatives formalism. In reference [44] it is presented a model using an operational procedure in the sense of evolution of fractional order chaotic economic system based on non-degenerative equilibrium points where have been obtained various chaotic scenarios such as doubling period, intermittence shown in bifurcation and time series diagrams. We note that our model offers more general description of the non-linear behavior of complex economics system based both on scale resolutions change and fractal dimensions choice of motion curves associated to the economical phenomenon.

In the present paper, the analogic/digital type behaviors are explained in the dynamics of complex economics systems and particularly, for economic systems, by using the SRT model in the form of Schrödinger and Madelung type scenarios. The holographic character in the description of the complex economics system dynamics will thus be highlighted.

2. Schrödinger and Madelung-type scenarios in the description of complex economics system dynamics

It is a known the fact that the dynamics of complex economics systems in the SRT [7-9] can be described through the multifractal Schrödinger equation – the Schrödinger-type scenario:

$$\lambda^2(dt)^{\left[\frac{4}{f(\alpha)}\right]-2}\partial_l\partial^l\Psi + i\lambda(dt)^{\left[\frac{2}{f(\alpha)}\right]-1}\partial_t\Psi = 0, \tag{1}$$

where:

$$\partial_t = \frac{\partial}{\partial_t}, \partial_l = \frac{\partial}{\partial x^l}, \partial_l\partial^l = \frac{\partial^2}{\partial x_l^2}, \tag{2}$$

In the above relations Ψ is the states function, dt is the scale resolution, x^l is the multifractal spatial coordinate, t is the non-multifractal temporal coordinate with the role of an affine parameter of the motion curves (it is mentioned that in SRT, the dynamics of the structural units belonging to any complex economics system are described through continuous and non-differentiable curves - fractal curves), λ is a parameter associated to the fractal/multifractal-non-fractal/non-multifractal scale transition, $f(\alpha)$ is the singularity spectrum with a singularity index of order $\alpha = \alpha(D_F)$ and D_F is the fractal dimension of the motion curves [3,4,6].

On the other hand, by choosing Ψ of the form

$$\Psi = \sqrt{\rho} e^{is}, \tag{3}$$

where $\sqrt{\rho}$ is the amplitude and s is the phase, and introducing the real velocity fields (V_D^i- differentiable velocity field, V_F^i-non-differentiable velocity field):

$$V_D^i = 2\lambda (dt)^{\left[\frac{2}{f(\alpha)}\right]-1} \partial^i s, \tag{4}$$

$$V_F^i = i\lambda (dt)^{\left[\frac{2}{f(\alpha)}\right]-1} \partial^i \ln \rho, \tag{5}$$

the multifractal Schrödinger equation is reduced to the multifractal hydrodynamic equation system – the Madelung-type scenario:

$$\partial_t V_D^i + V_D^l \partial_l V_D^i = -\partial^i Q, \tag{6}$$

$$\partial_t \rho + \partial_l \left(\rho V_D^l\right) = 0, \tag{7}$$

with Q the multifractal specific potential:

$$Q = -2\lambda^2 (dt)^{\left[\frac{4}{f(\alpha)}\right]-2} \frac{\partial^l \partial_l \sqrt{\rho}}{\sqrt{\rho}} = -V_F^i V_F^i - \frac{1}{2}\lambda (dt)^{\left[\frac{2}{f(\alpha)}\right]-1} \partial_l V_F^l. \tag{8}$$

Equation (6) corresponds to the multifractal specific momentum conservation law, while equation (7) corresponds to the multifractal state density conservation law. The multifractal specific potential (8) implies the multifractal specific force:

$$F^i = -\partial^i Q = -2\lambda^2 (dt)^{\left[\frac{4}{f(\alpha)}\right]-2} \partial^i \frac{\partial^l \partial_l \sqrt{\rho}}{\sqrt{\rho}}, \tag{9}$$

which is a measure of the multifractality of the motion curves of the dynamics. From the equations (6)-(8) the following meanings result:

- Any complex economics system structural units are in a permanent contact with a multifractal medium through the multifractal specific force;

- The multifractal medium can be assimilated with a multifractal fluid whose dynamics are characterized by the multifractal hydrodynamic equation system;
- The velocity field V_F^i is absent from the multifractal states density conservation laws. In a such context it induces non-manifest complex economics system dynamics facilitating the transmission of multifractal specific momentum and multifractal energy of focus;
- In the complex economics system dynamics the ”self – aspect” of the multifractal specific momentum, transfer the reversibility and existence of eigenstates are guaranteed by the conservation of multifractal energy and multifractal momentum. Using the tensor:

$$\hat{\tau}^{il} = 2\lambda^2 (dt)^{\left[\frac{4}{f(\alpha)}\right]-2} \rho \partial^i \partial^l \ln \rho, \tag{10}$$

equation (9) takes the form of a multifractal equilibrium equation:

$$\rho \partial^i Q = \partial_l \hat{\tau}^{il}. \tag{11}$$

Moreover, since the tensor $\hat{\tau}^{il}$ can also be written in the form:

$$\hat{\tau}^{il} = \eta\left(\partial_l V_F^i + \partial_i V_F^l\right), \tag{12}$$

with:

$$\eta = \lambda (dt)^{\left[\frac{2}{f(\alpha)}\right]-1} \rho. \tag{13}$$

a multifractal linear constitutive equation for a multifractal “viscous fluid”, becomes functionally offering in the same time the reason for an original interpretation of coefficient η as a multifractal dynamic viscosity of the multifractal fluid.

3. “Digital” type behavior in the dynamics of complex economics systems through Riccati type gauge

The multifractal Schrödinger equation admits, besides the clasical Galilei group proper, an extra set of symmetries [17] that, in general conditions, can be taken in a form involving just one space dimension and time, as a SL(2,R) type group in two variables with three parameters [18]. Limiting the general conditions, the space dimension can be chosen as the radial coordinate in a free fall, as in the case of Galilei kinematics, which can also be extended as such in general relativity [19,20], for instance in the case of free fall in a Schwarzschild field.

The essentials of the argument of Alicia Herrero's and Juan Antonio Morales' work just cited are delineated based on the fact that the radial motion in a Minkowski spacetime should be a conformal Killing field, which is a three-parameter realization of the SL(2R) algebra in time and the radial coordinate. This is a Riemannian manifold of the Bianchi type VIII (or even type IX, forcing the concepts a little) [21]. The bottom line here is that, as long as the general relativity is involved, the nonstationary Schrödinger equation describes the continuity of matter.

And since, as a universal instrument of knowledge, the multifractal Schrödinger equation is referring to free particles, we need to show what kind of freedom is this in classical terms.

For our current necessities it is best to start with the finite equations of the specific SL(2,R) group, and build gradually upon these [22,23], in order to discover the connotations we are seeking for. Working in the variables (t,r) as above, the finite equations of this group are given by the transformations:

$$t \to \frac{\alpha t + \beta}{\gamma t + \delta};\ r \to \frac{r}{\gamma t + \delta} \tag{14}$$

This transformation is a realization of the SL(2,R) structure in variables (t,r), with three essential parameters (one of the four constants α, β, γ and δ is superfluous here). Every vector in the tangent space SL(2R) is a linear combination of three fundamental vectors, the infinitesimal generators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = t\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r}, \qquad X_3 = t^2\frac{\partial}{\partial t} + tr\frac{\partial}{\partial r} \tag{15}$$

satisfying the basic structure equations:

$$[X_1, X_2] = X_1, \qquad [X_2, X_3] = X_3, \qquad [X_3, X_3] = -2X_2 \tag{16}$$

which we take as standard commutation relations for this type of algebraic structure, all along the present work. The group has an invariant function, which can be obtained as the solution of a partial differential equation:

$$(cX_1 + 2bX_2 + aX_3)f(t,r) = 0$$

$$(at^2 + 2bt + c)\frac{\partial f}{\partial t} + (at + b)r\frac{\partial f}{\partial r} = 0 \quad (17)$$

The general solution of this equation is a function of the constant values of the ratio:

$$\frac{r^2}{at^2 + 2bt + c} \quad (18)$$

which represents the different paths of transitivity of the action described by equation (4).

In order to draw some proper conclusions from these mathematical facts, let us go back to the transformation (14) and consider it from the point of view of multifractal physics.

Firstly, let it be observed that the multifractal specific force (9) is defined with the help of a gradient. As an immediate consequence, if ‘r’ denotes the distance of the moving complex economics system structural unit from the center of multifractal specific force, then

$$r^2 d\theta = \dot{a}dt \qquad \dot{a}\frac{dt}{d\theta} = r^2 \qquad \dot{a} = const. \quad (19)$$

where θ is the central angle of the position vector of the moving complex economics system structural unit with respect to the center of multifractal specific force and $\dot{a}$ is the constant of the multifractal kinetic moment.

Now, since according with (7)

$$\frac{r^2}{at^2+2bt+c} = L = const., \quad (20)$$

then from (20) and (19), through the substitutions

$$\frac{dt}{d\theta} = \dot{w}, \qquad \frac{Lat^2}{\dot{a}} = \frac{1}{M}w^2, \qquad \frac{2Lbt}{\dot{a}} = -2\frac{R}{M}w, \qquad \frac{Lc}{\dot{a}} = K \quad (21)$$

the following Riccati-type differential equation is satisfied (i.e., we operate here with a Riccati-type gauge):

$$\dot{w}-\frac{1}{M}w^2+2\frac{R}{M}w-K=0. \tag{22}$$

For obviously physical reasons, it is therefore important to find the most general solution of that equation. José Carineña et al. offer us a pass in short but modern and pertinent review of the integrability of Riccati's equation [24]. For our current needs it is enough to note that the complex numbers

$$w_0\equiv R+iM\Omega, w_0^*\equiv R-iM\Omega; \Omega^2=\frac{K}{M}-\left(\frac{R}{M}\right)^2 \tag{23}$$

roots of the quadratic polynomial on the right side of equation (22), are two solutions (constants, that's right) of the equation: being constants, their derivative is zero, being roots of the right-hand polynomial, it cancels. So, first we do the homographic transformation:

$$z=\frac{w-w_0}{w-w_0^*} \tag{24}$$

and now it can easily be seen by direct calculation that z is a solution of the linear and homogeneous equation of the first order

$$\dot{z}=2i\Omega z \therefore z(t)=z(0)e^{2i\Omega t}. \tag{25}$$

Therefore, if we conveniently express the initial condition $z(0)$, we can give the general solution of the equation (22) by simply inverting the transformation (24), with the result

$$w=\frac{w_0+re^{2i\Omega(t-t_r)}w_0^*}{1+re^{2i\Omega(t-t_r)}} \tag{26}$$

where r and t_r are two real constants that characterize the solution. Using equation (23) we can put this solution in real terms, i.e.

$$z = R + M\Omega\left(\frac{2r\sin[2\Omega(t-t_r)]}{1+r^2+2r\cos[2\Omega(t-t_r)]} + i\frac{1-r^2}{1+r^2+2r\cos[2\Omega(t-t_r)]}\right) \tag{27}$$

which highlights a frequency modulation through what we would call a Stoler transformation [22,23] which leads us to a complex form of this parameter. More than that, if we make the notation

$$r \equiv \coth\tau, \tag{28}$$

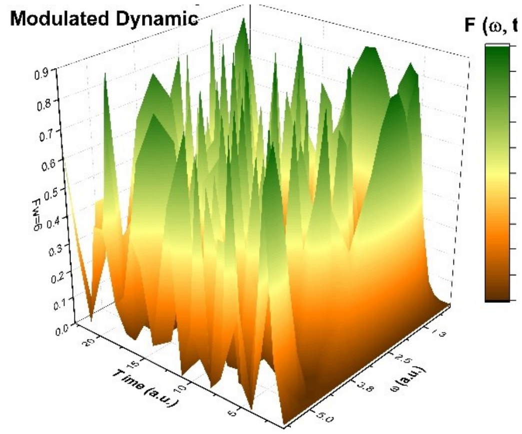
equation (27) becomes

$$z = R + M\Omega h \tag{29}$$

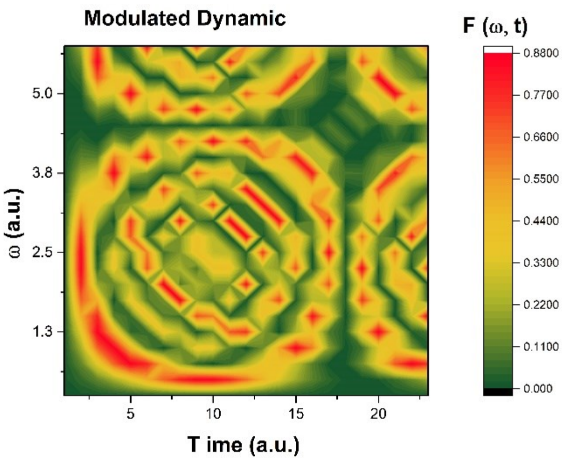
where h is given by

$$h = -i\frac{\cosh\tau - e^{-2i\Omega(t-t_m)}\sinh\tau}{\cosh\tau + e^{-2i\Omega(t-t_m)}\sinh\tau}. \tag{30}$$

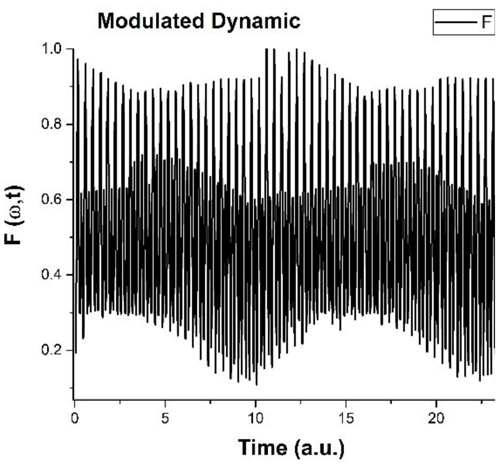
The meaning of this complex parameter will be clear a little later. For the moment, let's note that any dynamic process appears here as a frequency modulation process by means of a gauge invariance of a Riccati-type, imposed through the multifractal kinetic momentum conservation law.



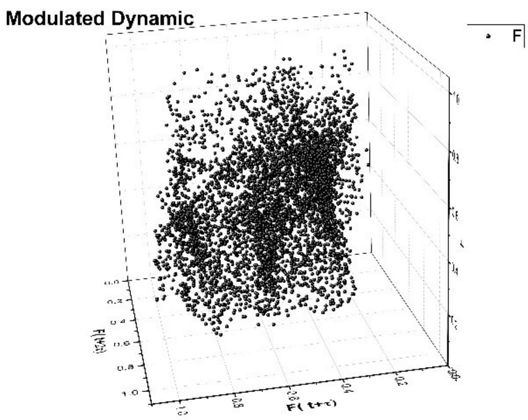
(a) 3D diagram



(b) contour diagram

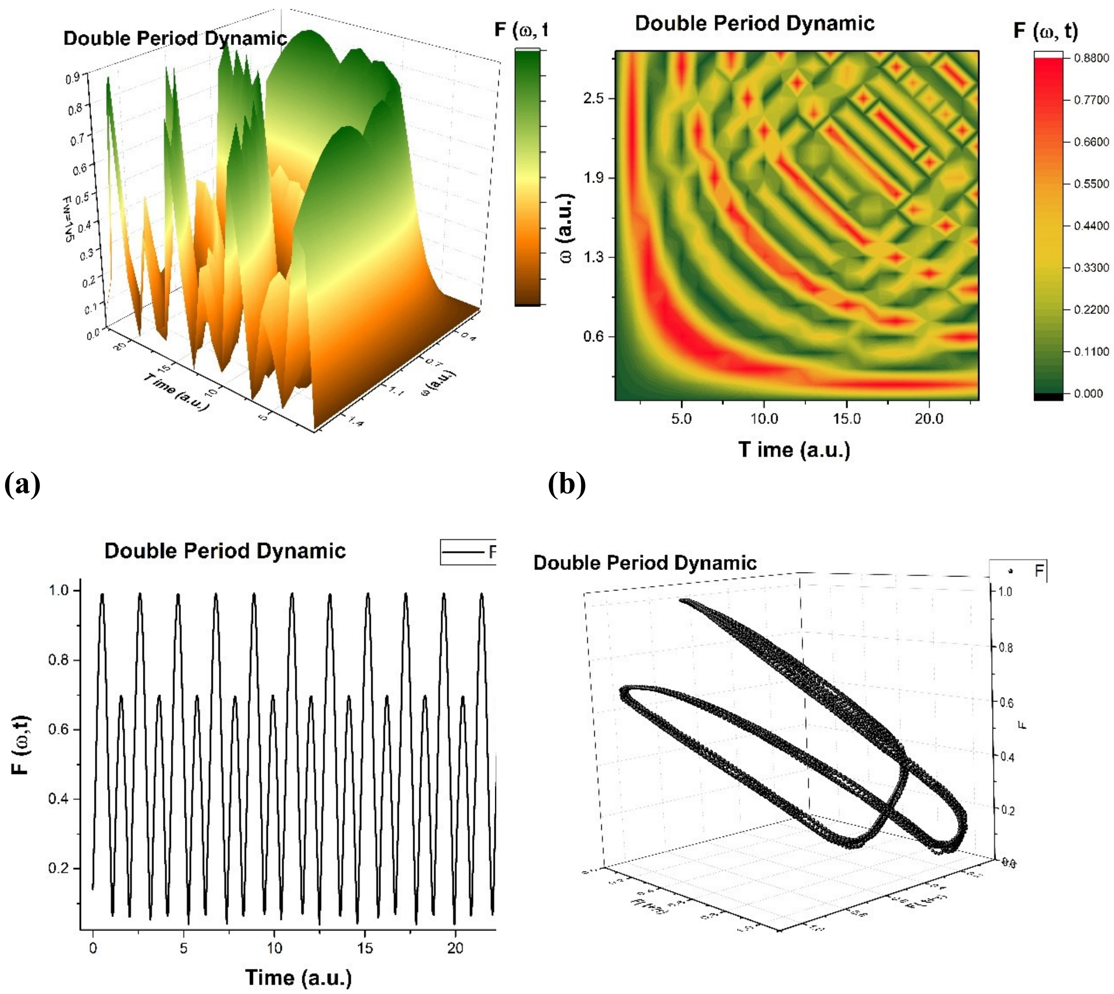


(c) time series



(d) reconstituted attractor

Fig. 1 a-d: The ”modulated dynamic modes” of the structural units of complex economics system dynamics are presented: **(a)** - 3D diagram, **(b)** - contour diagram, **(c)** - time series and **(d)** - reconstituted attractor for scale resolutions given by Ωmax.



(a) (b)

(c) (d)

Fig. 2 a-d: The ”double period dynamic modes” of the structural units of complex economics system dynamics are presented:(a) - 3D diagram, (b) - contour diagram, (c) - time series and (d) - reconstituted attractor for scale resolutions given by Ωmax.

In these figures, Real (h) (the amplitude at various scale resolutions given by the maximum value of Ω) is represented as functions of t and Ω for r=0,5.

As it can be observed in Figures 1 and 2 a-d, the natural transition of complex economics system dynamics is to evolve from normal period doubling state towards modulated dynamics. The complex economics system dynamics never reach chaotic state but they permanently evolve towards that state. All of the above highlight an “digital”-type behavior (double period and modulated dynamics). Similar behaviors are often seen in the dynamics of other complex

economics systems like transient plasmas[25-28] or other depositions system based on flow of multi-component fluid [29].
In particular, when referring to economics dynamics, the speculative bubbles, which can be the result of individual reaction to market signals through the price at a particular moment, can be assimilated to the digital signals [15, 16].

4. "Analogic" type behavior in the dynamics of complex economics systems through spontaneous symmetry breaking of Riccati type

Let it be admitted that relation (20) is not operable anymore (i.e. null constant). Then,

$$r^2 = x_1^2 + x_2^2 + x_3^2 = 0 \tag{31}$$

If we accept the null vectors as being the "complex economics system background", it follows that they have a major importance in "vacuum" problems.

Now, it is naturally to suppose that, the most general invariance group is the group with three rotation parameters, whose infinitesimal generators are:

$$\begin{aligned} M_1 &= -i\left(x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}\right), \\ M_2 &= -i\left(x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3}\right), \\ M_3 &= -i\left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right), \end{aligned} \tag{32}$$

This fact was put into evidence by [9], which highlights that the form (31) of infinitesimal generators is equivalent to the form resulting from it for:

$$x_1 = \rho\sin\omega, x_2 = -\rho\cos\omega, x_3 = -i\rho, \tag{33}$$

that is:

$$M_1 = \cos\omega\, \rho \frac{\partial}{\partial \rho} - \sin\omega \frac{\partial}{\partial \omega}, \tag{34}$$

$$M_2 = \sin\omega\, \rho\frac{\partial}{\partial\rho} - \cos\omega\, \frac{\partial}{\partial\omega}$$

$$M_3 = -i\frac{\partial}{\partial\omega}.$$

The action of the operators (33) on the spin “eigenfunctions”:

$$v_+ = \sqrt{\rho}e^{\frac{i\omega}{2}}, v_- = \sqrt{\rho}e^{\frac{-i\omega}{2}} \tag{35}$$

reproduces the action of the Pauli matrices, according to relations:

$$\begin{bmatrix} M_1 v_+ \\ M_1 v_- \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_+ \\ v_- \end{bmatrix},$$
$$\begin{bmatrix} M_2 v_+ \\ M_2 v_- \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\begin{bmatrix} v_+ \\ v_- \end{bmatrix}, \tag{36}$$
$$\begin{bmatrix} M_3 v_+ \\ M_3 v_- \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v_+ \\ v_- \end{bmatrix}.$$

It can be shown in a straight manner that the operators (34) satisfy the same algebras as Pauli’s matrices.

The infinitesimal transformations (34) and (32) don’t tell us very much about this group. To this end, we shall look for some finite transformations, generated by the infinitesimal ones. In this respect, we shall set operators (34) in a form capable to put into evidence its isomorphism to the group, considered in Section 2. The new operators are given by the linear combinations:

$$X_1 = M_1 - iM_2,$$
$$X_2 = M_3, \tag{37}$$
$$X_3 = -(M_1 + iM_2),$$

and satisfy the same structure relations as the infinitesimal generators of the group (15).

Taking α and $\bar{\alpha}$ as group variables, with α = v_+ from (35), the operators (37) also write:

$$Y_1 = \bar{\alpha}\frac{\partial}{\partial\bar{\alpha}}, Y_2 = \frac{1}{2}(\alpha\frac{\partial}{\partial\alpha} - \bar{\alpha}\frac{\partial}{\partial\bar{\alpha}}), Y_3 = -\alpha\frac{\partial}{\partial\alpha} \tag{38}$$

In order to find out the finite transformations generated by these infinitesimal transformations; we shall proceed to determine invariant functions of the operator:

$$\mathrm{U}=\mu Y_1 + \nu Y_2 + \lambda Y_3,\ \mu, \nu, \lambda - constants,$$

which are solutions of the equation $U\psi = 0$:

$$(\mu\bar{\alpha} + \frac{\nu}{2}\alpha)\frac{\partial\psi}{\partial\alpha} - (\frac{\nu}{2}\bar{\alpha} + \lambda\alpha)\frac{\partial\psi}{\partial\bar{\alpha}} = 0, \tag{39}$$

with the characteristic system:

$$\frac{d\alpha}{\mu\bar{\alpha} + \frac{\nu}{2}\alpha} = -\frac{d\bar{\alpha}}{\lambda\alpha + \frac{\nu}{2}\alpha}. \tag{40}$$

We now look for another two constants m and n, so that the linear combination resulting from (40):

$$\frac{md\alpha - nd\bar{\alpha}}{m(\frac{\nu}{2}\alpha + \mu\bar{\alpha}) - n(\lambda\alpha + \frac{\nu}{2}\bar{\alpha})} = d\tau \tag{41}$$

is a total differential. This can happen only under conditions:

$$\begin{aligned} &(\frac{\nu}{2} - \rho)m - \lambda n = 0, \\ &\mu m - (\frac{\nu}{2} - \rho)n = 0, \end{aligned} \tag{42}$$

where ρ is an arbitrary factor. This system is compatible only for the following values of ρ:

$$\rho_1 = \frac{\nu}{2} + \sqrt{\lambda\mu}, \rho_2 = \frac{\nu}{2} - \sqrt{\lambda\mu}. \tag{43}$$

For the first value, (41) yields the integral:

$$\sqrt{\lambda\alpha} + \sqrt{\mu\bar{\alpha}} = (\sqrt{\lambda}\alpha_0 + \sqrt{\mu}\bar{\alpha}_0)e^{(\frac{\nu}{2}+\sqrt{\lambda\mu})\tau}, \quad (44)$$

and for the second, the integral:

$$\sqrt{\lambda\alpha} + \sqrt{\mu\bar{\alpha}} = (\sqrt{\lambda}\alpha_0 + \sqrt{\mu}\bar{\alpha}_0)e^{(\frac{\nu}{2}-\sqrt{\lambda\mu})\tau}, \quad (45)$$

where α_0 and $\bar{\alpha}_0$ are the values of α and $\bar{\alpha}$ for τ = 0. Relations (43) and (44) give:

$$\alpha = e^{\frac{\nu}{2}}\alpha_0 cosh\sqrt{\lambda\mu}\tau + \sqrt{\frac{\mu}{\lambda}}\bar{\alpha}_0 sinh\sqrt{\lambda\mu}\tau,$$
$$\bar{\alpha} = e^{\frac{\nu}{2}}\alpha_0\sqrt{\frac{\lambda}{\mu}}sinh\sqrt{\lambda\mu}\tau + \bar{\alpha}_0 cosh\sqrt{\lambda\mu}\tau. \quad (46)$$

In order that (46) make sense, it is necessary that λ be the complex conjugate of μ: $\bar{\lambda}$ = μ=$\gamma e^{i\phi}$, while ν has to be real – and we take it as zero. Therefore, the finite transformations of the group (37), generated by the infinitesimal transformations, are given by the unimodular group:

$$\alpha = \alpha_0 cosh\,\gamma\tau + \bar{\alpha}_0 e^{i\phi} sinh\,\gamma\tau, \quad (47)$$
$$\bar{\alpha} = \alpha_0 e^{-i\phi} sinh\,\gamma\tau + \bar{\alpha}_0 sinh\,\gamma\tau$$

It is no accident that we have denoted by α the group variable. This notation is commonly used to denote the complex amplitude of the harmonic oscillators, that is the eigenvalue of the annihilation operator in the second quantization [24]. Transformation (47) was first given by Stoler [30], in connection with generalization of the eigenstates of the annihilation operator (coherent states). If α0 describes, for example, an oscillator in a state satisfying the minimum conditions of momentum- coordinate uncertainty, the state α is characterized by the uncertainty relation:

$$(\Delta p)^2(\Delta q)^2 = \frac{1}{4}(1 + sinh^2\gamma\tau\, sin^2\phi), \quad (48)$$

where (Δp) and (Δq) are variations of the momentum, – respectively, coordinate – of the oscillator, while the interaction constant (in particular, the Planck constant) has been taken as equal to unity.

In particular, when referring to economics dynamics, the “economics” complex economics system background, by means of coherence states of Stoler type, refers to an optimistic state induced by the behaviors of large investors and of the State, through banking or monetary policies. This aspect can be assimilated to analogical signals [15,16].

5. Conclusions

In a Schrödinger-type and Madelung-type scenarios for the description of complex economics system dynamics, SL(2R) symmetries are highlighted. The existence of such symmetries has several consequences: the existence of analogic-type behavior as a gauge invariance of Riccati type as well as the existence of digital-type behavior through the spontaneous symmetry breaking of the same gauge invariance.

When referring to economics dynamics, the markets have a fractal/multi-fractal structure on the long term, being characterized by a "self memory". In such a context, the individual reaction to market signals can be associated to period doubling and modulated dynamics (i.e. to the digital signals) while, the behaviors of large investors and of the State, through banking or monetary policies, can associated to the "complex economics system background" (i.e. analogical signals). The economic structures emphasize fluctuations but, they never reach the chaos state.

Moreover, when referring to biostructures, smart materials etc., the above mathematical formalism can be applied to a large collection of research domains [32-43]. Based on this model, many properties and characterizations can be explained.

As a final conclusion, a holographic approach on complex economics system dynamics becomes more natural than the standard approaches.

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