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QUANTUM MECHANICS AS A THEORY BASED ON THE GENERAL THEORY OF RELATIVITY

Eliade STEFANESCU[1]

Rezumat. *În această lucrare, obținem dinamica cuantică în cadrul teoriei generale a relativității, unde o particulă cuantică este descrisă de o distribuție de materie, prin funcții de amplitudine ale densității materiei, în cele două spații conjugate ale coordonatelor spațiale și impulsului, numite funcții de undă. Pentru o particulă liberă, aceste funcții de undă sunt pachete de unde conjugate în spațiile coordonatelor și impulsului, cu faze dependente de timp proporționale cu Lagrangeanul relativist, în timp ce vitezele undelor în spațiul coordonatelor sunt egale cu viteza distribuției descrise de pachetul de unde din acest spațiu. Din vitezele undelor funcțiilor de undă ale unei particule, obținem forța Lorentz și ecuațiile lui Maxwell. Pentru o particulă cuantică în câmp electromagnetic, obținem ecuații dinamice în spațiile coordonatelor și impulsului, și funcții de undă pentru particule și antiparticule. Obținem rata de împrăștiere într-un câmp electromagnetic, pentru cazurile posibile, cu conservarea, sau inversarea spinului.*

Abstract. In this paper, we obtain the quantum dynamics in the framework of the general theory of relativity, where a quantum particle is described by a distribution of matter, with amplitude functions of the matter density, in the two conjugate spaces of the spatial coordinates and of the momentum, called wave functions. For a free particle, these wave functions are conjugate wave packets in the coordinate and momentum spaces, with time dependent phases proportional to the relativistic lagrangian, as the wave velocities in the coordinate space are equal to the distribution velocity described by the wave packet in this space. From the wave velocities of the particle wave functions, we obtain lorentz's force and the maxwell equations. For a quantum particle in electromagnetic field, we obtain dynamic equations in the coordinate and momentum spaces, and the particle and antiparticle wave functions. We obtain the scattering or tunneling rate in an electromagnetic field, for the two possible cases, with the spin conservation, or inversion.

Keywords: Quantum particle, wave/group velocity, spinor, fermi's golden rule, scattering

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1. Introduction

The theory of relativity describes a continuous distribution of matter [1], in a 4-dimensional physical system with the time-space coordinates,

$$x^{\alpha} = x^{0}(ct), x^{1}, x^{2}, x^{3} = x^{0}(ct), x^{i}, \quad i = 1,2,3. \tag{1}$$

[1] Prof. PhD, Advanced Studies in Physics Centre of the Romanian Academy, Academy of Romanian Scientists, Bucharest, Romania (e-mail: eliadestefanescu@yahoo.fr).

Following Dirac [2], we consider a larger N-dimensional universe with the coordinates

$$z^n, \quad n = 1, 2, ..., N, \tag{2}$$

as a flat one, with any time-space interval $\mathrm{d}s$,

$$\mathrm{d}s^2 = h_{mn}\mathrm{d}z^m \mathrm{d}z^n, \tag{3}$$

determined by a constant metric tensor h_{mn}, where our universe is a hypersurface defined by $N-4$ equations,

$$z^n = z^n\left(x^\alpha\right), \quad n = 1, 2, ..., N-4. \tag{4}$$

In this universe, our physical hypersurface is curved by the presence of matter, according to the equivalence of the inertial and gravitational forces, as the time-space interval,

$$\mathrm{d}s^2 = h_{nm} z^n{}_{,\alpha} z^m{}_{,\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta, \qquad n, m = 1, 2, ..., N-4, \tag{5}$$

depends on the metric tensor, as a symmetric function of the coordinates,

$$g_{\alpha\beta} = g_{\beta\alpha} = h_{nm} z^n{}_{,\alpha}\left(x^\mu\right) z^m{}_{,\beta}\left(x^\nu\right) = z_{n,\alpha}\left(x^\mu\right) z^n{}_{,\beta}\left(x^\nu\right). \tag{6}$$

Essentially, this theory is based on the invariance of the time-space interval for any coordinate transformation, which, for the proper time τ defined by this interval

$$\mathrm{d}s = c\mathrm{d}\tau = \sqrt{g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta} = c\sqrt{g_{\alpha\beta}v^\alpha v^\beta}\,\mathrm{d}\tau, \;\; \alpha, \beta = 0, i\left(1,2,3\right), \;\; g_{00} > 0, \;\; g_{ii} < 0, \tag{7}$$

as a function of the velocities

$$v^\alpha = \frac{\mathrm{d}x^\alpha}{c\mathrm{d}\tau} = \frac{\dot{x}^\alpha}{c}, \tag{8}$$

leads to the fundamental equation:

$$\sqrt{g_{\alpha\beta}v^\alpha v^\beta} = 1. \tag{9}$$

For a quantum particle, we consider a distribution of matter in the space of the coordinates $x=\left(x^1,x^2,x^3\right)$, and in the space of the momentum $p=\left(p^1,p^2,p^3\right)$,

$$\begin{aligned}\rho_M\left(x,t\right)&=M_0\left|\psi\left(x,t\right)\right|^2\\ \rho_M\left(p,t\right)&=M_0\left|\varphi\left(p,t\right)\right|^2,\end{aligned}\tag{10}$$

with the amplitude functions, which we call the wave functions [3-6],

$$\begin{aligned}\psi\left(x^i,\tau\right)&=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\varphi\left(p^j,\tau\right)e^{\frac{\mathrm{i}}{\hbar}\left[p^jx^j-L\left(p^j,x^j\right)\tau\right]}\mathrm{d}^3p\\ \varphi\left(p^j,\tau\right)&=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\psi\left(x^i,\tau\right)e^{-\frac{\mathrm{i}}{\hbar}\left[p^jx^j-L\left(p^j,x^j\right)\tau\right]}\mathrm{d}^3x,\end{aligned}\tag{11}$$

satisfying the normalization conditions

$$\int\left|\psi\left(x,t\right)\right|^2\mathrm{d}^3x=\int\left|\varphi\left(p,t\right)\right|^2\mathrm{d}^3p=1.\tag{12}$$

For a free quantum particle, these wave functions depend on the relativistic Lagrangian

$$L\left(p^j,x^j\right)=-Mc^2\sqrt{g_{\alpha\beta}v^\alpha v^\beta},\tag{13}$$

and the momentum

$$\begin{aligned}p^j&=\frac{\partial L}{\partial\dot{x}^j}=\frac{\partial L}{\partial\left(cv^j\right)}=-\frac{Mc}{2\sqrt{g_{\alpha\beta}v^\alpha v^\beta}}\frac{\partial}{\partial\left(v^j\right)}\left(g_{j\beta}v^jv^\beta+g_{\alpha j}v^\alpha v^j\right)\\ &=-Mcg_{j\beta}v^\beta,\end{aligned}\tag{14}$$

with mass M as a characteristic of continuous matter dynamics. We note that, the velocity of any wave of the wave packet (11) in space, is equal to the velocity of the wave function defined by this wave packet,

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}x^j\right)_{\text{wave}} &= \frac{\partial L}{\partial p^j} = \frac{c}{2\sqrt{g_{\alpha\beta}v^\alpha v^\beta}}\frac{\partial}{\partial\left(g_{j\mu}v^\mu\right)}\left(g_{j\beta}v^j v^\beta + g_{\alpha j}v^\alpha v^j\right) \\ &= c\frac{2v^j}{2\sqrt{g_{\alpha\beta}v^\alpha v^\beta}} = cv^j = \dot{x}^j,\end{aligned} \tag{15}$$

as the time variation of any wave of the wave packet (11) in the momentum space takes the form of the gradient of a potential:

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}p^j\right)_{\text{wave}} &= \frac{\partial L}{\partial x^j} = \frac{\partial\left(-Mc^2\sqrt{g_{\alpha\beta}v^\alpha v^\beta}\right)}{\partial x^j} = \frac{-Mc^2 v^\alpha v^\beta}{2\sqrt{g_{\alpha\beta}v^\alpha v^\beta}}\frac{\partial}{\partial x^j}g_{\alpha\beta} \\ &= -\frac{1}{2}Mc^2 v^\alpha v^\beta g_{\alpha\beta,j}.\end{aligned} \tag{16}$$

For a non-relativistic velocity, with the kinetic energy

$$T\left(\vec{p}\right) = \frac{M\dot{\vec{r}}^2}{2} = \frac{\vec{p}^2}{2M}, \tag{17}$$

and a potential energy $U\left(\vec{r}\right)$, with the Hamiltonian

$$H\left(\vec{p},\vec{r}\right) = T\left(\vec{p}\right) + U\left(\vec{r}\right), \tag{18}$$

as the Lagrangian is

$$L\left(\vec{p},\vec{r}\right) = \vec{p}\dot{\vec{r}} - H\left(\vec{p},\vec{r}\right) = T\left(\vec{p}\right) - U\left(\vec{r}\right), \tag{19}$$

the wave functions (11) take the form

$$\begin{aligned}
\psi\left(\vec{r},t\right) &= \frac{1}{\left(2\pi\hbar\right)^{3/2}}\int e^{\frac{\mathrm{i}}{\hbar}\left[\vec{p}\vec{r}-L\left(\vec{p},\vec{r}\right)t\right]}\varphi\left(\vec{p},t\right)\mathrm{d}^3\vec{p} \\
&= \frac{1}{\left(2\pi\hbar\right)^{3/2}}\int e^{\frac{\mathrm{i}}{\hbar}\left\{\vec{p}\vec{r}-\left[\vec{p}\dot{\vec{r}}-H\left(\vec{p},\vec{r}\right)\right]t\right\}}\varphi\left(\vec{p},t\right)\mathrm{d}^3\vec{p} \\
&= \frac{1}{\left(2\pi\hbar\right)^{3/2}}\int e^{\frac{\mathrm{i}}{\hbar}\left\{\vec{p}\vec{r}-\left[T\left(\vec{p}\right)-U\left(\vec{r}\right)\right]t\right\}}\varphi\left(\vec{p},t\right)\mathrm{d}^3\vec{p} \\
\varphi\left(\vec{p},t\right) &= \frac{1}{\left(2\pi\hbar\right)^{3/2}}\int e^{-\frac{\mathrm{i}}{\hbar}\left[\vec{p}\vec{r}-L\left(\vec{p},\vec{r}\right)t\right]}\psi\left(\vec{r},t\right)\mathrm{d}^3\vec{r} \\
&= \frac{1}{\left(2\pi\hbar\right)^{3/2}}\int e^{-\frac{\mathrm{i}}{\hbar}\left\{\vec{p}\vec{r}-\left[\vec{p}\dot{\vec{r}}-H\left(\vec{p},\vec{r}\right)\right]t\right\}}\psi\left(\vec{r},t\right)\mathrm{d}^3\vec{r} \\
&= \frac{1}{\left(2\pi\hbar\right)^{3/2}}\int e^{-\frac{\mathrm{i}}{\hbar}\left\{\vec{p}\vec{r}-\left[T\left(\vec{p}\right)-U\left(\vec{r}\right)\right]t\right\}}\psi\left(\vec{r},t\right)\mathrm{d}^3\vec{r}.
\end{aligned} \tag{20}$$

In this case, the wave/group velocities of these wave packets are of the form of the Hamilton equations:

$$\begin{aligned}
\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_{\text{wave}} &= \frac{\partial T\left(\vec{p}\right)}{\partial\vec{p}} = \frac{\vec{p}}{M} = \dot{\vec{r}} = \frac{\partial}{\partial\vec{p}}H\left(\vec{p},\vec{r}\right) \\
\left(\frac{\mathrm{d}\vec{p}}{\mathrm{d}t}\right)_{\text{wave}} &= -\frac{\partial U\left(\vec{r}\right)}{\partial\vec{r}} = -\frac{\partial}{\partial\vec{r}}H\left(\vec{p},\vec{r}\right).
\end{aligned} \tag{21}$$

We conclude that the distribution functions (10), with the amplitude functions of the form (11), represent distributions of matter, as amplitude functions of the mass density, with the total mass M_0, obtained from the integrals of these functions, equal to the mass M as the dynamic characteristic of matter density described by these functions – the matter quantization rule:

$$\int\rho_M\left(x^i,\tau\right)\mathrm{d}^3x = M_0\int\left|\psi\left(x^i,\tau\right)\right|^2\mathrm{d}^3x = M_0\int\left|\varphi\left(p^i,\tau\right)\right|^2\mathrm{d}^3p = M_0 = M\ . \tag{22}$$

2. Quantum particle as a distribution of matter with finite density

In the conventional quantum mechanics, a quantum particle is a punctual entity, randomly occupying position coordinates with probabilities given by the particle wave function. This model raises two fundamental problems. The first problem is that of the wave function interference/diffraction: for instance in a two slit system, although a punctual particle can pass only through one slit, it reaches the screen according to interference model (see Figure 1), like as the particle knew somehow about the existence of the other slit.

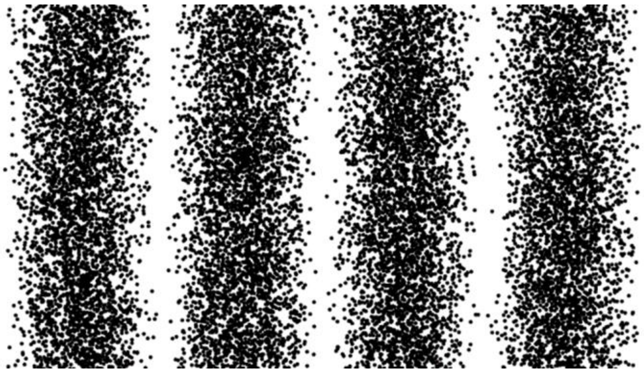

Fig. 1. Quantum particles, hitting a screen with a probability given by a wave function describing a two-slit interference.

In this way, the conventional quantum mechanics is a science of probabilities, not a science of the physical entities. The second problem is that the state of a conventional quantum particle, as a punctual entity, can be described by a system of time-space coordinates (t,x). According to the general theory of relativity any differential element of the time-space

$$\mathrm{d}s=\sqrt{c^2\mathrm{d}t^2-\mathrm{d}x^2} \tag{23}$$

is invariant for any change of coordinates. An interaction with an external/non-gravitational field leads to a time-space variation, which, for the simplicity of our inference, we suppose as a small one, $\delta t \ll \mathrm{d}t,\ \delta x \ll c\mathrm{d}t$. This means a variation of the time-space interval δs of the form:

$$\begin{aligned}
\underline{\mathrm{d}s}+\delta s &= \sqrt{c^2\left(\mathrm{d}t+\delta t\right)^2-\left(\mathrm{d}x+\delta x\right)^2} = \sqrt{\underbrace{c^2\mathrm{d}t^2-\mathrm{d}x^2}_{\mathrm{d}s^2}+2c^2\mathrm{d}t\delta t-2\mathrm{d}x\delta x} \\
&= \sqrt{\mathrm{d}s^2+2c^2\mathrm{d}t\delta t-2\mathrm{d}x\delta x}=\mathrm{d}s\sqrt{1+\frac{2c^2\mathrm{d}t\delta t-2\mathrm{d}x\delta x}{\mathrm{d}s^2}} \\
&= \underline{\mathrm{d}s}+\frac{c^2\mathrm{d}t\delta t-\mathrm{d}x\delta x}{\mathrm{d}s} \\
\Rightarrow\quad & \frac{\mathrm{d}x}{\mathrm{d}t}\cdot\underline{\frac{\delta x}{\delta t}}=c^2-\frac{\mathrm{d}s}{\mathrm{d}t}\cdot\frac{\delta s}{\delta t}=c^2-c\sqrt{1-\frac{\mathrm{d}x^2}{c^2\mathrm{d}t^2}}\cdot\frac{\delta s}{\delta x}\cdot\underline{\frac{\delta x}{\delta t}}.
\end{aligned} \tag{24}$$

Thus, by this interaction, we obtain a velocity increase

$$\frac{\delta x}{\delta t} = \frac{c}{\frac{\mathrm{d}x}{c\mathrm{d}t} + \frac{\delta s}{\delta x}\sqrt{1 - \frac{\mathrm{d}x^2}{c^2\mathrm{d}t^2}}}, \tag{25}$$

induced by the field to a particle with the velocity $\frac{\mathrm{d}x}{\mathrm{d}t}$. For a particle at rest,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0 \rightarrow \frac{\delta x}{\delta t} = c\frac{\delta x}{\delta s}, \tag{26}$$

we obtain a time-space additional time variation

$$\frac{\delta s}{\delta t} = c. \tag{27}$$

With this expression, equation (24) takes the form

$$\mathrm{d}s + \delta s = \underline{c\mathrm{d}t + c\delta t} = \sqrt{c^2\left(\mathrm{d}t + \delta t\right)^2 - \delta x^2} = \sqrt{c^2\mathrm{d}t^2 + 2c^2\mathrm{d}t\delta t + c^2\delta t^2 - \delta x^2}$$
$$= c\mathrm{d}t\left(\underline{1 + \frac{\delta t}{\mathrm{d}t}} + \frac{\delta t^2}{2\mathrm{d}t^2} - \frac{\delta x^2}{2c^2\mathrm{d}t^2}\right),$$

leading to a velocity equal to the light velocity,

$$\frac{\delta x}{\delta t} = c, \tag{28}$$

which means a null mass. This result is in agreement with the Standard Model, where the quantum particle mass is only obtained by coupling with a field of Higgs Bosons [7, 8]. However, here we notice that this null mass for any quantum particle is only a result of the general theory of relativity for a punctual particle, characterized only by a system of coordinates. A quantum particle as a distribution of matter described by wave functions of the form (11), or (20), has a non-null mass, according to the general theory of relativity. In this framework, the matter of a quantum particle in a two slit system, as it is represented in Figure 1, propagates through both slits, naturally leading to an interference of the wave components corresponding to these slits. A point in this figure does not represent a particle reaching the screen, but a localized molecular system interacting with the quantum particle matter, comprising all these systems.

3. Matter density and conservation

We consider the quantum matter in an arbitrary time-space volume Ω, as an integral of the matter density in this volume, for two arbitrary systems of time-space coordinates S and S':

$$\int_{\Omega}\rho\left(x^{\mu}\right)\ J\,\mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3=\int_{\Omega}\rho'\left(x^{\mu'}\right)\mathrm{d}x^{0'}\mathrm{d}x^{1'}\mathrm{d}x^{2'}\mathrm{d}x^{3'}\ , \tag{29}$$

depending on the Jacobian

$$J=\det\left(x^{\mu'}_{,\alpha}\right). \tag{30}$$

From the tensorial transformation of the metric tensor

$$g_{\alpha\beta}=x^{\mu'}_{,\alpha}x^{\nu'}_{,\beta}g_{\mu'\nu'}\ , \tag{31}$$

we obtain the Jacobian as a function of the metric tensor determinants g and g',

$$J=\frac{\sqrt{-g}}{\sqrt{-g'}}\ . \tag{32}$$

Taking into account that the matter density is scalar,

$$\rho'\left(x^{\mu'}\right)=\rho\left(x^{\mu}\right),$$

the relation (29) takes the form of the invariance relation

$$\int_{\Omega}\rho\left(x^{\mu}\right)\sqrt{-g}\,\mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3=\int_{\Omega}\rho'\left(x^{\mu'}\right)\sqrt{-g'}\mathrm{d}x^{0'}\mathrm{d}x^{1'}\mathrm{d}x^{2'}\mathrm{d}x^{3'}\ , \tag{33}$$

as an integral of the scalar density $\rho\left(x^{\mu}\right)\sqrt{-g}$.

We consider the matter flow density

$$J^{\mu}=\rho v^{\mu}\ , \tag{34}$$

as the product of the matter density ρ with the velocity in the local time, v^{μ}, with the null covariant divergence:

$$J^{\mu}{}_{;\mu}=0\ . \tag{35}$$

From this covariant divergence

$$J^{\mu}{}_{:\mu} = J^{\mu}_{,\mu} + \Gamma^{\mu}_{\nu\mu} J^{\nu} = J^{\nu}_{,\nu} + \Gamma^{\mu}_{\nu\mu} J^{\nu}, \tag{36}$$

as a function of the Christoffel symbol [2, 3]

$$\begin{aligned} \Gamma^{\mu}_{\nu\mu} &= \frac{1}{2} g^{\mu\lambda} \left(\underline{g_{\lambda\nu,\mu}} + g_{\lambda\mu,\nu} - \underline{g_{\mu\nu,\lambda}} \right) = \frac{1}{2} g^{\mu\lambda} g_{\lambda\mu,\nu} \\ &= \frac{1}{2} g^{-1} g_{,\nu} = \frac{1}{2} (-g)^{-1} (-g)_{,\nu} = \frac{\left(\sqrt{-g}\right)_{,\nu}}{\sqrt{-g}}, \end{aligned} \tag{37}$$

we obtain the ordinary divergence of the scalar density of the matter flow density, as the scalar density of the covariant divergence of the matter flow density, which according to (35) is null,

$$\left(J^{\nu} \sqrt{-g} \right)_{,\nu} = J^{\mu}{}_{:\mu} \sqrt{-g} = 0. \tag{38}$$

By integrating this equation with the expression (34) over the three-dimensional, spatial volume V , with the Gauss theorem, we obtain the conservation equation

$$\frac{\partial}{\partial t} \int_{V} \rho v^{0} \sqrt{-g} \mathrm{d}^{3} x = -\oiint_{\Sigma_V} \rho v^{m} \sqrt{-g} \mathrm{d}^{2} x, \tag{39}$$

of the time variation of the matter density integral over this volume, by the matter flow through the surface of this volume. In the non-relativistic case, $v^{m} << v^{0} \approx 1$,

$$J^{0} = \rho v^{0} \approx \rho, \quad J^{m} = \rho v^{m} \approx \rho \frac{\vec{v}}{c}, \tag{40}$$

we obtain the classical form of the matter conservation,

$$\frac{\partial}{\partial t} \int_{V} \rho \mathrm{d}^{3} \vec{r} = -\oiint_{\Sigma_V} \rho \vec{v} \mathrm{d}^{2} \vec{r}. \tag{41}$$

4. Quantum dynamics, density function, and density operator

We consider the general wave functions (20), with the Lagrangian (19) as a function of the Hamiltonian,

$$\psi\left(\vec{r},t\right)=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\varphi\left(\vec{p},t\right)e^{\frac{i}{\hbar}\left\{\vec{p}\vec{r}-\left[\vec{p}\dot{\vec{r}}-H\left(\vec{p},\vec{r}\right)\right]t\right\}}\mathrm{d}^3\vec{p}=e^{\frac{i}{\hbar}\vec{p}\vec{r}}\psi_t\left(\vec{r},t\right)$$
$$\varphi\left(\vec{p},t\right)=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\psi\left(\vec{r},t\right)e^{-\frac{i}{\hbar}\left\{\vec{p}\vec{r}-\left[\vec{p}\dot{\vec{r}}-H\left(\vec{p},\vec{r}\right)\right]t\right\}}\mathrm{d}^3\vec{r}=e^{-\frac{i}{\hbar}\vec{p}\vec{r}}\varphi_t\left(\vec{p},t\right), \tag{42}$$

of the form of the propagation operators $e^{\frac{i}{\hbar}\vec{p}\vec{r}}$, $e^{-\frac{i}{\hbar}\vec{p}\vec{r}}$ applied to the time dependent wave functions $\psi_t\left(\vec{r},t\right)$ and $\varphi_t\left(\vec{p},t\right)$, as solutions of the dynamic equations

$$i\hbar\frac{\partial}{\partial t}\psi_t\left(\vec{r},t\right)=\left[\vec{p}\dot{\vec{r}}-H\left(\vec{p},\vec{r}\right)\right]\psi_t\left(\vec{r},t\right)$$
$$i\hbar\frac{\partial}{\partial t}\varphi_t\left(\vec{p},t\right)=-\left[\vec{p}\dot{\vec{r}}-H\left(\vec{p},\vec{r}\right)\right]\varphi_t\left(\vec{p},t\right). \tag{43}$$

From the first expression (10) with the first wave function (42), we obtain the matter density

$$\rho_M\left(\vec{r},t\right)=M\left|\psi\left(\vec{r},t\right)\right|^2=M\psi_t^*\left(\vec{r},t\right)e^{-\frac{i}{\hbar}\vec{p}\vec{r}}e^{\frac{i}{\hbar}\vec{p}\vec{r}}\psi_t\left(\vec{r},t\right)$$
$$=M\psi_t^*\left(\vec{r},t\right)\psi_t\left(\vec{r},t\right)=M\rho\left(\vec{r},t\right), \tag{44}$$

depending on the density function

$$\rho\left(\vec{r},t\right)=\psi_t^*\left(\vec{r},t\right)\psi_t\left(\vec{r},t\right)=\psi_t\left(\vec{r},t\right)\psi_t^*\left(\vec{r},t\right)=\left\langle\vec{r}\,|\psi_t\right\rangle\left\langle\psi_t\,|\vec{r}\right\rangle$$
$$=\left\langle\vec{r}\,|\rho(t)|\,\vec{r}\right\rangle, \tag{45}$$

as the diagonal element of the density operator

$$\rho(t)=\left|\psi_t\right\rangle\left\langle\psi_t\right|. \tag{46}$$

At the same time, for the first dynamic equation (43), with operators applied to the wave functions,

$$\frac{\partial}{\partial t}\left\langle\vec{r}\left|\psi_t(t)\right.\right\rangle=-\frac{i}{\hbar}\left[\vec{p}\dot{\vec{r}}-H\left(\vec{p},\vec{r}\right)\right]\left\langle\vec{r}\left|\psi_t(t)\right.\right\rangle, \tag{47}$$

we consider the form with operators applied to state vectors,

$$\left\langle\vec{r}\,\left|\frac{\partial}{\partial t}\right|\psi_t(t)\right\rangle=-\frac{i}{\hbar}\left\langle\vec{r}\,\left|\left[\vec{p}\dot{\vec{r}}-H\left(\vec{p},\vec{r}\right)\right]\right|\psi_t(t)\right\rangle. \tag{48}$$

Thus, for the time dependent state vectors we obtain the dynamic equation

$$\frac{\partial}{\partial t}\left|\psi_t(t)\right\rangle = -\frac{\mathrm{i}}{\hbar}L(\vec{p},\vec{r})\left|\psi_t(t)\right\rangle = -\frac{\mathrm{i}}{\hbar}\left[\vec{p}\dot{\vec{r}} - H(\vec{p},\vec{r})\right]\left|\psi_t(t)\right\rangle. \tag{49}$$

With this equation, we derive the dynamic equation for the density operator (46),

$$\begin{aligned}\frac{\partial}{\partial t}\rho(t) &= \frac{\partial}{\partial t}\left|\psi_t(t)\right\rangle\left\langle\psi_t(t)\right| + \left|\psi_t(t)\right\rangle\frac{\partial}{\partial t}\left\langle\psi_t(t)\right| \\ &= -\frac{\mathrm{i}}{\hbar}L(\vec{p},\vec{r})\left|\psi_t(t)\right\rangle\left\langle\psi_t(t)\right| + \frac{\mathrm{i}}{\hbar}\left|\psi_t(t)\right\rangle\left\langle\psi_t(t)\right|L(\vec{p},\vec{r}).\end{aligned} \tag{50}$$

We obtain the dynamic equation of the density operator

$$\frac{\partial}{\partial t}\rho(t) = -\frac{\mathrm{i}}{\hbar}\left[L(\vec{p},\vec{r}),\rho(t)\right] = -\frac{\mathrm{i}}{\hbar}\left[\vec{p}\dot{\vec{r}} - H(\vec{p},\vec{r}),\rho(t)\right]. \tag{51}$$

5. Quantum dynamics and matter conservation

We consider the first dynamic equation (43) of a quantum particle with an energy $H(\vec{p},\vec{r}) = E$, with the explicit momentum operator,

$$\frac{\partial}{\partial t}\psi_t(\vec{r},t) = -\frac{\mathrm{i}}{\hbar}\left(-\mathrm{i}\hbar\dot{\vec{r}}\frac{\partial}{\partial\vec{r}} - E\right)\psi_t(\vec{r},t), \tag{52}$$

which, with its complex conjugate form

$$\frac{\partial}{\partial t}\psi_t^*(\vec{r},t) = \frac{\mathrm{i}}{\hbar}\left(\mathrm{i}\hbar\dot{\vec{r}}\frac{\partial}{\partial\vec{r}} - E\right)\psi_t^*(\vec{r},t), \tag{53}$$

leads to the dynamic equation of the density function (45),

$$\begin{aligned}\frac{\partial}{\partial t}\rho(\vec{r},t) &= \frac{\partial}{\partial t}\psi_t^*(\vec{r},t)\psi_t(\vec{r},t) + \psi_t^*(\vec{r},t)\frac{\partial}{\partial t}\psi_t(\vec{r},t) \\ &= \frac{\mathrm{i}}{\hbar}\left(\mathrm{i}\hbar\dot{\vec{r}}\frac{\partial}{\partial\vec{r}} - \cancel{E}\right)\psi_t^*(\vec{r},t)\psi_t(\vec{r},t) - \psi_t^*(\vec{r},t)\frac{\mathrm{i}}{\hbar}\left(-\mathrm{i}\hbar\dot{\vec{r}}\frac{\partial}{\partial\vec{r}} - \cancel{E}\right)\psi_t(\vec{r},t) \\ &= -\dot{\vec{r}}\frac{\partial}{\partial\vec{r}}\rho(\vec{r},t).\end{aligned} \tag{54}$$

Thus, for the density function, we obtain the conservation equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho\left(\vec{r},t\right)=\frac{\partial}{\partial t}\rho\left(\vec{r},t\right)+\dot{\vec{r}}\,\frac{\partial}{\partial\vec{r}}\rho\left(\vec{r},t\right)=0, \tag{55}$$

with the integral form

$$\frac{\partial}{\partial t}\int_V\rho\left(\vec{r},t\right)\mathrm{d}^3\vec{r}=-\oint_{\Sigma_V}\rho\left(\vec{r},t\right)\dot{\vec{r}}\mathrm{d}^2\vec{r}=-\oint_{\Sigma_V}\vec{J}\left(\vec{r},\dot{\vec{r}},t\right)\mathrm{d}^2\vec{r} \tag{56}$$

of the time variation of the matter density $\rho\left(\vec{r},t\right)$ in an arbitrary volume V, as a flow with a flux density

$$\vec{J}\left(\vec{r},\dot{\vec{r}},t\right)=\rho\left(\vec{r},t\right)\dot{\vec{r}} \tag{57}$$

through the surface Σ_V of this volume. The null total time derivative of the density function (55) describes a quantum particle propagating in space as a constant/invariant distribution of matter.

In the case of a non-relativistic velocity, for a particle in a potential of energy $U\left(\vec{r}\right)$, the first dynamic equation (43) and its complex conjugate equation take the form

$$\begin{aligned}\frac{\partial}{\partial t}\psi_t\left(\vec{r},t\right)&=\left[-\dot{\vec{r}}\,\frac{\partial}{\partial\vec{r}}-\frac{\mathrm{i}\hbar}{2M}\frac{\partial^2}{\partial\vec{r}^{\,2}}+\frac{\mathrm{i}}{\hbar}U\left(\vec{r}\right)\right]\psi_t\left(\vec{r},t\right)\\ \frac{\partial}{\partial t}\psi_t^*\left(\vec{r},t\right)&=\left[-\dot{\vec{r}}\,\frac{\partial}{\partial\vec{r}}+\frac{\mathrm{i}\hbar}{2M}\frac{\partial^2}{\partial\vec{r}^{\,2}}-\frac{\mathrm{i}}{\hbar}U\left(\vec{r}\right)\right]\psi_t^*\left(\vec{r},t\right).\end{aligned} \tag{58}$$

By multiplying the first equation with the complex conjugate wave function, and the second equation with the wave function, and integrating over an arbitrary spatial volume V, we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int_V \rho(\vec{r},t)\mathrm{d}^3\vec{r} = \frac{\partial}{\partial t}\int_V \psi_t^*(\vec{r},t)\psi_t(\vec{r},t)\mathrm{d}^3\vec{r} \\
&\quad = \int_V \left[\psi_t(\vec{r},t)\frac{\partial}{\partial t}\psi_t^*(\vec{r},t) + \psi_t^*(\vec{r},t)\frac{\partial}{\partial t}\psi_t(\vec{r},t)\right]\mathrm{d}^3\vec{r} \\
&\quad = \int_V \left\{\begin{aligned} &-\dot{\vec{r}}\frac{\partial}{\partial \vec{r}}\left[\psi_t^*(\vec{r},t)\psi_t(\vec{r},t)\right] \\ &-\frac{\mathrm{i}\hbar}{2M}\frac{\partial}{\partial \vec{r}}\left[\psi_t^*(\vec{r},t)\frac{\partial}{\partial \vec{r}}\psi_t(\vec{r},t) - \psi_t(\vec{r},t)\frac{\partial}{\partial \vec{r}}\psi_t^*(\vec{r},t)\right]\end{aligned}\right\}\mathrm{d}^3\vec{r} \\
&\quad = \oint_{\Sigma_V} \left\{\begin{aligned} &-\dot{\vec{r}}\psi_t^*(\vec{r},t)\psi_t(\vec{r},t) \\ &-\frac{\mathrm{i}\hbar}{2M}\left[\psi_t^*(\vec{r},t)\frac{\partial}{\partial \vec{r}}\psi_t(\vec{r},t) - \psi_t(\vec{r},t)\frac{\partial}{\partial \vec{r}}\psi_t^*(\vec{r},t)\right]\end{aligned}\right\}\mathrm{d}^2\vec{r} \\
&\quad = \oint_{\Sigma_V} \left\{-\dot{\vec{r}}\rho(\vec{r},t) - \frac{\mathrm{i}\hbar}{2M}\left[\psi_t^*(\vec{r},t)\frac{\partial}{\partial \vec{r}}\psi_t(\vec{r},t) - \psi_t(\vec{r},t)\frac{\partial}{\partial \vec{r}}\psi_t^*(\vec{r},t)\right]\right\}\mathrm{d}^2\vec{r}.
\end{aligned}
$$

This equation is of the form

$$\frac{\partial}{\partial t}\int_V \rho(\vec{r},t)\mathrm{d}^3\vec{r} = -\oint_{\Sigma_V} \vec{J}_\psi\left(\vec{r},\dot{\vec{r}},t\right)\mathrm{d}^2\vec{r}\ , \tag{59}$$

depending on the quantum matter flow density

$$\vec{J}_\psi\left(\vec{r},\dot{\vec{r}},t\right) = \dot{\vec{r}}\rho(\vec{r},t) + \frac{\mathrm{i}\hbar}{2M}\left[\psi_t^*(\vec{r},t)\frac{\partial}{\partial \vec{r}}\psi_t(\vec{r},t) - \psi_t(\vec{r},t)\frac{\partial}{\partial \vec{r}}\psi_t^*(\vec{r},t)\right], \tag{60}$$

which, besides the drift term (57) proportional to the velocity, contains a diffusion term, proportional to the gradient of the time dependent wave function.

6. Fermi's golden rule for a quantum particle as a distribution of matter

For a quantum particle with a Hamiltonian $H_0(\vec{p},\vec{r})$, we consider a scattering /tunneling process described by a perturbing potential εV with a strength coefficient ε [9], described by the dynamic equation (51),

$$\frac{\partial}{\partial t}\rho(t) = -\frac{\mathrm{i}}{\hbar}\left[\vec{p}\dot{\vec{r}} - H_0(\vec{p},\vec{r}) - \varepsilon V, \rho(t)\right] = -\frac{\mathrm{i}}{\hbar}\left[L(\vec{p},\vec{r}) - \varepsilon V, \rho(t)\right]. \tag{61}$$

For the interaction picture operators

$$\tilde{\rho}(t)=e^{\frac{\mathrm{i}}{\hbar}L(\vec{p},\vec{r})t}\rho(t)e^{-\frac{\mathrm{i}}{\hbar}L(\vec{p},\vec{r})t}$$
$$\tilde{V}(t)=e^{\frac{\mathrm{i}}{\hbar}L(\vec{p},\vec{r})t}Ve^{-\frac{\mathrm{i}}{\hbar}L(\vec{p},\vec{r})t}, \tag{62}$$

this equation takes the interaction picture form

$$\frac{\partial}{\partial t}\tilde{\rho}(t)=\frac{\mathrm{i}}{\hbar}\left[\varepsilon\tilde{V}(t),\tilde{\rho}(t)\right]. \tag{63}$$

With a series expansion of the density operator

$$\tilde{\rho}(t)=\tilde{\rho}^{(0)}(t)+\varepsilon\tilde{\rho}^{(1)}(t)+\varepsilon^2\tilde{\rho}^{(2)}(t)+\ldots, \tag{64}$$

this equation takes the form

$$\frac{\partial}{\partial t}\left[\tilde{\rho}^{(0)}(t)+\varepsilon\tilde{\rho}^{(1)}(t)+\varepsilon^2\tilde{\rho}^{(2)}(t)+\ldots\right]$$
$$=\frac{\mathrm{i}}{\hbar}\left[\varepsilon\tilde{V}(t),\tilde{\rho}^{(0)}(t)+\varepsilon\tilde{\rho}^{(1)}(t)+\varepsilon^2\tilde{\rho}^{(2)}(t)+\ldots\right]$$

with the components

$$\frac{\partial}{\partial t}\tilde{\rho}^{(0)}(t)=0$$
$$\frac{\partial}{\partial t}\tilde{\rho}^{(1)}(t)=\frac{\mathrm{i}}{\hbar}\left[\tilde{V}(t),\tilde{\rho}^{(0)}(t)\right]$$
$$\frac{\partial}{\partial t}\tilde{\rho}^{(2)}(t)=\frac{\mathrm{i}}{\hbar}\left[\tilde{V}(t),\tilde{\rho}^{(1)}(t)\right] \tag{65}$$
$$\ldots .$$

As solutions of the first two equations, we consider the time independent zeroth order tem

$$\tilde{\rho}^{(0)}(t)=|0\rangle\langle 0|, \tag{66}$$

and the time dependent first order term

$$\tilde{\rho}^{(1)}(t)=\frac{\mathrm{i}}{\hbar}\int_0^t\left[\tilde{V}(t'),\tilde{\rho}^{(0)}\right]\mathrm{d}t', \tag{67}$$

as the third equation takes the form

$$\frac{\partial}{\partial t}\tilde{\rho}^{(2)}(t)=\frac{\mathrm{i}}{\hbar}\left[\tilde{V}(t),\tilde{\rho}^{(1)}(t)\right]=-\frac{1}{\hbar^2}\int_0^t\left[\tilde{V}(t),\left[\tilde{V}(t'),\tilde{\rho}^{(0)}\right]\right]\mathrm{d}t'. \tag{68}$$

From these equations we obtain the decay rate of the particle, by a scattering/tunneling process from the initial state $|0\rangle$ to the final state $|i\rangle$,

$$\Gamma_i=\frac{\partial}{\partial t}\langle i|\tilde{\rho}(t)|i\rangle=\frac{1}{\hbar^2}\int_0^t\left[\tilde{V}_{i0}(t)\tilde{V}_{0i}(t')+\tilde{V}_{i0}(t')\tilde{V}_{0i}(t)\right]\mathrm{d}t'. \tag{69}$$

From the expression (62) of the scattering/tunneling operator, we obtain the matrix elements of the form

$$\begin{aligned}\tilde{V}_{i0}(t)&=\langle i|\tilde{V}(t)|0\rangle=e^{\frac{\mathrm{i}}{\hbar}(L_i-L_0)t}V_{i0}, \quad \tilde{V}_{0i}(t')=\langle 0|\tilde{V}(t')|i\rangle=e^{-\frac{\mathrm{i}}{\hbar}(L_i-L_0)t'}V_{0i}\\ \tilde{V}_{i0}(t')&=\langle i|\tilde{V}(t')|0\rangle=e^{\frac{\mathrm{i}}{\hbar}(L_i-L_0)t'}V_{i0}, \quad \tilde{V}_{0i}(t)=\langle 0|\tilde{V}(t)|i\rangle=e^{-\frac{\mathrm{i}}{\hbar}(L_i-L_0)t}V_{0i}.\end{aligned} \tag{70}$$

With the notation

$$\omega_i=\frac{L_i-L_0}{\hbar}, \tag{71}$$

the decay rate (69) takes the explicit form

$$\begin{aligned}\Gamma_i&=\frac{\partial}{\partial t}\tilde{\rho}_{ii}(t)=\frac{1}{\hbar^2}\int_0^t|V_{i0}|^2\left(e^{\mathrm{i}\omega_i(t-t')}+e^{-\mathrm{i}\omega_i(t-t')}\right)\mathrm{d}t'\\ &=\frac{2}{\hbar^2}\int_0^t|V_{i0}|^2\cos\left[\omega_i(t-t')\right]\mathrm{d}t'\\ &=-\frac{2}{\hbar^2}|V_{i0}|^2\left.\frac{\sin\left[\omega_i(t-t')\right]}{\omega_i}\right|_0^t=\frac{2}{\hbar^2}|V_{i0}|^2\frac{\sin(\omega_i t)}{\omega_i}.\end{aligned} \tag{72}$$

Fermi's golden rule refers to the scattering/tunneling rate of a particle from an initial state $|0\rangle$ to a final state of a channel $(i)=(|i_1\rangle,|i_2\rangle,...\ |i_N\rangle)$, with a density $g(\omega_i)$,

$$\Gamma_{(i)}=\frac{\partial}{\partial t}\sum_i\tilde{\rho}_{ii}(t)=\frac{2}{\hbar^2}|V_{i0}|^2\,g(\omega_i)\int_{-\infty}^{\infty}\frac{\sin(\omega_i t)}{\omega_i}\mathrm{d}\omega_i=\frac{2\pi}{\hbar^2}|V_{i0}|^2\,g(\omega_i). \tag{73}$$

A non-relativistic particle in a potential $U(\vec{r})$, in an initial time dependent state $\psi_{t0}(\vec{r},t)$, of energy E_0, under the action of a scattering/tunneling operator V [9], reaching the final state $\psi_{ti}(\vec{r},t)$, is described by the dynamic equations

$$\begin{aligned}&(H_0+V)\psi_{t0}(\vec{r},t)=\left[-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial\vec{r}^2}+U(\vec{r})+V\right]\psi_{t0}(\vec{r},t)=E_0\psi_{t0}(\vec{r},t)\\&H_0\psi_{ti}^*(\vec{r},t)=\left[-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial\vec{r}^2}+U(\vec{r})\right]\psi_{ti}^*(\vec{r},t)=E_i\psi_{ti}^*(\vec{r},t)\end{aligned} \tag{74}$$

From the difference of these equations multiplied with $\psi_{ti}^*(\vec{r},t)$ and $\psi_{t0}(\vec{r},t)$, integrated over the volume Ω mainly occupied by the initial state, we obtain the matrix element

$$\begin{aligned}V_{i0}&=\frac{\hbar^2}{2M}\int_\Omega\left[\psi_{ti}^*(\vec{r},t)\frac{\partial^2}{\partial\vec{r}^2}\psi_{t0}(\vec{r},t)-\psi_{t0}(\vec{r},t)\frac{\partial^2}{\partial\vec{r}^2}\psi_{ti}^*(\vec{r},t)\right]\mathrm{d}^3\vec{r}\\&=\frac{\hbar^2}{2M}\oint_{\Sigma_\Omega}\left[\psi_{ti}^*(\vec{r},t)\frac{\partial}{\partial\vec{r}}\psi_{t0}(\vec{r},t)-\psi_{t0}(\vec{r},t)\frac{\partial}{\partial\vec{r}}\psi_{ti}^*(\vec{r},t)\right]\mathrm{d}^2\vec{r}.\end{aligned} \tag{75}$$

7. The Maxwell-Lorentz equations as characteristics of a field interacting with a quantum particle

We consider a quantum particle with the wave functions of the form (20),

$$\begin{aligned}\psi(\vec{r},t)&=\frac{1}{(2\pi\hbar)^{3/2}}\int\varphi(\vec{P},t)e^{\frac{\mathrm{i}}{\hbar}\left[\vec{P}\cdot\vec{r}-L(\vec{r},\dot{\vec{r}})t\right]}\mathrm{d}^3\vec{P}\\\varphi(\vec{P},t)&=\frac{1}{(2\pi\hbar)^{3/2}}\int\psi(\vec{r},t)e^{-\frac{\mathrm{i}}{\hbar}\left[\vec{P}\cdot\vec{r}-L(\vec{r},\dot{\vec{r}})t\right]}\mathrm{d}^3\vec{r},\end{aligned} \tag{76}$$

with a Lagrangian which, besides the relativistic term includes additional terms for the interaction with a field, with a vector potential $\vec{A}(\vec{r},t)$ conjugated to the coordinates, as in the Aharonov-Bohm effect [10], and a scalar potential $U(\vec{r})$ conjugated to time, that we call the electric potential [3-6],

$$L(\vec{r},\dot{\vec{r}},t) = -Mc^2\sqrt{1-\frac{\dot{\vec{r}}^2}{c^2}} - eU(\vec{r}) + e\vec{A}(\vec{r},t)\dot{\vec{r}} \ . \tag{77}$$

Here we considered only the vector potential as a function of time, as the electric potential is only a function of the spatial coordinates, as in the problems of interest of the atom field interaction [11-14]. From this expression, we obtain the canonical momentum

$$\vec{P} = \frac{\partial}{\partial\dot{\vec{r}}} L(\vec{r},\dot{\vec{r}},t) = \frac{M\dot{\vec{r}}}{\sqrt{1-\frac{\dot{\vec{r}}^2}{c^2}}} + e\vec{A}(\vec{r},t) = \vec{p} + e\vec{A}(\vec{r},t), \tag{78}$$

as a sum of the mechanical momentum

$$\vec{p} = \frac{M\dot{\vec{r}}}{\sqrt{1-\frac{\dot{\vec{r}}^2}{c^2}}}, \tag{79}$$

with the electromagnetic momentum $e\vec{A}(\vec{r},t)$. With these expressions, we obtain the Hamiltonian

$$\begin{aligned} H(\vec{P},\vec{r},t) &= \vec{P}\dot{\vec{r}} - L(\vec{r},\dot{\vec{r}},t) \\ &= \left[\frac{M\dot{\vec{r}}}{\sqrt{1-\frac{\dot{\vec{r}}^2}{c^2}}} + \underline{e\vec{A}(\vec{r},t)}\right]\dot{\vec{r}} - \left[-\mathrm{M}c^2\sqrt{1-\frac{\dot{\vec{r}}^2}{c^2}} - eU(\vec{r}) + \underline{e\vec{A}(\vec{r},t)}\dot{\vec{r}}\ \right] \\ &= \frac{Mc^2}{\sqrt{1-\frac{\dot{\vec{r}}^2}{c^2}}} + eU(\vec{r}) = cE, \end{aligned} \tag{80}$$

as constant of motion cE, which includes the mechanical energy of the particle and the potential energy of this particle in the electromagnetic field. With (78), (79) and the algebraic relation

$$\frac{M^2c^2}{1-\frac{\dot{\vec{r}}^2}{c^2}} = \frac{M^2\dot{\vec{r}}^2}{1-\frac{\dot{\vec{r}}^2}{c^2}} + M^2c^2 = \vec{p}^2 + M^2c^2, \tag{81}$$

one obtains the Hamiltonian

$$\begin{aligned} H\left(\vec{P},\vec{r}\right) &= c\sqrt{M^2c^2+\vec{p}^2} + eU\left(\vec{r}\right) = c\sqrt{M^2c^2+\left[\vec{P}-e\vec{A}\left(\vec{r},t\right)\right]^2} + eU\left(\vec{r}\right) \\ &= cE, \end{aligned} \tag{82}$$

as the wave functions (76), with this Hamiltonian as a constant of motion E, take a form

$$\begin{aligned} \psi\left(\vec{r},t\right) &= \frac{1}{\left(2\pi\hbar\right)^{3/2}}\int \varphi\left(\vec{P},t\right) e^{\frac{\mathrm{i}}{\hbar}\left\{\vec{P}\cdot\vec{r}-\left[\vec{P}\dot{\vec{r}}-H\left(\vec{P},\vec{r}\right)\right]t\right\}} \mathrm{d}^3\vec{P} \\ &= \frac{1}{\left(2\pi\hbar\right)^{3/2}}\int \varphi\left(\vec{P},t\right) e^{\frac{\mathrm{i}}{\hbar}\left[\vec{P}\cdot\vec{r}-\left(\vec{P}\dot{\vec{r}}-cE\right)t\right]} \mathrm{d}^3\vec{P} \\ \varphi\left(\vec{P},t\right) &= \frac{1}{\left(2\pi\hbar\right)^{3/2}}\int \psi\left(\vec{r},t\right) e^{-\frac{\mathrm{i}}{\hbar}\left\{\vec{P}\cdot\vec{r}-\left[\vec{P}\dot{\vec{r}}-H\left(\vec{P},\vec{r}\right)\right]t\right\}} \mathrm{d}^3\vec{r} \\ &= \frac{1}{\left(2\pi\hbar\right)^{3/2}}\int \psi\left(\vec{r},t\right) e^{-\frac{\mathrm{i}}{\hbar}\left[\vec{P}\cdot\vec{r}-\left(\vec{P}\dot{\vec{r}}-cE\right)t\right]} \mathrm{d}^3\vec{r}. \end{aligned} \tag{83}$$

From the first wave function, we obtain wave velocities in the coordinate space,

$$\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_{\text{wave}} = \frac{\partial}{\partial\vec{P}}\left(\vec{P}\dot{\vec{r}}-E\right) = \dot{\vec{r}}, \tag{84}$$

equal to velocity $\dot{\vec{r}}$ of the matter distribution described by this wave function. From the second wave function (76) with the Lagrangian (77), we obtain the time variation of the momentum (78), as the velocity of this wave in the momentum space

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{P} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{p} + e\frac{\mathrm{d}}{\mathrm{d}t}\vec{A}\left(\vec{r},t\right) = \frac{\partial}{\partial\vec{r}}L\left(\vec{r},\dot{\vec{r}},t\right) = -e\frac{\partial}{\partial\vec{r}}U\left(\vec{r}\right) + e\frac{\partial}{\partial\vec{r}}\left[\vec{A}\left(\vec{r},t\right)\dot{\vec{r}}\right]. \tag{85}$$

With the second term of this expression from algebraic relation

$$\dot{\vec{r}} \times \left[\frac{\partial}{\partial \vec{r}} \times \vec{A}(\vec{r},t) \right] = \frac{\partial}{\partial \vec{r}} \left[\dot{\vec{r}} \vec{A}(\vec{r},t) \right] - \left(\dot{\vec{r}} \frac{\partial}{\partial \vec{r}} \right) \vec{A}(\vec{r},t), \tag{86}$$

depending on the magnetic induction

$$\vec{B}(\vec{r},t) = \frac{\partial}{\partial \vec{r}} \times \vec{A}(\vec{r},t), \tag{87}$$

we obtain Lorentz's force

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \vec{p} &= -e \frac{\partial}{\partial \vec{r}} U(\vec{r}) - e \frac{\mathrm{d}}{\mathrm{d}t} \vec{A}(\vec{r},t) + \dot{\vec{r}} \times \vec{B}(\vec{r},t) + \left(\dot{\vec{r}} \frac{\partial}{\partial \vec{r}} \right) \vec{A}(\vec{r},t) \\ &= -e \frac{\partial}{\partial \vec{r}} U(\vec{r}) - e \frac{\partial}{\partial t} \vec{A}(\vec{r},t) + \dot{\vec{r}} \times \vec{B}(\vec{r},t) \\ &= e\vec{E}(\vec{r},t) + \dot{\vec{r}} \times \vec{B}(\vec{r},t), \end{aligned} \tag{88}$$

as a function of the magnetic induction $\vec{B}(\vec{r},t)$, and the electric field

$$\vec{E}(\vec{r},t) = -\frac{\partial}{\partial \vec{r}} U(\vec{r}) - \frac{\partial}{\partial t} \vec{A}(\vec{r},t). \tag{89}$$

From (87), we obtain the Gauss-Maxwell law of the magnetic flux

$$\frac{\partial}{\partial \vec{r}} \vec{B}(\vec{r},t) = 0. \tag{90}$$

From the divergence of the electric field (89), with the gouge condition

$$\frac{\partial}{\partial \vec{r}} \vec{A}(\vec{r},t) = 0, \tag{91}$$

we obtain the Gauss-Maxwell law of the electric flux,

$$\frac{\partial}{\partial \vec{r}} \vec{E}(\vec{r},t) = -\frac{\partial^2}{\partial \vec{r}^{\,2}} U(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_0}, \tag{92}$$

with the electric charge density as a source of the electric field,

$$\rho(\vec{r}) = \varepsilon_0 \frac{\partial}{\partial \vec{r}} \vec{E}(\vec{r},t) = -\varepsilon_0 \frac{\partial^2}{\partial \vec{r}^2} U(\vec{r}), \tag{93}$$

depending on the universal constant ε_0 called electric permittivity. From the curl of the electric field (89) with (87), we obtain the Faraday-Maxwell law of the electromagnetic induction,

$$\frac{\partial}{\partial \vec{r}} \times \vec{E}(\vec{r},t) = -\frac{\partial}{\partial t} \vec{B}(\vec{r},t). \tag{94}$$

From (87), we obtain the curl of the magnetic field

$$\begin{aligned} \frac{\partial}{\partial \vec{r}} \times \vec{B}(\vec{r},t) &= \frac{\partial}{\partial \vec{r}} \times \left[\frac{\partial}{\partial \vec{r}} \times \vec{A}(\vec{r},t) \right] = \frac{\partial}{\partial \vec{r}} \left[\frac{\partial}{\partial \vec{r}} \vec{A}(\vec{r},t) \right] - \frac{\partial^2}{\partial \vec{r}^2} \vec{A}(\vec{r},t) \\ &= -\frac{\partial^2}{\partial \vec{r}^2} \vec{A}(\vec{r},t), \end{aligned} \tag{95}$$

as a function of the Laplacian of the vector potential $\vec{A}(\vec{r},t)$. We consider this vector potential, interacting with a quantum particle, as a wave propagating with the maximum matter velocity c, in a dissipative environment with a decay rate γ,

$$\frac{\partial^2}{\partial \vec{r}^2} \vec{A}(\vec{r},t) = \frac{1}{c^2} \left[\frac{\partial^2}{\partial t^2} \vec{A}(\vec{r},t) + \gamma \frac{\partial}{\partial t} \vec{A}(\vec{r},t) \right]. \tag{96}$$

From the expression (95) with this Laplacian,

$$\frac{\partial}{\partial \vec{r}} \times \vec{B}(\vec{r},t) = \frac{1}{c^2} \left[\frac{\partial}{\partial t} \vec{E}(\vec{r},t) - \gamma \frac{\partial}{\partial t} \vec{A}(\vec{r},t) \right], \tag{97}$$

we obtain the Ampère-Maxwell law of the magnetic circuit,

$$\frac{1}{\mu_0} \frac{\partial}{\partial \vec{r}} \times \vec{B}(\vec{r},t) = \vec{j} + \varepsilon_0 \frac{\partial}{\partial t} \vec{E}(\vec{r},t), \tag{98}$$

depending on the electric permittivity ε_0, the magnetic permeability μ_0,

$$\varepsilon_0 \mu_0 = \frac{1}{c^2}, \tag{99}$$

and the current density

$$\vec{j} = -\varepsilon_0 \gamma \frac{\partial}{\partial t} \vec{A}(\vec{r}, t) = \varepsilon_0 \gamma \left[\vec{E}(\vec{r}, t) + \frac{\partial}{\partial \vec{r}} U(\vec{r}) \right]. \tag{100}$$

With the electric field $\vec{E}(\vec{r}, t)$, and the magnetic field

$$\vec{H}(\vec{r}, t) = \frac{1}{\mu_0} \vec{B}(\vec{r}, t), \tag{101}$$

the Maxwell equations (94), (98), (92), and (90), take the form

$$\begin{aligned}
&\frac{\partial}{\partial \vec{r}} \times \vec{E}(\vec{r}, t) = -\mu_0 \frac{\partial}{\partial t} \vec{H}(\vec{r}, t) \\
&\frac{\partial}{\partial \vec{r}} \times \vec{H}(\vec{r}, t) = \vec{j}(\vec{r}, t) + \varepsilon_0 \frac{\partial}{\partial t} \vec{E}(\vec{r}, t) \\
&\frac{\partial}{\partial \vec{r}} \vec{E}(\vec{r}, t) = \frac{\rho(\vec{r}, t)}{\varepsilon_0} \\
&\frac{\partial}{\partial \vec{r}} \vec{H}(\vec{r}, t) = 0.
\end{aligned} \tag{102}$$

8. Dynamics of a quantum particle in electromagnetic field

We consider a quantum particle in electromagnetic field $U(\vec{r}), \vec{A}(\vec{r}, t)$, described by the wave functions (83),

$$\begin{aligned}
\psi(\vec{r}, t) &= \frac{1}{(2\pi\hbar)^{3/2}} \int e^{\frac{i}{\hbar}\left\{\vec{P}\cdot\vec{r} - \left[\vec{P}\dot{\vec{r}} - H(\vec{P}, \vec{r})\right]t\right\}} \varphi(\vec{P}, t) \mathrm{d}^3\vec{P} \\
\varphi(\vec{P}, t) &= \frac{1}{(2\pi\hbar)^{3/2}} \int e^{-\frac{i}{\hbar}\left\{\vec{P}\cdot\vec{r} - \left[\vec{P}\dot{\vec{r}} - H(\vec{P}, \vec{r})\right]t\right\}} \psi(\vec{r}, t) \mathrm{d}^3\vec{r},
\end{aligned} \tag{103}$$

with a Hamiltonian of the form (82)

$$H\left(\vec{P},\vec{r}\right)=c\sqrt{M^2c^2+\vec{p}^2}+eU\left(\vec{r}\right)=c\sqrt{M^2c^2+\left[\vec{P}-e\vec{A}\left(\vec{r},t\right)\right]^2}+eU\left(\vec{r}\right). \tag{104}$$

For the first term of this Hamiltonian, we consider Dirac's form

$$\begin{aligned} H &= c\left\{\begin{matrix} \alpha_0 Mc+\alpha_1\left[P^1-eA^1\left(\vec{r},t\right)\right] \\ +\alpha_2\left[P^2-eA^2\left(\vec{r},t\right)\right]+\alpha_3\left[P^3-eA^3\left(\vec{r},t\right)\right]\end{matrix}\right\}+eU\left(\vec{r}\right) \\ &= \alpha_0 Mc^2+eU\left(\vec{r}\right)+c\alpha_j\left[P^j-eA^j\left(\vec{r},t\right)\right], \end{aligned} \tag{105}$$

depending on Dirac's operators

$$\alpha_0=\begin{pmatrix}\hat{1} & 0\\ 0 & -\hat{1}\end{pmatrix},\quad \alpha_i=\begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix},\quad i=1,2,3, \tag{106}$$

as functions of the Pauli spin operators

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{107}$$

with the algebra

$$\sigma_i\sigma_j=\mathrm{i}\varepsilon_{ijk}\sigma_k, \tag{108}$$

where ε_{ijk} is the Levi-Civita symbol, the anticommutation relations

$$\left\{\sigma_i,\sigma_j\right\}=2\delta_{ij}\hat{1},\qquad \hat{1}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{109}$$

and the normalization relations

$$\sigma_i^2=\hat{1}. \tag{110}$$

With these relations, we obtain a similar algebra for the Dirac operators,

$$\alpha_i\alpha_j=\mathrm{i}\varepsilon_{ijk}\alpha_k, \tag{111}$$

with the anticommutation relations

$$\left\{\alpha_\mu,\alpha_\nu\right\}=2\delta_{\mu\nu}\hat{\hat{1}},\qquad \hat{\hat{1}}\,\square\begin{pmatrix}\hat{1} & 0\\ 0 & \hat{1}\end{pmatrix},\qquad \mu,\nu=0,1,2,3\,, \tag{112}$$

and the normalization relations

$$\alpha_\mu{}^2=\hat{\hat{1}}. \tag{113}$$

With the Hamiltonian (105), the wave function (103) in the coordinate space,

$$\begin{aligned}\psi\left(\vec{r},t\right)&=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int e^{\frac{\mathrm{i}}{\hbar}\left\{\vec{P}\cdot\vec{r}-\left[\vec{P}\dot{\vec{r}}-\left(\alpha_0Mc^2+eU(\vec{r})+c\alpha_j\left(P^j-eA^j(\vec{r},t)\right)\right)\right]t\right\}}\varphi\left(\vec{P},t\right)\mathrm{d}^3\vec{P}\\ &=e^{\frac{\mathrm{i}}{\hbar}\left\{\vec{P}\cdot\vec{r}-\left[-eU(\vec{r})+ec\alpha_jA^j(\vec{r},t)\right]t\right\}}\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int e^{-\frac{\mathrm{i}}{\hbar}\left[\vec{P}\dot{\vec{r}}-\left(\alpha_0Mc^2+c\alpha_jP^j\right)\right]t}\varphi\left(\vec{P},t\right)\mathrm{d}^3\vec{P}\\ &=e^{\frac{\mathrm{i}}{\hbar}\left\{\vec{P}\cdot\vec{r}-\left[-eU(\vec{r})+ec\alpha_jA^j(\vec{r},t)\right]t\right\}}\psi_t\left(\vec{r},t\right)=\mathcal{P}_f\psi_t\left(\vec{r},t\right),\end{aligned} \tag{114}$$

takes the form of the time dependent wave function

$$\psi_t\left(\vec{r},t\right)=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int e^{-\frac{\mathrm{i}}{\hbar}\left[\vec{P}\dot{\vec{r}}-\left(\alpha_0Mc^2+c\alpha_jP^j\right)\right]t}\varphi\left(\vec{P},t\right)\mathrm{d}^3\vec{P}\,, \tag{115}$$

with the propagation operator of the quantum particle in the electromagnetic field,

$$\mathcal{P}_f\left[U\left(\vec{r}\right),A^j\left(\vec{r},t\right)\right]=e^{\frac{\mathrm{i}}{\hbar}\left\{\vec{P}\cdot\vec{r}-\left[-eU(\vec{r})+ec\alpha_jA^j(\vec{r},t)\right]t\right\}}, \tag{116}$$

which, besides the coordinate dependent phase $\frac{\mathrm{i}}{\hbar}\vec{P}\cdot\vec{r}$ for a free particle, where $\vec{P}$ is an operator and $\vec{r}$ is a variable, also includes a time dependent phase $-\frac{\mathrm{i}}{\hbar}\left[-eU\left(\vec{r}\right)+ec\alpha_jA^j\left(\vec{r},t\right)\right]t$, depending on the electromagnetic potentials. With these expressions, and $\vec{P}$ as a variable and $\vec{r}$ as an operator, the wave function (103) in the momentum space,

$$\begin{aligned}
\varphi\left(\vec{P},t\right) &= \frac{1}{\left(2\pi\hbar\right)^{3/2}} e^{-\frac{\mathrm{i}}{\hbar}\vec{P}\cdot\vec{r}} \int e^{\frac{\mathrm{i}}{\hbar}\left[\vec{P}\dot{\vec{r}}-H\left(\vec{P},\vec{r},t\right)\right]t} \psi\left(\vec{r},t\right)\mathrm{d}^3\vec{r} \\
&= \frac{1}{\left(2\pi\hbar\right)^{3/2}} e^{-\frac{\mathrm{i}}{\hbar}\vec{P}\cdot\vec{r}} \int \left[\begin{array}{l} e^{\frac{\mathrm{i}}{\hbar}\left\{\vec{P}\dot{\vec{r}}-\left[\alpha_0 Mc^2+\underline{eU\left(\vec{r}\right)}+c\alpha_j\left(P^j-\underline{\underline{eA^j\left(\vec{r},t\right)}}\right)\right]\right\}t} \\ e^{\frac{\mathrm{i}}{\hbar}\left\{\vec{P}\cdot\vec{r}-\left[-\underline{eU\left(\vec{r}\right)}+\underline{\underline{ec\alpha_j A^j\left(\vec{r},t\right)}}\right]t\right\}} \psi_t\left(\vec{r},t\right) \end{array}\right] \mathrm{d}^3\vec{r} \\
&= \frac{1}{\left(2\pi\hbar\right)^{3/2}} e^{-\frac{\mathrm{i}}{\hbar}\vec{P}\cdot\vec{r}} \int e^{\frac{\mathrm{i}}{\hbar}\left[\vec{P}\dot{\vec{r}}-\left(\alpha_0 Mc^2+c\alpha_j P^j\right)\right]t} \left[e^{\frac{\mathrm{i}}{\hbar}\vec{P}\cdot\vec{r}} \psi_t\left(\vec{r},t\right)\right]\mathrm{d}^3\vec{r} \\
&= e^{-\frac{\mathrm{i}}{\hbar}\vec{P}\cdot\hat{\vec{r}}} \varphi_t\left(\vec{P},t\right) = \mathcal{P}^{-1}\varphi_t\left(\vec{P},t\right),
\end{aligned} \tag{117}$$

takes the form of the inverse propagation operator

$$\mathcal{P}^{-1} = e^{-\frac{\mathrm{i}}{\hbar}\vec{P}\cdot\vec{r}}, \tag{118}$$

applied to a time dependent wave function in the momentum space, as an integral over the coordinate space,

$$\begin{aligned}
\varphi_t\left(\vec{P},t\right) &= \frac{1}{\left(2\pi\hbar\right)^{3/2}} \int e^{\frac{\mathrm{i}}{\hbar}\left[\vec{P}\dot{\vec{r}}-\left(\alpha_0 Mc^2+c\alpha_j P^j\right)\right]t} \left[e^{\frac{\mathrm{i}}{\hbar}\vec{P}\cdot\vec{r}} \psi_t\left(\vec{r},t\right)\right]\mathrm{d}^3\vec{r} \\
&= \frac{1}{\left(2\pi\hbar\right)^{3}} \int e^{\frac{\mathrm{i}}{\hbar}\left[\vec{P}\dot{\vec{r}}-\left(\alpha_0 Mc^2+c\alpha_j P^j\right)\right]t} \left[e^{\frac{\mathrm{i}}{\hbar}\left\{\vec{P}'\cdot\vec{r}-\left[\vec{P}'\dot{\vec{r}}-\left(\alpha_0 Mc^2+c\alpha_j P'^j\right)\right]t\right\}} \varphi\left(\vec{P}',t\right)\right]\mathrm{d}^3\vec{P}'\mathrm{d}^3\vec{r} \\
&= \frac{1}{\left(2\pi\hbar\right)^{3}} e^{\frac{\mathrm{i}}{\hbar}\underline{\vec{P}\cdot\vec{r}}} \int e^{\frac{\mathrm{i}}{\hbar}\left[\vec{P}\dot{\vec{r}}-\left(\alpha_0 Mc^2+c\alpha_j P^j\right)\right]t} \left[\begin{array}{l} e^{\frac{\mathrm{i}}{\hbar}\left\{\underbrace{\vec{P}'\cdot\vec{r}-\underline{\vec{P}\cdot\vec{r}}}_{\to\delta\left(\vec{P}'-\vec{P}\right)}-\left[\vec{P}'\dot{\vec{r}}-\left(\alpha_0 Mc^2+c\alpha_j P'^j\right)\right]t\right\}} \\ \varphi\left(\vec{P}',t\right) \end{array}\right]\mathrm{d}^3\vec{P}'\mathrm{d}^3\vec{r} \\
&= e^{\frac{\mathrm{i}}{\hbar}\vec{P}\cdot\vec{r}} \int e^{\frac{\mathrm{i}}{\hbar}\left[\vec{P}\dot{\vec{r}}-\left(\alpha_0 Mc^2+c\alpha_j P^j\right)\right]t} \left[e^{-\frac{\mathrm{i}}{\hbar}\left[\vec{P}'\dot{\vec{r}}-\left(\alpha_0 Mc^2+c\alpha_j P'^j\right)\right]t} \varphi\left(\vec{P}',t\right)\right]\delta\left(\vec{P}'-\vec{P}\right)\mathrm{d}^3\vec{P}' \\
&= e^{\frac{\mathrm{i}}{\hbar}\vec{P}\cdot\vec{r}} \varphi\left(\vec{P},t\right) = \mathcal{P}\varphi\left(\vec{P},t\right).
\end{aligned} \tag{119}$$

With (114), the two coupled wave functions (103) with the Hamiltonian (104), of a quantum particle in an electromagnetic field $U\left(\vec{r}\right), \vec{A}\left(\vec{r},t\right)$, take the form

$$\begin{aligned}\psi\left(\vec{r},t\right)&=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int e^{\frac{\mathrm{i}}{\hbar}\left\{\vec{P}\cdot\vec{r}-\left[\vec{P}\dot{\vec{r}}-H\left(\vec{P},\vec{r}\right)\right]t\right\}}\varphi\left(\vec{P},t\right)\mathrm{d}^3\vec{P}\\&=\mathcal{P}e^{\frac{\mathrm{i}}{\hbar}\left[eU\left(\vec{r}\right)-ec\alpha_j A^j\left(\vec{r},t\right)\right]t}\psi_t\left(\vec{r},t\right)=\mathcal{P}_f\psi_t\left(\vec{r},t\right)\\\varphi\left(\vec{P},t\right)&=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int e^{-\frac{\mathrm{i}}{\hbar}\left\{\vec{P}\cdot\vec{r}-\left[\vec{P}\dot{\vec{r}}-H\left(\vec{P},\vec{r}\right)\right]t\right\}}\psi\left(\vec{r},t\right)\mathrm{d}^3\vec{r}=\mathcal{P}^{-1}\varphi_t\left(\vec{P},t\right),\end{aligned}\tag{120}$$

as functions of the propagation operators in vacuum,

$$\mathcal{P}=e^{\frac{\mathrm{i}}{\hbar}\vec{P}\cdot\vec{r}},\tag{121}$$

and in the electromagnetic field,

$$\mathcal{P}_f\left[U\left(\vec{r}\right),A^j\left(\vec{r},t\right)\right]=\mathcal{P}e^{\frac{\mathrm{i}}{\hbar}\left[eU\left(\vec{r}\right)-ec\alpha_j A^j\left(\vec{r},t\right)\right]t}=e^{\frac{\mathrm{i}}{\hbar}\left\{\vec{P}\cdot\vec{r}-\left[-eU\left(\vec{r}\right)+ec\alpha_j A^j\left(\vec{r},t\right)\right]t\right\}},\tag{122}$$

and the two coupled time dependent wave functions,

$$\begin{aligned}\psi_t\left(\vec{r},t\right)&=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int e^{-\frac{\mathrm{i}}{\hbar}\left[\vec{P}\dot{\vec{r}}-\left(\alpha_0 Mc^2+c\alpha_j P^j\right)\right]t}e^{-\frac{\mathrm{i}}{\hbar}\vec{P}\cdot\vec{r}}\varphi_t\left(\vec{P},t\right)\mathrm{d}^3\vec{P}\\\varphi_t\left(\vec{P},t\right)&=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int e^{\frac{\mathrm{i}}{\hbar}\left[\vec{P}\dot{\vec{r}}-\left(\alpha_0 Mc^2+c\alpha_j P^j\right)\right]t}e^{\frac{\mathrm{i}}{\hbar}\vec{P}\cdot\vec{r}}\psi_t\left(\vec{r},t\right)\mathrm{d}^3\vec{r},\end{aligned}\tag{123}$$

with contravariant components of the momentum. However, to solve these equations it is more convenient to use also the similar equations with the covariant components of the momentum. In this case, the Hamiltonian (104)-(105) is of the form

$$\begin{aligned}H&=c\sqrt{M^2c^2+\vec{p}^2\left(\vec{r},t\right)}+eU\left(\vec{r}\right)=c\sqrt{M^2c^2+\left[\vec{P}-e\vec{A}\left(\vec{r},t\right)\right]^2}+eU\left(\vec{r}\right)\\&=c\left\{\begin{matrix}\alpha_0 Mc+\alpha_1\left[P_1-eA_1\left(\vec{r},t\right)\right]\\+\alpha_2\left[P_2-eA_2\left(\vec{r},t\right)\right]+\alpha_3\left[P_3-eA_3\left(\vec{r},t\right)\right]\end{matrix}\right\}+eU\left(\vec{r}\right)\\&=\alpha_0 Mc^2+eU\left(\vec{r}\right)+c\alpha_j\left[P_j-eA_j\left(\vec{r},t\right)\right].\end{aligned}\tag{124}$$

In this case, the first wave function (103) of a particle in the electromagnetic field is of the form

$$\begin{aligned}\psi\left(\vec{r},t\right)&=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int e^{\frac{\mathrm{i}}{\hbar}\left[\vec{P}\cdot\vec{r}-\left(\vec{P}\dot{\vec{r}}-\left\{\alpha_0Mc^2+eU\left(\vec{r}\right)+c\alpha_j\left[P_j-eA_j\left(\vec{r},t\right)\right]\right\}\right)t\right]}\varphi\left(\vec{P},t\right)\mathrm{d}^3\vec{P}\\&=e^{\frac{\mathrm{i}}{\hbar}\left\{\vec{P}\cdot\vec{r}-\left[-eU\left(\vec{r}\right)+ec\alpha_jA_j\left(\vec{r},t\right)\right]t\right\}}\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int e^{-\frac{\mathrm{i}}{\hbar}\left[\vec{P}\dot{\vec{r}}-\left(\alpha_0Mc^2+c\alpha_jP_j\right)\right]t}\varphi\left(\vec{P},t\right)\mathrm{d}^3\vec{P}\\&=e^{\frac{\mathrm{i}}{\hbar}\left\{\hat{\vec{P}}\cdot\vec{r}-\left[-eU\left(\vec{r}\right)+ec\alpha_jA_j\left(\vec{r},t\right)\right]t\right\}}\psi_t\left(\vec{r},t\right),\end{aligned}\tag{125}$$

of the propagation operator of the particle in electromagnetic field

$$\mathcal{P}\left[U\left(\vec{r}\right),A_j\left(\vec{r},t\right)\right]=e^{\frac{\mathrm{i}}{\hbar}\left\{\vec{P}\cdot\vec{r}-\left[-eU\left(\vec{r}\right)+ec\alpha_jA_j\left(\vec{r},t\right)\right]t\right\}},\tag{126}$$

applied to the time dependent wave function

$$\psi_t\left(\vec{r},t\right)=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int e^{-\frac{\mathrm{i}}{\hbar}\left[\vec{P}\dot{\vec{r}}-\left(\alpha_0Mc^2+c\alpha_jP_j\right)\right]t}\varphi\left(\vec{P},t\right)\mathrm{d}^3\vec{P},\tag{127}$$

which, with (119), is

$$\psi_t\left(\vec{r},t\right)=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int e^{-\frac{\mathrm{i}}{\hbar}\left[\vec{P}\dot{\vec{r}}-\left(\alpha_0Mc^2+c\alpha_jP_j\right)\right]t}e^{-\frac{\mathrm{i}}{\hbar}\vec{P}\cdot\vec{r}}\varphi_t\left(\vec{P},t\right)\mathrm{d}^3\vec{P}\,.\tag{128}$$

We obtain a time dependent wave function in the coordinate space similar (123), but with covariant momentum components,

$$\begin{aligned}\psi_t\left(\vec{r},t\right)&=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int e^{-\frac{\mathrm{i}}{\hbar}\left[\vec{P}\dot{\vec{r}}-\left(\alpha_0Mc^2+c\alpha_jP_j\right)\right]t}e^{-\frac{\mathrm{i}}{\hbar}\vec{P}\cdot\vec{r}}\varphi_t\left(\vec{P},t\right)\mathrm{d}^3\vec{P}\\\varphi_t\left(\vec{P},t\right)&=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int e^{\frac{\mathrm{i}}{\hbar}\left[\vec{P}\dot{\vec{r}}-\left(\alpha_0Mc^2+c\alpha_jP^j\right)\right]t}e^{\frac{\mathrm{i}}{\hbar}\vec{P}\cdot\vec{r}}\psi_t\left(\vec{r},t\right)\mathrm{d}^3\vec{r}.\end{aligned}\tag{129}$$

From these time dependent wave functions, we obtain the two coupled equations

$$\begin{aligned}
\mathrm{i}\hbar\frac{\partial}{\partial t}\psi_t\left(\vec r,t\right) &= \left[\vec P\dot{\vec r}-\left(\alpha_0 Mc^2+c\alpha_j P_j\right)\right]\psi_t\left(\vec r,t\right)\\
&\quad+\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\limits_{\Delta^3\vec P} e^{-\frac{\mathrm{i}}{\hbar}\left[\vec P\dot{\vec r}-\left(\alpha_0 Mc^2+c\alpha_j P_j\right)\right]t}e^{-\frac{\mathrm{i}}{\hbar}\vec P\cdot\vec r}\,\mathrm{i}\hbar\frac{\partial}{\partial t}\varphi_t\left(\vec P,t\right)\mathrm{d}^3\vec P\\
\mathrm{i}\hbar\frac{\partial}{\partial t}\varphi_t\left(\vec P,t\right) &= -\left[\vec P\dot{\vec r}-\left(\alpha_0 Mc^2+c\alpha_j P^j\right)\right]\varphi_t\left(\vec P,t\right)\\
&\quad+\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\limits_{\Delta^3\vec r} e^{\frac{\mathrm{i}}{\hbar}\left[\vec P\dot{\vec r}-\left(\alpha_0 Mc^2+c\alpha_j P^j\right)\right]t}e^{\frac{\mathrm{i}}{\hbar}\vec P\cdot\vec r}\,\mathrm{i}\hbar\frac{\partial}{\partial t}\psi_t\left(\vec r,t\right)\mathrm{d}^3\vec r,
\end{aligned}\tag{130}$$

where $\Delta^3\vec P$ and $\Delta^3\vec r$ are the particle dimensions in the two spaces, of the momentum and of the coordinates. Due to the translational symmetry of these equations in the coordinate and momentum spaces, the integral terms with the exponential factors $e^{-\frac{\mathrm{i}}{\hbar}\vec P\cdot\vec r}$ and $e^{\frac{\mathrm{i}}{\hbar}\vec P\cdot\vec r}$ reduce to null, as this system of equations takes the form

$$\begin{aligned}
\mathrm{i}\hbar\frac{\partial}{\partial t}\psi_t\left(\vec r,t\right) &= \left[\vec P\dot{\vec r}-\left(\alpha_0 Mc^2+c\alpha_j P_j\right)\right]\psi_t\left(\vec r,t\right)\\
\mathrm{i}\hbar\frac{\partial}{\partial t}\varphi_t\left(\vec P,t\right) &= -\left[\vec P\dot{\vec r}-\left(\alpha_0 Mc^2+c\alpha_j P^j\right)\right]\varphi_t\left(\vec P,t\right).
\end{aligned}\tag{131}$$

With the momentum operators

$$P_0=\mathrm{g}_{00}P^0=P^0=\mathrm{i}\hbar\frac{\partial}{\partial x^0}=\mathrm{i}\hbar\frac{\partial}{c\partial t},\qquad P_i=g_{ii}P^i=-P^i=\mathrm{i}\hbar\frac{\partial}{\partial x^i},\tag{132}$$

these equations take the form

$$\begin{aligned}
\mathrm{i}\hbar\frac{\partial}{\partial t}\psi_t\left(\vec r,t\right) &= \left[\vec P\dot{\vec r}-\left(\alpha_0 Mc^2+\mathrm{i}\hbar c\alpha_j\frac{\partial}{\partial x^j}\right)\right]\psi_t\left(\vec r,t\right)\\
\mathrm{i}\hbar\frac{\partial}{\partial t}\varphi_t\left(\vec P,t\right) &= -\left[\vec P\dot{\vec r}-\left(\alpha_0 Mc^2-c\alpha_j P_j\right)\right]\varphi_t\left(\vec P,t\right).
\end{aligned}\tag{133}$$

Multiplying these equations with α_0, we obtain the dynamic equations

$$\begin{aligned} & \mathrm{i}\hbar\left(\gamma^{\mu}\frac{\partial}{\partial x^{\mu}}+Mc-\gamma^{0}\vec{P}\frac{\dot{\vec{r}}}{c}\right)\psi_{t}\left(\vec{r},t\right)=0 \\ & \left(\gamma^{\mu}P_{\mu}-Mc+\gamma^{0}\vec{P}\frac{\dot{\vec{r}}}{c}\right)\varphi_{t}\left(\vec{P},t\right)=0, \end{aligned} \tag{134}$$

as functions of the Dirac matrices

$$\gamma^{0}=\alpha_{0}=\begin{pmatrix}\hat{1} & 0\\ 0 & -\hat{1}\end{pmatrix},\quad \gamma^{i}=\alpha_{0}\alpha_{i}=\begin{pmatrix}\hat{1} & 0\\ 0 & -\hat{1}\end{pmatrix}\begin{pmatrix}0 & \sigma_{i}\\ \sigma_{i} & 0\end{pmatrix}=\begin{pmatrix}0 & \sigma_{i}\\ -\sigma_{i} & 0\end{pmatrix}, \tag{135}$$

which satisfy the relations

$$\begin{aligned} & \gamma^{0}\gamma^{i}=\alpha_{0}\alpha_{0}\alpha_{i}=\alpha_{i} \\ & \gamma^{i}\gamma^{0}=\alpha_{0}\alpha_{i}\alpha_{0}=-\alpha_{0}\alpha_{0}\alpha_{i}=-\alpha_{i} \\ & \gamma^{i}\gamma^{j}=\alpha_{0}\alpha_{i}\alpha_{0}\alpha_{j}=-\alpha_{i}\alpha_{0}\alpha_{0}\alpha_{j}=-\mathrm{i}\varepsilon_{ijk}\alpha_{k}. \end{aligned} \tag{136}$$

For a flat space,

$$g^{00}=1,\quad g^{11}=g^{22}=g^{33}=-1,\quad g^{\mu\nu}\Big|_{\mu\neq\nu}=0, \tag{137}$$

these matrices are elements of the Clifford algebra

$$\left\{\gamma^{\mu},\gamma^{\nu}\right\}=2g^{\mu\nu},\qquad g^{\mu\nu}\Big|_{\mu\neq\nu}=0,\quad g^{00}=1,\quad g^{ii}=-1, \tag{138}$$

with the normalization relations

$$\gamma^{0^{2}}=\hat{\hat{1}},\quad \gamma^{i^{2}}=-\hat{\hat{1}}. \tag{139}$$

With the conventional notations of the quantum field theory [7],

$$\begin{aligned} & \not{\partial}=\gamma^{\mu}\partial_{\mu} \\ & \not{P}=\gamma^{\mu}P_{\mu}, \end{aligned} \tag{140}$$

the dynamic equations (134) are of the form

$$\begin{aligned}&\left[-\mathrm{i}\hbar\not{\partial}-m\left(1-\gamma^{0}\eta\right)\right]\psi_{t}\left(\vec{r},t\right)=0\\&\left[\not{P}-m\left(1-\gamma^{0}\eta\right)\right]\varphi_{t}\left(\vec{P},t\right)=0,\end{aligned}\tag{141}$$

depending on the rest momentum

$$m=Mc.\tag{142}$$

These equations, for the particle dynamics in the conjugate spaces of the coordinates and momentum, can be compared with the similar equations for a free quantum particle in [7],

$$\begin{aligned}&\left(\mathrm{i}\not{\partial}-m\right)\psi=\left(\mathrm{i}\gamma^{\mu}p_{\mu}-m\right)\psi=0 && (2.2)\text{ in }[7]\\&\left(\not{p}-m\right)\psi=\left(\gamma^{\mu}p_{\mu}-m\right)\psi=0 && (2.1)\text{ in }[7]\end{aligned}\tag{143}$$

describing the wave function of a quantum particle $\psi\left(\vec{r},\vec{p},t\right)$ in the two spaces of the coordinates and of the momentum. Unlike these equations, our dynamic equations (134) or (141) describe the two conjugate time dependent wave functions $\psi_{t}\left(\vec{r},t\right)$ and $\varphi_{t}\left(\vec{P},t\right)$, in the two conjugate spaces, of the coordinates, and of the momentum. In our dynamic equations (141), the interaction with the electromagnetic field is described by the dynamic function

$$\begin{aligned}\eta\left(\vec{r},t\right)&=\frac{\vec{P}\dot{\vec{r}}}{mc}=\frac{\vec{P}\vec{p}}{M^{2}c^{2}}\sqrt{1-\frac{\dot{\vec{r}}^{2}}{c^{2}}}=\frac{\vec{P}\vec{p}}{Mc\sqrt{M^{2}c^{2}+\vec{p}^{2}}}\\&=\frac{\vec{P}\left[\vec{P}-e\vec{A}\left(\vec{r},t\right)\right]}{m\sqrt{m^{2}+\left[\vec{P}-e\vec{A}\left(\vec{r},t\right)\right]^{2}}},\end{aligned}\tag{144}$$

where $\vec{P}$ is a constant of motion. We notice that, for a free particle, $\vec{A}\left(\vec{r},t\right)=0$, this dynamic function takes the value

$$\eta_{0}=\frac{\vec{p}^{2}}{m\sqrt{m^{2}+\vec{p}^{2}}},\tag{145}$$

depending on the particle momentum

$$\vec{p} = \frac{M\dot{\vec{r}}}{\sqrt{1-\frac{\dot{\vec{r}}^2}{c^2}}}. \tag{146}$$

We note that for the ultra-relativistic case, $\dot{\vec{r}}^2 \to c^2$, this dynamic function takes very large values, $\eta_0 >> 1$.

9. Wave function of a quantum particle in electromagnetic field

We consider the algebraic dynamic equation (134), with a solution of the form

$$\varphi_t\left(\vec{P}\right) \square\, u\left(\vec{P}\right) = \begin{pmatrix} \tilde{u}\left(\vec{P}\right) \\ \tilde{v}\left(\vec{P}\right) \end{pmatrix} = \begin{pmatrix} u_1\left(\vec{P}\right) \\ u_2\left(\vec{P}\right) \\ u_3\left(\vec{P}\right) \\ u_4\left(\vec{P}\right) \end{pmatrix} \tag{147}$$

We obtain the dynamic equation

$$\left[\gamma^0 P_0 + \gamma^1 P_1 + \gamma^2 P_2 + \gamma^3 P_3 - m\left(1-\gamma^0\eta\right)\right] \begin{pmatrix} u_1\left(\vec{P}\right) \\ u_2\left(\vec{P}\right) \\ u_3\left(\vec{P}\right) \\ u_4\left(\vec{P}\right) \end{pmatrix} = 0, \tag{148}$$

depending on the Dirac matrices (135) with (107),

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$
$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{149}$$

We consider a coordinate system with the x^3-axis in the direction of the momentum $\vec{P}$, which is a constant of motion, $P_3 = |\vec{P}|$, $P_1 = P_2 = 0$. In this case, the dynamic equation (148) takes the explicit form

$$\begin{pmatrix} P_0 + m\eta - m & 0 & P_3 & 0 \\ 0 & P_0 + m\eta - m & 0 & -P_3 \\ -P_3 & 0 & -(P_0 + m\eta + m) & 0 \\ 0 & P_3 & 0 & -(P_0 + m\eta + m) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = 0. \quad (150)$$

This homogeneous system of equations has a nontrivial solution for a null determinant,

$$\begin{aligned} &\begin{vmatrix} P_0 + m\eta - m & 0 & P_3 & 0 \\ 0 & P_0 + m\eta - m & 0 & -P_3 \\ -P_3 & 0 & -(P_0 + m\eta + m) & 0 \\ 0 & P_3 & 0 & -(P_0 + m\eta + m) \end{vmatrix} \\ &= [P_0 - m(1-\eta)]\left\{[P_0 - m(1-\eta)][P_0 + m(1+\eta)]^2 - P_3^2[P_0 + m(1+\eta)]\right\} \\ &\quad + P_3\left[P_3^3 - [P_0 + m(1+\eta)][P_0 - m(1-\eta)]P_3\right] \\ &= \left[(P_0 + m\eta)^2 - P_3^2 - m^2\right]^2 = 0. \end{aligned} \quad (151)$$

In this way, we obtain the expression

$$P_0 + m\eta = \pm\tilde{E}(\vec{P}), \quad (152)$$

as a function of the particle-photon energy

$$\tilde{E}(\vec{P}) = \sqrt{m^2 + |\vec{P}|^2}. \quad (153)$$

For a particle, with a positive energy (152), and a wave function of the form

$$u_+(\vec{P}) = \begin{pmatrix} \tilde{u}_+(\vec{P}) \\ \tilde{v}_+(\vec{P}) \end{pmatrix}, \quad (154)$$

the dynamic equation (150) is

$$\begin{pmatrix} \tilde{E}-m & P_3\sigma_3 \\ -P_3\sigma_3 & -\left(\tilde{E}+m\right) \end{pmatrix} \begin{pmatrix} \tilde{u}_+\left(\vec{P}\right) \\ \tilde{v}_+\left(\vec{P}\right) \end{pmatrix} = 0 . \tag{155}$$

We obtain the two dimensional equations

$$\begin{aligned} &\left(\tilde{E}-m\right)\tilde{u}_+\left(\vec{P}\right)+\sigma_3 P_3 \tilde{v}_+\left(\vec{P}\right)=0 \\ &\sigma_3 P_3 \tilde{u}_+\left(\vec{P}\right)+\left(\tilde{E}+m\right)\tilde{v}_+\left(\vec{P}\right)=0. \end{aligned} \tag{156}$$

From the second equation we obtain the solution

$$\tilde{v}_+\left(\vec{P}\right)=-\frac{\sigma_3 P_3}{\tilde{E}+m}\tilde{u}_+\left(\vec{P}\right)=\frac{\sigma_3 P^3}{\tilde{E}+m}\tilde{u}_+\left(\vec{P}\right)=\frac{\vec{\sigma}\vec{P}}{\tilde{E}+m}\tilde{u}_+\left(\vec{P}\right), \tag{157}$$

which is also obtained from the first equation multiplied with σ_3,

$$\tilde{v}_+\left(\vec{P}\right)=-\frac{\tilde{E}-m}{P_3}\sigma_3\tilde{u}_+\left(\vec{P}\right)=-\frac{\left(\tilde{E}-m\right)\left(\tilde{E}+m\right)}{P_3\left(\tilde{E}+m\right)}\sigma_3\tilde{u}_+\left(\vec{P}\right)=\frac{\vec{\sigma}\vec{P}}{\tilde{E}+m}\tilde{u}_+\left(\vec{P}\right). \tag{158}$$

With this relation, the wave function (154) is of the form

$$u_+\left(\vec{P}\right)=\begin{pmatrix} \tilde{u}_+\left(\vec{P}\right) \\ \dfrac{\vec{\sigma}\vec{P}}{\tilde{E}+m}\tilde{u}_+\left(\vec{P}\right) \end{pmatrix}. \tag{159}$$

For an antiparticle, with a negative energy (152), and a wave function of the form

$$u_-\left(\vec{P}\right)=\begin{pmatrix} \tilde{u}_-\left(\vec{P}\right) \\ \tilde{v}_-\left(\vec{P}\right) \end{pmatrix}, \tag{160}$$

the dynamic equation (150) is

$$\begin{pmatrix} -\left(\tilde{E}+m\right) & P_3\sigma_3 \\ -P_3\sigma_3 & \tilde{E}-m \end{pmatrix}\begin{pmatrix} \tilde{u}_-\left(\vec{P}\right) \\ \tilde{v}_-\left(\vec{P}\right) \end{pmatrix}=0\,. \tag{161}$$

We obtain the two-dimensional equations

$$\begin{aligned} &-\left(\tilde{E}+m\right)\tilde{u}_-\left(\vec{P}\right)+\sigma_3 P_3\tilde{v}_-\left(\vec{P}\right)=0 \\ &-\sigma_3 P_3\tilde{u}_-\left(\vec{P}\right)+\left(\tilde{E}-m\right)\tilde{v}_-\left(\vec{P}\right)=0. \end{aligned} \tag{162}$$

From the first equation, we obtain the solution

$$\tilde{u}_-\left(\vec{P}\right)=\frac{\sigma_3 P_3}{\tilde{E}+m}\tilde{v}_-\left(\vec{P}\right)=-\frac{\sigma_3 P^3}{\tilde{E}+m}\tilde{v}_-\left(\vec{P}\right)=-\frac{\vec{\sigma}\vec{P}}{\tilde{E}+m}\tilde{v}_-\left(\vec{P}\right), \tag{163}$$

which can be also obtained from the second equation multiplied with σ_3,

$$\tilde{u}_-\left(\vec{P}\right)=\frac{\tilde{E}-m}{P_3}\sigma_3\tilde{v}_-\left(\vec{P}\right)=\frac{\left(\tilde{E}-m\right)\left(\tilde{E}+m\right)}{P_3\left(\tilde{E}+m\right)}\sigma_3\tilde{v}_-\left(\vec{P}\right)=-\frac{\vec{\sigma}\vec{P}}{\tilde{E}+m}\tilde{v}_-\left(\vec{P}\right). \tag{164}$$

With this expression, we obtain the wave function (160) of an antiparticle

$$u_-\left(\vec{P}\right)=\begin{pmatrix} -\dfrac{\vec{\sigma}\vec{P}}{\tilde{E}+m}\tilde{v}_-\left(\vec{P}\right) \\ \tilde{v}_-\left(\vec{P}\right) \end{pmatrix}. \tag{165}$$

The two wave functions (159) of a particle and (165) of an antiparticle, depend on the matter-field momentum (78),

$$\vec{P}=\vec{p}\left(\vec{r},t\right)+e\vec{A}\left(\vec{r},t\right), \tag{166}$$

and the matter-field energy (153),

$$\tilde{E}=\sqrt{m^2+\vec{P}^2}\ , \tag{167}$$

as constants of motion, including the particle momentum $\vec{p}(\vec{r},t)$ and the electromagnetic vector potential $\vec{A}(\vec{r},t)$ as correlated variables. We note that (159) and (165) are orthogonal wave functions:

$$u_+^\dagger(\vec{P})u_-(\vec{P}) = \begin{pmatrix} \tilde{u}_+^\dagger(\vec{P}) & \tilde{u}_+^\dagger(\vec{P})\dfrac{\vec{P}\vec{\sigma}}{\tilde{E}+m} \end{pmatrix} \begin{pmatrix} -\dfrac{\vec{\sigma}\vec{P}}{\tilde{E}+m}\tilde{v}_-(\vec{P}) \\ \tilde{v}_-(\vec{P}) \end{pmatrix}$$
$$= -\tilde{u}_+^\dagger(\vec{P})\frac{\vec{\sigma}\vec{P}}{\tilde{E}+m}\tilde{v}_-(\vec{P}) + \tilde{u}_+^\dagger(\vec{P})\frac{\vec{P}\vec{\sigma}}{\tilde{E}+m}\tilde{v}_-(\vec{P}) = 0. \tag{168}$$

With the spin eigenfunctions of a particle,

$$\tilde{u}_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \tilde{u}_+^\dagger \tilde{u}_+ = 1, \tag{169}$$

we obtain the normalization relation of a particle wave function (159),

$$u_+^\dagger(\vec{P})u_+(\vec{P}) = \begin{pmatrix} \tilde{u}_+^\dagger(\vec{P}) & \tilde{u}_+^\dagger(\vec{P})\dfrac{\vec{P}\vec{\sigma}}{\tilde{E}+m} \end{pmatrix} \begin{pmatrix} \tilde{u}_+(\vec{P}) \\ \dfrac{\vec{\sigma}\vec{P}}{E+m}\tilde{u}_+(\vec{P}) \end{pmatrix}$$
$$= \tilde{u}_+^\dagger(\vec{P})\tilde{u}_+(\vec{P}) + \tilde{u}_+^\dagger(\vec{P})\frac{\vec{P}\vec{\sigma}}{E+m}\frac{\vec{\sigma}\vec{P}}{E+m}\tilde{u}_+(\vec{P}) \tag{170}$$
$$= \left(1 + \frac{\tilde{E}^2 - m^2}{(\tilde{E}+m)^2}\right)\tilde{u}_+^\dagger(\vec{P})\tilde{u}_+(\vec{P}) = \left(1 + \frac{\tilde{E}-m}{\tilde{E}+m}\right)\tilde{u}_+^\dagger(\vec{P})\tilde{u}_+(\vec{P}) = \frac{2\tilde{E}}{\tilde{E}+m}.$$

With the spin eigenfunction of an antiparticle,

$$\tilde{v}_- = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \tilde{v}_-^\dagger \tilde{v}_- = 1, \tag{171}$$

we obtain the same normalization relation of an antiparticle wave function (165),

$$u_-^{\dagger}\left(\vec{P}\right)u_-\left(\vec{P}\right)=\begin{pmatrix}-\tilde{v}_-^{\dagger}\left(\vec{P}\right)\dfrac{\vec{P}\vec{\sigma}}{\tilde{E}+m} & \tilde{v}_-^{\dagger}\left(\vec{P}\right)\end{pmatrix}\begin{pmatrix}-\dfrac{\vec{\sigma}\vec{P}}{\tilde{E}+m}\tilde{v}_-\left(\vec{P}\right)\\ \tilde{v}_-\left(\vec{P}\right)\end{pmatrix}$$

$$=\tilde{v}_-^{\dagger}\left(\vec{P}\right)\frac{\vec{P}\vec{\sigma}}{\tilde{E}+m}\frac{\vec{\sigma}\vec{P}}{\tilde{E}+m}\tilde{v}_-\left(\vec{P}\right)+\tilde{v}_-^{\dagger}\left(\vec{P}\right)\tilde{v}_-\left(\vec{P}\right) \tag{172}$$

$$=\left(\frac{\tilde{E}^2-m^2}{\left(\tilde{E}+m\right)^2}+1\right)\tilde{v}_-^{\dagger}\left(\vec{P}\right)\tilde{v}_-\left(\vec{P}\right)=\left(\frac{\tilde{E}-m}{\tilde{E}+m}+1\right)\tilde{v}_-^{\dagger}\left(\vec{P}\right)\tilde{v}_-\left(\vec{P}\right)=\frac{2\tilde{E}}{\tilde{E}+m}.$$

We consider the wave function of a particle-antiparticle system

$$u\left(\vec{P}\right)=\alpha u_+\left(\vec{P}\right)+\beta u_-\left(\vec{P}\right)=\alpha\begin{pmatrix}\tilde{u}_+\left(\vec{P}\right)\\ \dfrac{\vec{\sigma}\vec{P}}{\tilde{E}+m}\tilde{u}_+\left(\vec{P}\right)\end{pmatrix}+\beta\begin{pmatrix}-\dfrac{\vec{\sigma}\vec{P}}{\tilde{E}+m}\tilde{v}_-\left(\vec{P}\right)\\ \tilde{v}_-\left(\vec{P}\right)\end{pmatrix}, \tag{173}$$

with the coefficients α and β satisfying the normalization conditions, for a particle,

$$\left|\alpha\right|^2 u_+^{\dagger}\left(\vec{P}\right)u_+\left(\vec{P}\right)=\left|\alpha\right|^2\frac{2\tilde{E}}{\tilde{E}+m}=\frac{\tilde{E}}{m}, \tag{174}$$

and an antiparticle

$$\left|\beta\right|^2 u_-^{\dagger}\left(\vec{P}\right)u_-\left(\vec{P}\right)=\left|\beta\right|^2\frac{2\tilde{E}}{\tilde{E}+m}=\frac{\tilde{E}}{m}. \tag{175}$$

We obtain the normalization conditions

$$\left|\alpha\right|^2=\left|\beta\right|^2=\frac{\tilde{E}+m}{2m}. \tag{176}$$

For the wave function (173) of a particle-antiparticle system, we obtain the normalization condition

$$
\begin{aligned}
u^{\dagger}\left(\vec{P}\right)u\left(\vec{P}\right) &= \left|\alpha\right|^{2} u_{+}^{\dagger}\left(\vec{P}\right)u_{+}\left(\vec{P}\right)+\left|\beta\right|^{2} u_{-}^{\dagger}\left(\vec{P}\right)u_{-}\left(\vec{P}\right) \\
&= \left(\left|\alpha\right|^{2}+\left|\beta\right|^{2}\right)\frac{2\tilde{E}}{\tilde{E}+m}=\frac{\tilde{E}+m}{m}\frac{2\tilde{E}}{\tilde{E}+m}=2\frac{\tilde{E}}{m}.
\end{aligned}
\tag{177}
$$

We consider the wave function (147), as a solution of the dynamic equation (134) in the momentum space, of the form

$$
\varphi_{t}\left(\vec{p}\right)=\frac{1}{L_{P}^{3/2}}u\left(\vec{P}\right), \tag{178}
$$

with the normalization length L_{p}. We determine this length from the normalization condition of the particle-antiparticle wave function (117) or (119),

$$
\varphi\left(\vec{P},t\right)=e^{-\frac{\mathrm{i}}{\hbar}\vec{P}\cdot\hat{\vec{r}}}\varphi_{t}\left(\vec{P}\right)=\frac{1}{L_{P}^{3/2}}e^{-\frac{\mathrm{i}}{\hbar}\vec{P}\cdot\hat{\vec{r}}}u\left(\vec{P}\right)=\varphi\left(\vec{P}\right), \tag{179}
$$

as a constant distribution function in the momentum space $\vec{P}$. With this distribution function, we obtain the time dependent wave function of the particle (127), or (128),

$$
\psi_{t}\left(\vec{r},t\right)=\frac{1}{\left(2\pi\hbar L_{P}\right)^{3/2}}\int\limits_{\Delta^{3}\vec{P}} e^{-\frac{\mathrm{i}}{\hbar}\left[\vec{P}\dot{\vec{r}}\mp\left(\alpha_{0}Mc^{2}+c\alpha_{j}P_{j}\right)\right]t}e^{-\frac{\mathrm{i}}{\hbar}\vec{P}\vec{r}}u\left(\vec{P}\right)\mathrm{d}^{3}\vec{P}, \tag{180}
$$

with the $\mp$ sign, the $-$sign for a positive energy (particle), and the $+$sign for a negative energy (antiparticle), as an integral over the momentum domain $\Delta^{3}\vec{P}$, giving the finite particle dimensions in the coordinate space. With this time dependent wave function, the particle wave function (125) in the coordinate space is

$$
\begin{aligned}
\psi\left(\vec{r},t\right) &= e^{\frac{\mathrm{i}}{\hbar}\left\{\vec{P}\cdot\vec{r}\mp\left[-eU\left(\vec{r}\right)+ec\alpha_{j}A_{j}\left(\vec{r},t\right)\right]t\right\}}\psi_{t}\left(\vec{r},t\right) \\
&= e^{\frac{\mathrm{i}}{\hbar}\left\{\cancel{\vec{P}\cdot\vec{r}}\mp\left[-eU\left(\vec{r}\right)+ec\alpha_{j}A_{j}\left(\vec{r},t\right)\right]t\right\}}\frac{1}{\left(2\pi\hbar L_{P}\right)^{3/2}}\int\limits_{\Delta^{3}\vec{P}} e^{-\frac{\mathrm{i}}{\hbar}\left[\vec{P}\dot{\vec{r}}\mp\left(\alpha_{0}Mc^{2}+c\alpha_{j}P_{j}\right)\right]t}e^{-\frac{\mathrm{i}}{\hbar}\cancel{\vec{P}\vec{r}}}u\left(\vec{P}\right)\mathrm{d}^{3}\vec{P} \\
&= e^{\pm\frac{\mathrm{i}}{\hbar}e\left[U\left(\vec{r}\right)-c\alpha_{j}A_{j}\left(\vec{r},t\right)\right]t}\frac{1}{\left(2\pi\hbar L_{P}\right)^{3/2}}\int\limits_{\Delta^{3}\vec{P}} e^{-\frac{\mathrm{i}}{\hbar}\left[\vec{P}\dot{\vec{r}}\mp\left(\alpha_{0}Mc^{2}+c\alpha_{j}P_{j}\right)\right]t}u\left(\vec{P}\right)\mathrm{d}^{3}\vec{P}.
\end{aligned}
\tag{181}
$$

The upper signs in this expression is for a particle, $u\left(\vec{P}\right)=u_{+}\left(\vec{P}\right)$, as the lower sign is for an antiparticle, $u\left(\vec{P}\right)=u_{-}\left(\vec{P}\right)$. With the wave function (173) and the normalization condition (176), we obtain the wave function of a particle-antiparticle system in an electromagnetic field,

$$\psi\left(\vec{r},t\right)=\sqrt{\frac{\tilde{E}+m}{2m}}\left[\begin{array}{l} e^{\frac{\mathrm{i}}{\hbar}e\left[U(\vec{r})-c\alpha_j A_j(\vec{r},t)\right]t} \\ \dfrac{1}{\left(2\pi\hbar L_P\right)^{3/2}}\displaystyle\int_{\Delta^3\vec{P}} e^{-\frac{\mathrm{i}}{\hbar}\left[\vec{P}\dot{\vec{r}}-\left(\alpha_0 Mc^2+c\alpha_j P_j\right)\right]t}\begin{pmatrix}\tilde{u}_{+}\left(\vec{P}\right)\\ \dfrac{\vec{\sigma}\vec{P}}{\tilde{E}+m}\tilde{u}_{+}\left(\vec{P}\right)\end{pmatrix}\mathrm{d}^3\vec{P} \\ +e^{-\frac{\mathrm{i}}{\hbar}e\left[U(\vec{r})-c\alpha_j A_j(\vec{r},t)\right]t} \\ \dfrac{1}{\left(2\pi\hbar L_P\right)^{3/2}}\displaystyle\int_{\Delta^3\vec{P}} e^{-\frac{\mathrm{i}}{\hbar}\left[\vec{P}\dot{\vec{r}}+\left(\alpha_0 Mc^2+c\alpha_j P_j\right)\right]t}\begin{pmatrix}-\dfrac{\vec{\sigma}\vec{P}}{\tilde{E}+m}\tilde{v}_{-}\left(\vec{P}\right)\\ \tilde{v}_{-}\left(\vec{P}\right)\end{pmatrix}\mathrm{d}^3\vec{P}\end{array}\right]. \tag{182}$$

From the normalization condition of a particle-antiparticle system,

$$\int\psi^{\dagger}\left(\vec{r},t\right)\psi\left(\vec{r},t\right)\mathrm{d}^3\vec{r}=2, \tag{183}$$

with the normalization relations (170) and (172), we obtain the normalization length as a function of the particle domain in the momentum space, with the relativistic factor depending on the amplitude of this momentum,

$$\begin{aligned} L_P{}^3 &= \frac{1}{2}\int_{\Delta^3\vec{P}}\frac{\tilde{E}+m}{2m}\cdot 2\frac{2\tilde{E}}{\tilde{E}+m}\mathrm{d}^3\vec{P}=\int_{\Delta^3\vec{P}}\frac{E}{m}\mathrm{d}^3\vec{P} \\ &= \int_{\Delta^3\vec{P}}\sqrt{1+\frac{\vec{P}^2}{m^2}}\mathrm{d}^3\vec{P}. \end{aligned} \tag{184}$$

10. Scattering/tunneling of a quantum particle in electromagnetic field

The scattering/tunneling rate (73) of a quantum particle essentially depends on the matrix element of the scattering/tunneling operator V . This operator for a particle in an electromagnetic field, is defined as a Hamiltonian perturbation of the dynamic equation (133) in the coordinate space, which determines a transition from the initial state $\psi_{t0}\left(\vec{r},t\right)$ to the final state $\psi_{ti}\left(\vec{r},t\right)$,

$$\begin{aligned}&\left[\mathrm{i}\hbar\left(\frac{\partial}{\partial t}+\alpha_j c\frac{\partial}{\partial x^j}\right)+\alpha_0 Mc^2+\alpha_0 V-\vec{P}\dot{\vec{r}}\right]\psi_{t0}\left(\vec{r},t\right)=0 \qquad /\overrightarrow{\psi_{ti}^{\dagger}\left(\vec{r},t\right)\alpha_0}\\ &\psi_{ti}^{\dagger}\left(\vec{r},t\right)\left[-\mathrm{i}\hbar\left(\frac{\partial}{\partial t}+\alpha_j c\frac{\partial}{\partial x^j}\right)+\alpha_0 Mc^2-\vec{P}\dot{\vec{r}}\right]=0 \qquad /\overleftarrow{\alpha_0\psi_{t0}\left(\vec{r},t\right)}.\end{aligned} \tag{185}$$

From the difference between the first equation multiplied with $\psi_{ti}^{\dagger}\left(\vec{r},t\right)\alpha_0$ to left, and the second equation multiplied with $\alpha_0\psi_{t0}\left(\vec{r},t\right)$ to right

$$\begin{aligned}&\mathrm{i}\hbar\psi_{ti}^{\dagger}\left(\vec{r},t\right)\alpha_0\frac{\partial}{\partial t}\psi_{t0}\left(\vec{r},t\right)+\underline{\underline{\mathrm{i}\hbar c\psi_{ti}^{\dagger}\left(\vec{r},t\right)\alpha_0\alpha_j\frac{\partial}{\partial x^j}\psi_{t0}\left(\vec{r},t\right)}}\\ &\quad+\underline{Mc^2\psi_{ti}^{\dagger}\left(\vec{r},t\right)\alpha_0\alpha_0\psi_{t0}\left(\vec{r},t\right)}+\psi_{ti}^{\dagger}\left(\vec{r},t\right)\underbrace{\alpha_0\alpha_0}_{1}V\psi_{t0}\left(\vec{r},t\right)\\ &\qquad-\underline{\underline{\vec{P}\dot{\vec{r}}\psi_{ti}^{\dagger}\left(\vec{r},t\right)\alpha_0\psi_{t0}\left(\vec{r},t\right)}}=0\\ &-\mathrm{i}\hbar\frac{\partial}{\partial t}\psi_{ti}^{\dagger}\left(\vec{r},t\right)\alpha_0\psi_{t0}\left(\vec{r},t\right)-\underline{\underline{\mathrm{i}\hbar c\frac{\partial}{\partial x^j}\psi_{ti}^{\dagger}\left(\vec{r},t\right)\alpha_j\alpha_0\psi_{t0}\left(\vec{r},t\right)}}\\ &\quad+\underline{Mc^2\psi_{ti}^{\dagger}\left(\vec{r},t\right)\alpha_0\alpha_0\psi_{t0}\left(\vec{r},t\right)}-\underline{\underline{\vec{P}\dot{\vec{r}}\psi_{ti}^{\dagger}\left(\vec{r},t\right)\alpha_0\psi_{t0}\left(\vec{r},t\right)}}=0,\end{aligned}$$

we obtain the expression

$$\begin{aligned}\psi_{ti}^{\dagger}\left(\vec{r},t\right)V\psi_{t0}\left(\vec{r},t\right)=&-\mathrm{i}\hbar\frac{\partial}{\partial t}\left[\psi_{ti}^{\dagger}\left(\vec{r},t\right)\gamma^0\psi_{t0}\left(\vec{r},t\right)\right]\\ &-\mathrm{i}\hbar c\frac{\partial}{\partial x^j}\left[\psi_{ti}^{\dagger}\left(\vec{r},t\right)\gamma^j\psi_{t0}\left(\vec{r},t\right)\right],\end{aligned} \tag{186}$$

depending on the Dirac operators $\gamma^0=\alpha_0,\ \gamma^j=\alpha_0\alpha_j$. For the first term of this equation, with the wave function (180) for a particle,

$$\psi_t\left(\vec{r},t\right)=\frac{1}{\left(2\pi\hbar L_P\right)^{3/2}}\int e^{-\frac{\mathrm{i}}{\hbar}\left[\vec{P}\dot{\vec{r}}-\left(\alpha_0 Mc^2+c\alpha_j P_j\right)\right]t}e^{-\frac{\mathrm{i}}{\hbar}\vec{P}\vec{r}}u\left(\vec{P}\right)\mathrm{d}^3\vec{P}, \tag{187}$$

we obtain the expression

$$\begin{aligned}
&\psi_{ti}^{\dagger}\left(\vec{r},t\right)\gamma^{0}\psi_{t0}\left(\vec{r},t\right)\\
&=\frac{1}{\left(2\pi\hbar L_P\right)^3}\int u^{\dagger}\left(\vec{P}\right)e^{\frac{\mathrm{i}}{\hbar}\vec{P}\vec{r}}e^{\frac{\mathrm{i}}{\hbar}\left[\vec{P}\dot{\vec{r}}-\left(\alpha_0 Mc^2+c\alpha_j P_j\right)\right]t}\gamma^0 e^{-\frac{\mathrm{i}}{\hbar}\left[\vec{P}'\dot{\vec{r}}-\left(\alpha_0 Mc^2+c\alpha_j P_j'\right)\right]t}e^{-\frac{\mathrm{i}}{\hbar}\vec{P}'\vec{r}}u\left(\vec{P}'\right)\\
&\qquad\qquad\mathrm{d}^3\vec{P}\mathrm{d}^3\vec{P}'\\
&=\frac{1}{\left(2\pi\hbar L_P\right)^3}\int u^{\dagger}\left(\vec{P}\right)e^{\frac{\mathrm{i}}{\hbar}\vec{P}\vec{r}}e^{\frac{\mathrm{i}}{\hbar}\left[\vec{P}\dot{\vec{r}}-c\sqrt{M^2c^2+\vec{P}^2}\right]t}\gamma^0 e^{-\frac{\mathrm{i}}{\hbar}\left[\vec{P}'\dot{\vec{r}}-c\sqrt{M^2c^2+\vec{P}'^2}\right]t}e^{-\frac{\mathrm{i}}{\hbar}\vec{P}'\vec{r}}u\left(\vec{P}'\right)\\
&\qquad\qquad\mathrm{d}^3\vec{P}\mathrm{d}^3\vec{P}'\\
&=\frac{1}{\left(2\pi\hbar L_P\right)^3}\int e^{\frac{\mathrm{i}}{\hbar}\left(\vec{P}-\vec{P}'\right)\vec{r}}e^{\frac{\mathrm{i}}{\hbar}\left[\left(\vec{P}-\vec{P}'\right)\dot{\vec{r}}-c\left(\sqrt{M^2c^2+\vec{P}^2}-\sqrt{M^2c^2+\vec{P}'^2}\right)\right]t}u^{\dagger}\left(\vec{P}\right)\gamma^0 u\left(\vec{P}'\right)\mathrm{d}^3\vec{P}\mathrm{d}^3\vec{P}',
\end{aligned} \tag{188}$$

depending on the spinor (173) with the coefficient (176) for a particle,

$$u\left(\vec{P}\right)=\sqrt{\frac{\tilde{E}+m}{2m}}\begin{pmatrix}\tilde{u}\left(\vec{P}\right)\\ \dfrac{\vec{\sigma}\vec{P}}{\tilde{E}+m}\tilde{u}\left(\vec{P}\right)\end{pmatrix}. \tag{189}$$

We note that in the coefficient $\tilde{E}\left(\vec{P}\right)+m=\sqrt{m^2+\vec{P}^2}+m$, the small momentum variation $\Delta\vec{P}<<m=Mc$, which gives finite dimensions of the particle in the coordinate space, is negligible. We obtain the expression

$$\begin{aligned}
\psi_{ti}^{\dagger}\left(\vec{r},t\right)\gamma^{0}\psi_{t0}\left(\vec{r},t\right)&=\frac{1}{\left(2\pi\hbar L_P\right)^3}\frac{\tilde{E}+m}{2m}\int e^{\frac{\mathrm{i}}{\hbar}\left(\vec{P}-\vec{P}'\right)\vec{r}}e^{\frac{\mathrm{i}}{\hbar}\left[\left(\vec{P}-\vec{P}'\right)\dot{\vec{r}}-c\left(\sqrt{M^2c^2+\vec{P}^2}-\sqrt{M^2c^2+\vec{P}'^2}\right)\right]t}\\
&\quad\begin{pmatrix}\tilde{u}^{\dagger}\left(\vec{P}\right) & \tilde{u}^{\dagger}\left(\vec{P}\right)\dfrac{\vec{\sigma}\vec{P}}{\tilde{E}+m}\end{pmatrix}\begin{pmatrix}\hat{1} & 0\\ 0 & -\hat{1}\end{pmatrix}\begin{pmatrix}\tilde{u}\left(\vec{P}'\right)\\ \dfrac{\vec{\sigma}\vec{P}'}{\tilde{E}+m}\tilde{u}\left(\vec{P}'\right)\end{pmatrix}\mathrm{d}^3\vec{P}\mathrm{d}^3\vec{P}'\\
&=\frac{1}{\left(2\pi\hbar L_P\right)^3}\frac{\tilde{E}+m}{2m}\int e^{\frac{\mathrm{i}}{\hbar}\left(\vec{P}-\vec{P}'\right)\vec{r}}e^{\frac{\mathrm{i}}{\hbar}\left[\left(\vec{P}-\vec{P}'\right)\dot{\vec{r}}-c\left(\sqrt{M^2c^2+\vec{P}^2}-\sqrt{M^2c^2+\vec{P}'^2}\right)\right]t}\\
&\quad\begin{pmatrix}\tilde{u}^{\dagger}\left(\vec{P}\right) & \tilde{u}^{\dagger}\left(\vec{P}\right)\dfrac{\vec{\sigma}\vec{P}}{\tilde{E}+m}\end{pmatrix}\begin{pmatrix}\tilde{u}\left(\vec{P}'\right)\\ -\dfrac{\vec{\sigma}\vec{P}'}{\tilde{E}+m}\tilde{u}\left(\vec{P}'\right)\end{pmatrix}\mathrm{d}^3\vec{P}\mathrm{d}^3\vec{P}',
\end{aligned}$$

which, with the formula

$$\left(\vec{\sigma}\vec{A}\right)\left(\vec{\sigma}\vec{B}\right)=\vec{A}\vec{B}+\mathrm{i}\vec{\sigma}\left(\vec{A}\times\vec{B}\right), \tag{190}$$

is

$$\psi_{ti}^{\dagger}\left(\vec{r},t\right)\gamma^{0}\psi_{t0}\left(\vec{r},t\right)=\frac{\tilde{E}+m}{2m\left(2\pi\hbar L_{P}\right)^{3}}\int e^{\frac{\mathrm{i}}{\hbar}\left(\vec{P}-\vec{P}'\right)\vec{r}}e^{\frac{\mathrm{i}}{\hbar}\left[\left(\vec{P}-\vec{P}'\right)\dot{\vec{r}}-c\left(\sqrt{M^{2}c^{2}+\vec{P}^{2}}-\sqrt{M^{2}c^{2}+\vec{P}'^{2}}\right)\right]t}$$
$$\left[1-\frac{\vec{P}\vec{P}'+\mathrm{i}\vec{\sigma}\left(\vec{P}\times\vec{P}'\right)}{\left(\tilde{E}+m\right)^{2}}\right]\tilde{u}^{\dagger}\left(\vec{P}\right)\tilde{u}\left(\vec{P}'\right)\mathrm{d}^{3}\vec{P}\mathrm{d}^{3}\vec{P}'. \tag{191}$$

For the second term of (186), we consider the expression

$$\psi_{ti}^{\dagger}\left(\vec{r},t\right)\gamma^{j}\psi_{t0}\left(\vec{r},t\right)$$
$$=\frac{1}{\left(2\pi\hbar L_{P}\right)^{3}}\int u^{\dagger}\left(\vec{P}\right)e^{\frac{\mathrm{i}}{\hbar}\vec{P}\vec{r}}e^{\frac{\mathrm{i}}{\hbar}\left[\vec{P}\dot{\vec{r}}-\left(\alpha_{0}Mc^{2}+c\alpha_{j}P_{j}\right)\right]t}\gamma^{j}e^{-\frac{\mathrm{i}}{\hbar}\left[\vec{P}'\dot{\vec{r}}-\left(\alpha_{0}Mc^{2}+c\alpha_{j}P_{j}'\right)\right]t}e^{-\frac{\mathrm{i}}{\hbar}\vec{P}'\vec{r}}u\left(\vec{P}'\right)\mathrm{d}^{3}\vec{P}\mathrm{d}^{3}\vec{P}'$$
$$=\frac{1}{\left(2\pi\hbar L_{P}\right)^{3}}\int e^{\frac{\mathrm{i}}{\hbar}\left(\vec{P}-\vec{P}'\right)\vec{r}}e^{\frac{\mathrm{i}}{\hbar}\left[\left(\vec{P}-\vec{P}'\right)\dot{\vec{r}}-c\left(\sqrt{M^{2}c^{2}+\vec{P}^{2}}-\sqrt{M^{2}c^{2}+\vec{P}'^{2}}\right)\right]t}u^{\dagger}\left(\vec{P}\right)\gamma^{j}u\left(\vec{P}'\right)\mathrm{d}^{3}\vec{P}\mathrm{d}^{3}\vec{P}',$$

which, with the particle spinor (189), is

$$\psi_{ti}^{\dagger}\left(\vec{r},t\right)\gamma^{j}\psi_{t0}\left(\vec{r},t\right)=\frac{\tilde{E}+m}{2m\left(2\pi\hbar L_{P}\right)^{3}}\int e^{\frac{\mathrm{i}}{\hbar}\left(\vec{P}-\vec{P}'\right)\vec{r}}e^{\frac{\mathrm{i}}{\hbar}\left[\left(\vec{P}-\vec{P}'\right)\dot{\vec{r}}-c\left(\sqrt{M^{2}c^{2}+\vec{P}^{2}}-\sqrt{M^{2}c^{2}+\vec{P}'^{2}}\right)\right]t}$$
$$\begin{pmatrix}\tilde{u}_{+}^{\dagger}\left(\vec{P}\right) & \tilde{u}_{+}^{\dagger}\left(\vec{P}\right)\frac{\vec{\sigma}\vec{P}}{\tilde{E}+m}\end{pmatrix}\begin{pmatrix}0 & \sigma_{j}\\ -\sigma_{j} & 0\end{pmatrix}\begin{pmatrix}\tilde{u}_{+}\left(\vec{P}'\right)\\ \frac{\vec{\sigma}\vec{P}'}{\tilde{E}+m}\tilde{u}_{+}\left(\vec{P}'\right)\end{pmatrix}\mathrm{d}^{3}\vec{P}\mathrm{d}^{3}\vec{P}'$$
$$=\frac{\tilde{E}+m}{2m\left(2\pi\hbar L_{P}\right)^{3}}\int e^{\frac{\mathrm{i}}{\hbar}\left(\vec{P}-\vec{P}'\right)\vec{r}}e^{\frac{\mathrm{i}}{\hbar}\left[\left(\vec{P}-\vec{P}'\right)\dot{\vec{r}}-c\left(\sqrt{M^{2}c^{2}+\vec{P}^{2}}-\sqrt{M^{2}c^{2}+\vec{P}'^{2}}\right)\right]t} \tag{192}$$
$$\begin{pmatrix}\tilde{u}_{+}^{\dagger}\left(\vec{P}\right) & \tilde{u}_{+}^{\dagger}\left(\vec{P}\right)\frac{\vec{\sigma}\vec{P}}{\tilde{E}+m}\end{pmatrix}\begin{pmatrix}\frac{\sigma_{j}\sigma_{k}P'^{k}}{\tilde{E}+m}\tilde{u}_{+}\left(\vec{P}'\right)\\ -\sigma_{j}\tilde{u}_{+}\left(\vec{P}'\right)\end{pmatrix}\mathrm{d}^{3}\vec{P}\mathrm{d}^{3}\vec{P}'.$$

With the expression

$$\sigma_i = \mathrm{i}\varepsilon_{ijk}\sigma_j\sigma_k\,, \tag{193}$$

where ε_{ijk} is the Levi-Civita symbol, we obtain

$$\begin{aligned}\psi_{ti}^{\dagger}(\vec{r},t)\gamma^j\psi_{t0}(\vec{r},t) &= \frac{\tilde{E}+m}{2m(2\pi\hbar L_P)^3}\int e^{\frac{\mathrm{i}}{\hbar}(\vec{P}-\vec{P}')\vec{r}} e^{\frac{\mathrm{i}}{\hbar}\left[(\vec{P}-\vec{P}')\dot{\vec{r}}-c\left(\sqrt{M^2c^2+\vec{P}^2}-\sqrt{M^2c^2+\vec{P}'^2}\right)\right]t} \\ &\quad \left(\tilde{u}_+^{\dagger}(\vec{P})\frac{\sigma_j\sigma_k P'^k}{\tilde{E}+m}\tilde{u}_+(\vec{P}')-\tilde{u}_+^{\dagger}(\vec{P})\frac{\sigma_k\sigma_j P^k}{\tilde{E}+m}\tilde{u}_+(\vec{P}')\right)\mathrm{d}^3\vec{P}\mathrm{d}^3\vec{P}' \\ &= \frac{1}{2m(2\pi\hbar L_P)^3}\int e^{\frac{\mathrm{i}}{\hbar}(\vec{P}-\vec{P}')\vec{r}} e^{\frac{\mathrm{i}}{\hbar}\left[(\vec{P}-\vec{P}')\dot{\vec{r}}-c\left(\sqrt{M^2c^2+\vec{P}^2}-\sqrt{M^2c^2+\vec{P}'^2}\right)\right]t} \\ &\quad \mathrm{i}\varepsilon_{jkl}\left(P'^k+P^k\right)\tilde{u}_+^{\dagger}(\vec{P})\sigma_l\tilde{u}_+(\vec{P}')\mathrm{d}^3\vec{P}\mathrm{d}^3\vec{P}'.\end{aligned} \tag{194}$$

From (186) with (191) and (194), and the algebraic formula

$$\delta\left(\vec{P}-\vec{P}'\right) = \frac{1}{(2\pi\hbar)^3}\int e^{\frac{\mathrm{i}}{\hbar}(\vec{P}-\vec{P}')\vec{r}}\mathrm{d}^3\vec{r}\,, \tag{195}$$

for a momentum conservation, $\vec{P}-\vec{P}'=0$, when, in fact, no scattering/tunneling exists, we obtain a null matrix element of the scattering/tunneling operator,

$$\begin{aligned}V_{i0} &= \int\psi_{ti}^{\dagger}(\vec{r},t)V\psi_{t0}(\vec{r},t)\mathrm{d}^3\vec{r} \\ &= -\mathrm{i}\hbar\int\left[\frac{\partial}{\partial t}\left[\psi_{ti}^{\dagger}(\vec{r},t)\gamma^0\psi_{t0}(\vec{r},t)\right]+c\frac{\partial}{\partial x^j}\psi_{ti}^{\dagger}(\vec{r},t)\gamma^j\psi_{t0}(\vec{r},t)\right]\mathrm{d}^3\vec{r} \\ &= \frac{1}{2m(2\pi\hbar L_P)^3}\int\mathrm{d}^3\vec{P}\mathrm{d}^3\vec{P}'\underline{e^{\frac{\mathrm{i}}{\hbar}(\vec{P}-\vec{P}')\vec{r}}}e^{\frac{\mathrm{i}}{\hbar}\left[(\vec{P}-\vec{P}')\dot{\vec{r}}-c\left(\sqrt{M^2c^2+\vec{P}^2}-\sqrt{M^2c^2+\vec{P}'^2}\right)\right]t} \\ &\quad \left\{\left(\tilde{E}+m\right)\left[\left(\vec{P}-\vec{P}'\right)\dot{\vec{r}}-c\left(\sqrt{M^2c^2+\vec{P}^2}-\sqrt{M^2c^2+\vec{P}'^2}\right)\right]\right. \\ &\quad \left[1-\frac{\vec{P}\vec{P}'+\mathrm{i}\vec{\sigma}\left(\vec{P}\times\vec{P}'\right)}{\left(\tilde{E}+m\right)^2}\right]\tilde{u}^{\dagger}(\vec{P})\tilde{u}(\vec{P}') \\ &\quad \left.+\mathrm{i}\varepsilon_{jkl}c\left(P^j-P'^j\right)\left(P'^k+P^k\right)\tilde{u}^{\dagger}(\vec{P})\sigma_l\tilde{u}(\vec{P}')\right\}\underline{\mathrm{d}^3\vec{r}} = 0.\end{aligned} \tag{196}$$

In a scattering/tunneling process, with a momentum variation $\vec{P}_0 - \vec{P}_i'$, the matrix element takes the form

$$\begin{aligned} V_{i0} &= \frac{1}{\left(2\pi\hbar L_P\right)^3} \int \mathrm{d}^3\vec{P}\mathrm{d}^3\vec{P}' e^{\frac{\mathrm{i}}{\hbar}\left[\left(\vec{P}-\vec{P}_0\right)-\left(\vec{P}'-\vec{P}_i'\right)\right]\vec{r}} e^{\frac{\mathrm{i}}{\hbar}\left[\left(\vec{P}-\vec{P}'\right)\dot{\vec{r}}-c\left(\sqrt{M^2c^2+\vec{P}^2}-\sqrt{M^2c^2+\vec{P}'^2}\right)\right]t} \\ &\qquad f\left(\vec{P},\vec{P}'\right)\mathrm{d}^3\vec{r} \\ &= \frac{1}{L_P{}^3} \int \mathrm{d}^3\vec{P}\mathrm{d}^3\vec{P}' e^{\frac{\mathrm{i}}{\hbar}\left[\left(\vec{P}-\vec{P}'\right)\dot{\vec{r}}-c\left(\sqrt{M^2c^2+\vec{P}^2}-\sqrt{M^2c^2+\vec{P}'^2}\right)\right]t} f\left(\vec{P},\vec{P}'\right) \\ &\qquad \delta\left[\left(\vec{P}-\vec{P}_0\right)-\left(\vec{P}'-\vec{P}_i'\right)\right] \\ &= \frac{1}{L_P{}^3} \int \mathrm{d}^3\vec{P} e^{\frac{\mathrm{i}}{\hbar}\left[\vec{P}_0\dot{\vec{r}}_0-\vec{P}_i'\dot{\vec{r}}_i-c\left(\sqrt{M^2c^2+\vec{P}_0^2}-\sqrt{M^2c^2+\vec{P}_i'^2}\right)\right]t} f\left(\vec{P}_0,\vec{P}_i'\right) \\ &= e^{\frac{\mathrm{i}}{\hbar}\left[\vec{P}_0\dot{\vec{r}}_0-\vec{P}_i'\dot{\vec{r}}_i-c\left(\sqrt{M^2c^2+\vec{P}_0^2}-\sqrt{M^2c^2+\vec{P}_i'^2}\right)\right]t} f\left(\vec{P}_0,\vec{P}_i'\right), \end{aligned} \tag{197}$$

depending on the transition function, which, with (184), is of the form

$$\begin{aligned} f\left(\vec{P}_0,\vec{P}_i'\right) &= \frac{1}{2m\sqrt{1+\frac{\left|\vec{P}\right|_{0i}^2}{m^2}}}\left\{\left(\tilde{E}+m\right)\right. \\ &\left[\vec{P}_0\dot{\vec{r}}_0-\vec{P}_i'\dot{\vec{r}}_i-c\left(\sqrt{M^2c^2+\vec{P}_0^2}-\sqrt{M^2c^2+\vec{P}_i'^2}\right)\right] \\ &\left[1-\frac{\vec{P}_0\vec{P}_i'+\mathrm{i}\vec{\sigma}\left(\vec{P}_0\times\vec{P}_i'\right)}{\left(\tilde{E}+m\right)^2}\right]\tilde{u}^\dagger\left(\vec{P}_0\right)\tilde{u}\left(\vec{P}_i'\right) \\ &\left.+\mathrm{i}\varepsilon_{jkl}c\left(P_0{}^j-P_i'^j\right)\left(P_0{}^k+P_i'^k\right)\tilde{u}^\dagger\left(\vec{P}_0\right)\sigma_l\tilde{u}\left(\vec{P}_i'\right)\right\}, \end{aligned} \tag{198}$$

where $\left|\vec{P}\right|_{0i}$ is the mean value of the amplitude of the canonical momentum $\vec{P}$, considered approximately constant in a collision process. From Fermi's golden rule, we obtain the scattering/tunneling rate in a channel $(i)=\left(\left|i_1\right\rangle,\left|i_2\right\rangle,\ldots\left|i_N\right\rangle\right)$,

$$\Gamma_{(i)} = \frac{2\pi}{\hbar^2} f\left(\vec{P}_0,\vec{P}_i'\right) f^*\left(\vec{P}_0,\vec{P}_i'\right) g\left(\omega_i\right). \tag{199}$$

In a collisional process with the spin conservation,

$$\tilde{u}\left(\vec{P}_0\right) \to \tilde{u}\left(\vec{P}_i'\right), \qquad \tilde{u}\left(\vec{P}_0\right) = \tilde{u}\left(\vec{P}_i'\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{200}$$

with the spin factor of the first term

$$\tilde{u}^\dagger\left(\vec{P}_0\right)\tilde{u}\left(\vec{P}_i'\right) = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 + 0 = 1, \tag{201}$$

and the spin factor of the second term

$$\begin{aligned} \varepsilon_{jkl}\tilde{u}^\dagger\left(\vec{P}_0\right)\sigma_l \tilde{u}\left(\vec{P}_i'\right) &= \tilde{u}^\dagger\left(\vec{P}_0\right)\left(\varepsilon_{jk1}\sigma_1 + \varepsilon_{jk2}\sigma_2 + \varepsilon_{jk3}\sigma_3 \tilde{u}_+\right)\left(\vec{P}_i'\right) \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \varepsilon_{jk3} & \varepsilon_{jk1} - \mathrm{i}\varepsilon_{jk2} \\ \varepsilon_{jk1} + \mathrm{i}\varepsilon_{jk2} & -\varepsilon_{jk3} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \varepsilon_{jk3} & \varepsilon_{jk1} - \mathrm{i}\varepsilon_{jk2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \varepsilon_{jk3} , \end{aligned} \tag{202}$$

this transition function (198) is

$$\begin{aligned} f\left(\vec{P}_0, \vec{P}_i'\right) = \frac{1}{2m\sqrt{1 + \frac{\left|\vec{P}\right|_{0i}^2}{m^2}}} &\Big\{\left(\tilde{E} + m\right) \\ &\left[\vec{P}_0\dot{\vec{r}}_0 - \vec{P}_i\dot{\vec{r}}_i - c\left(\sqrt{M^2c^2 + \vec{P}_0^2} - \sqrt{M^2c^2 + \vec{P}_i^2}\right)\right] \\ &\left[1 - \frac{\vec{P}_0\vec{P}_i' + \mathrm{i}\vec{\sigma}\left(\vec{P}_0 \times \vec{P}_i'\right)}{\left(\tilde{E} + m\right)^2}\right] + \mathrm{i}\varepsilon_{jk3}c\left(P_0^{\,j} - P_i'^{\,j}\right)\left(P_0^{\,k} + P_i'^{\,k}\right)\Big\}. \end{aligned} \tag{203}$$

In a collisional process with the spin inversion,

$$\tilde{u}\left(\vec{P}_0\right) \to \tilde{u}\left(\vec{P}_i'\right), \qquad \tilde{u}\left(\vec{P}_0\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{u}\left(\vec{P}_i'\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{204}$$

with the spin factor of the first term

$$\tilde{u}^{\dagger}\left(\vec{P}_0\right)\tilde{u}\left(\vec{P}_i'\right)=\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=0, \tag{205}$$

and the spin factor of the second term,

$$\begin{aligned}\mathrm{i}\varepsilon_{jkl}\tilde{u}^{\dagger}\left(\vec{P}_0\right)\sigma_l\tilde{u}\left(\vec{P}_i'\right)&=\mathrm{i}\tilde{u}^{\dagger}\left(\vec{P}_0\right)\left(\varepsilon_{jk1}\sigma_1+\varepsilon_{jk2}\sigma_2+\varepsilon_{jk3}\sigma_3\right)\tilde{u}\left(\vec{P}_i'\right)\\&=\mathrm{i}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\varepsilon_{jk3} & \varepsilon_{jk1}-\mathrm{i}\varepsilon_{jk2}\\ \varepsilon_{jk1}+\mathrm{i}\varepsilon_{jk2} & -\varepsilon_{jk3}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\\&=\mathrm{i}\left(\varepsilon_{jk1}-\mathrm{i}\varepsilon_{jk2}\right)=\varepsilon_{jk2}+\mathrm{i}\varepsilon_{jk1},\end{aligned} \tag{206}$$

the transition function (198) is

$$f\left(\vec{P}_0,\vec{P}_i'\right)=\frac{c}{2m\sqrt{1+\frac{\left|\vec{P}\right|_{0i}^{2}}{m^2}}}\left(\varepsilon_{jk2}+\mathrm{i}\varepsilon_{jk1}\right)\left(P_0^{\,j}-P_i'^{\,j}\right)\left(P_0^{\,k}+P_i'^{\,k}\right) \tag{207}$$

Conclusions

In this paper, we described a quantum particle as a distribution of matter in a motion according to the general theory of relativity. Compared to the conventional quantum mechanics, where a quantum particle is described by a single wave function depending on coordinates and momentum, we described a quantum particle by two conjugate wave functions, in the two conjugate spaces of the coordinates and of the momentum. It is remarkable that, in this framework, the velocities of the waves in the coordinate space are equal to the velocity of the total distribution described by these waves – such a wave function in the coordinate space can be considered as a Fourier series expansion of a distribution moving in space, as the density amplitude of an entity called matter. In this framework, the continuous matter dynamics is described by a an inertial quantity of the time dependent phases of the Fourier components of the matter density, called mass, which, we take equal with the mass of the quantum particle given by the integral of this density in the coordinate space, or in the momentum space, as a quantization rule. We showed that, in the framework of the general relativity, a quantum particle described only by coordinates and momentum has no mass, in agreement with the Standard Model, and with the simple fact that such a particle has no volume – no volume, no mass. Thus, according to the general theory of relativity, we avoid the mass description of the Standard Model, as a coupling of the quantum particle with the Higgs field. By the coupling of a quantum particle with a field described by a scalar potential conjugated to time, called electric potential, and a vector potential conjugated to the coordinates, we obtain the Lorentz force and the Maxwell equations. With the Dirac Hamiltonian, we derived the dynamic equations for the two wave functions in the two conjugated spaces, of the coordinates and of the momentum, which, compared to the similar equations in the quantum field theory, includes an additional term, depending on velocity and the electromagnetic potentials. We obtained the wave function for a particle-antiparticle system in an electromagnetic field, depending on spinors, the electromagnetic potentials, and the canonical momentum as a constant of motion coupling the mechanical momentum of the particle with the momentum of the electromagnetic field. With these wave functions we obtain the scattering/tunneling rate in a collision process for the two possible cases, with the spin conservation, or with the spin inversion.

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