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COLLECTIVE EXCITATIONS IN ATOMIC NUCLEI WITH ENERGY-DEPENDENT POTENTIALS

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Abstract

The analytical solutions for various realizations of the Bohr model Hamiltonian with energy-dependent Kratzer and Davidson potentials are presented. The domain of applicability for the associated solutions are determined from the analysis of the parameter dependence of selected spectral characteristics. Special cases of hyperbolic and harmonic oscillator potentials are considered to ascertain the isolated effect of the energy dependence on the energy spectrum. The theoretical formalism is validated by offering suitable experimental realizations. A systemized model description of nuclear collective spectra revealed a correlation between energy dependence of the potential and critical phenomena associated with shape phase transitions.

keywords: Geometric models, energy-dependent potentials, Collective energy levels

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1 Introduction

Quantum models, whose associated Schrödinger equations can be solved in a fully algebraical manner are referred to as exactly solvable. Exactly solvable models play a very important role in quantum physics, due to their underlying symmetries. The direct relation between the analytical structure of the exactly solvable models and the symmetry properties of the modeled systems allows an algebraic group theoretical description of the problem in terms of associated quantum numbers. This useful feature of quantum systems can be extended by considering quasi-exactly solvable potentials [1], or inducing an energy dependence into the usually exactly solvable potentials [2, 3, 4]. Both these approaches were successfully used in various instances of the Bohr-Mottelson model [5, 6] to describe the collective energy levels in even-even nuclei [7, 8, 9, 10, 11, 12]. Here one will focus on the Bohr model solutions with energy-dependent potentials. The energy dependence will be considered only for the β shape variable, whose differential equations is a priori separated from the γ-angular degrees of freedom. The analytic structure of an exactly solvable differential Schrödinger equation is partially retained when an energy-dependence is introduced. Although the proceedings for finding the eigenvalues are similar to the energy-independent case, the final result is not a formula for the energy but an algebraic equation for determining the energy. The later has in general many solutions, which must be screened for a correct physical behavior.

A BohrMottelson Hamiltonian with an energy-dependent potential was first considered in Ref.[8], where a stiffening spherical vibrator model was simulated by spherical harmonic potential with an energy-dependent string constant. This application opened a new chapter in the geometrical description of nuclear collective excitation, offering new varieties of spectral features. To make use of the analytical properties of the exactly solvable models and to have a reasonable degree of generality, one will consider here Bohr model solutions for energy-dependent Kratzer [13] and Davidson [14] potentials, whose associated local (with energy-independent potentials) differential equations have a well known algebraic group structure.

This study will commence by the following plan: First of all, one will present the quantum theoretical implications of the energy-dependent potentials in Section 2. Before applying this formalism for the description of collective excitations in nuclei, one will survey in Section 3, some reference collective conditions and their exactly separable realization of the Bohr Hamiltonian. Section 3 is devoted to the analytical formalism associated to the selected collective Hamiltonians with energy-dependent Kratzer and Davidson potentials. Numerical applications for general model characteristics as well as its experimental realizations will be presented in Section 4. Finally, the last section contains a summary of the theoretical formalism and the performed calculations.

2 Properties of quantum systems with energy-dependent potentials

For the sake of simplicity, the discussion in this section will be made on a one dimensional problem. The conclusions' extension to a space of any dimension is straightforward. Lets start with a time independent wave equation

$$H\psi(x) = \left[\frac{1}{2}p^2 + V(x,E)\right]\psi(x) = E\psi(x), \tag{2.1}$$

where $p = -i\partial/\partial x$. One of the most obvious modification of the usual quantum problem is that now the Hamiltonian contains its eigenvalue. This has an immediate effect on the commutation relations, where the eigenvalue must be replaced with the corresponding operator. For example

$$[H,x] = -ip + [V(x,H),x]. \tag{2.2}$$

In this situation, p no longer necessarily represents a momentum operator for the system modeled by H, because it is no longer a canonically conjugate variable for x.

A major change in the usual rules of quantum mechanics, noted also by the early studies on the energy-dependent potentials [2, 3], is the amendment of the density probability, or the scalar product, necessary for satisfying the continuity equation. To illustrate this, let us consider the time-dependent wave equation

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = H\Psi(x,t), \tag{2.3}$$

and two solutions for energies E and E':

$$\Psi(x,t) = e^{-iEt}\psi(x), \quad \Psi'(x,t) = e^{-iE't}\psi'(x). \tag{2.4}$$

Combining Eqs.(2.1) and (2.3), one obtains the following equation:

$$\frac{\partial}{\partial t}(P + P_a) = -\frac{\partial}{\partial x}j, \tag{2.5}$$

where

$$P = \Psi'^{*}(x,t)\Psi(x,t), \quad j = -\frac{i}{2}\left[\Psi'^{*}(x,t)\frac{\partial\Psi(x,t)}{\partial x} - \frac{\partial\Psi'^{*}(x,t)}{\partial x}\Psi(x,t)\right], \tag{2.6}$$

and the additional term in respect to the usual continuity equation $\partial P/\partial t = -\partial j/\partial x$ is

$$P_a = -\Psi'^{*}(x,t)\left[\frac{V(x,E') - V(x,E)}{E' - E}\right]\Psi(x,t). \tag{2.7}$$

Then scalar product is redefined as

$$(\Psi'|\Psi) = \int_{-\infty}^{\infty} \psi'^{*}(x) \left[1 - \frac{V(x,E') - V(x,E)}{E' - E}\right] \psi(x) dx, \tag{2.8}$$

where "(|)" denotes the modified scalar product which is to be distinguished from the regular one "$\langle|\rangle$". When $E' \to E$, one obtains the norm of the wave-function Ψ as:

$$N = \int_{-\infty}^{\infty} \psi^{*}(x) \left[1 - \frac{\partial V}{\partial E}\right] \psi(x) dx. \tag{2.9}$$

Considering now that the stationary states are normalized to unity and specified by a global quantum number n, one can write down the orthogonality condition in the following form:

$$\int_{-\infty}^{\infty} \psi^{*}_{n'}(x) \left[1 - \phi_{n'n}\right] \psi_n(x) dx = \delta_{n'n}, \tag{2.10}$$

where

$$\phi_{n'n} = \begin{cases} \frac{\partial V}{\partial E}, & n = n', \\ \frac{V(x,E_{n'}) - V(x,E_n)}{E_{n'} - E_n}, & n \neq n'. \end{cases} \tag{2.11}$$

It must be emphasized here that the stationary functions $\psi_n(x)$ and $\psi_{n'}(x)$ are eigenstates of distinct self-adjoint operators, *i.e.* they span different Hilbert spaces. As a consequence, the usual completeness relation no longer holds,

$$\sum_n \psi_n(x') \psi^{*}_n(x) \neq \delta(x - x'). \tag{2.12}$$

Using a similar recipe as in the case of the scalar product

$$\sum_n \psi_n(x)(1 - \phi_{nn}) \psi^{*}_n(x') = \delta(x - x'), \tag{2.13}$$

one obtains

$$\begin{aligned} \psi_{n'}(x') &= \int \psi_{n'}(x) \delta(x - x') dx \\ &= \int \psi_{n'}(x) \sum_n \psi_n(x')(1 - \phi_{nn}) \psi^{*}_n(x) dx \\ &= \psi_{n'}(x') - \sum_n \psi_n(x') \int \psi_{n'}(x) \psi^{*}_n(x)(\phi_{nn} - \phi_{nn'}) dx. \end{aligned} \tag{2.14}$$

The last term vanishes only when $\phi_{nn}(x) = \phi_{nn'}(x)$, that is when both of them are state-independent. This is true only for the linear energy dependence. Moreover, for

this special case of linear energy dependence, the problem can be reformulated in the usual quantum mechanics with a standard scalar product [4, 15]. To show this, one considers a potential of the form $V(x,E) = V_0(x) + EV_1(x)$. The Hamiltonian operator for this potential can be written as:

$$H = H_0 + V_1(x)H = \frac{1}{1-V_1(x)}H_0 = \zeta(x)^2 H_0, \tag{2.15}$$

where H_0 is its energy-independent part, that is involving only V_0. With this, the time-dependent Schrödinger equation (2.3) can be expressed as:

$$i\frac{\partial}{\partial t}\frac{1}{\zeta(x)^2}\Psi(x,t) = H_0\Psi(x,t), \tag{2.16}$$

which can be brought to the usual form of the time-dependent wave equation

$$i\frac{\partial}{\partial t}\tilde{\Psi}(x,t) = \tilde{H}\tilde{\Psi}(x,t) \tag{2.17}$$

for the wave-function $\tilde{\Psi}(x,t) = \Psi(x,t)/\zeta(x)$ with the associated modified Hamiltonian operator:

$$\begin{aligned}\tilde{H} &= \frac{1}{\zeta(x)}H\zeta(x) = \zeta(x)H_0\zeta(x) \\ &= \frac{1}{1-V_1(x)}\left\{-\frac{\partial^2}{\partial x^2} - \frac{1}{1-V_1(x)}\frac{\partial V_1(x)}{\partial x}\frac{\partial}{\partial x} + V_0(x)\right. \\ &\quad \left. -\frac{1}{2[1-V_1(x)]}\frac{\partial^2 V_1(x)}{\partial x^2} - \frac{3}{4[1-V_1(x)]}\left[\frac{\partial V_1(x)}{\partial x}\right]^2\right\}.\end{aligned} \tag{2.18}$$

This change is similar to the Darboux transformation [16, 17]. It is worth to mention, that although $\tilde{H}$ and H have the same eigenvalue E, they act within the constraints of distinct scalar products. This comes however with a price, which is the emergence of a non-local term expressed as a first order derivative (linear in momentum).

3 Varieties of the Bohr model Hamiltonian

The Bohr Hamiltonian [5, 6] for the quadrupole degrees of freedom acts in a five-dimensional space spanned by three Euler angles describing the rotational motion and two shape variables β and γ characterizing the deviation of the nuclear shape from spherical symmetry and respectively from axial symmetry. For a constant mass, the usual Bohr Hamiltonian has the following expression:

$$H = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} - \frac{\Lambda^2}{\beta^2}\right] + U(\beta,\gamma), \tag{3.1}$$

where B is a constant mass parameter, while

$$\Lambda^2 = -\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + \sum_{k=1}^{3}\frac{Q_k^2}{4\sin^2\left(\gamma-\frac{2}{3}\pi k\right)} \tag{3.2}$$

with $Q_k (k = 1, 2, 3)$ being the operators of the angular momentum projections on the axes of the body-fixed reference frame. The scaling property of the Hamiltonian, allows a simplified analytical formalism in terms of a reduced energy and potential:

$$\epsilon = \frac{2B}{\hbar^2}E, \;\; u(\beta,\gamma) = \frac{2B}{\hbar^2}U(\beta,\gamma). \tag{3.3}$$

The collective conditions described by the Hamiltonian (3.1) are usually referred to as γ-soft, which basically means that γ shape variable is active. The numerical applications are further greater simplified if the β variable and the γ-rotational ones are separated. This is usually done by emplyong a potential $u(\beta,\gamma) = v(\beta) + w(\gamma)/\beta^2$ [18, 19]. Then, considering a factorized total wave-function $\Psi(\beta,\gamma,\Omega) = R(\beta)\Phi(\gamma,\Omega)$, one obtains the following decoupled equations:

$$\left[-\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{W}{\beta^2} + v(\beta)\right]R(\beta) = \epsilon R(\beta), \tag{3.4}$$

$$\left[-\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + \frac{1}{4}\sum_{k=1}^{3}\frac{Q_k^2}{\sin^2\left(\gamma-\frac{2}{3}\pi k\right)}\right.$$
$$\left.+w(\gamma) - W\right]\Phi(\gamma,\Omega) = 0. \tag{3.5}$$

The solution of the γ-angular coupled equation for a general $w(\gamma)$ potential usually involves a diagonalization procedure in a fairly complicated basis [20, 21]. There are however some solvable limiting cases. One such case, referred as γ-unstable, is when $w(\gamma) = 0$. In this situation, the γ-angular wave-function is a $SO(5)$ spherical harmonic $|\tau\alpha LM\rangle = \mathcal{Y}_{\tau\alpha LM}(\gamma,\Omega)$ [22] indexed by the seniority τ [23], α order distinguishing multiple occurrences of the same angular momentum within a multiplet of fixed seniority, and by angular momentum and its projection on the third intrinsic axis. The kinetic operator Λ^2 is actually the Casimir operator of the $SO(5)$ symmetry group, such that its eigenvalue problem can be represented as [24]:

$$\hat{\Lambda}^2|\tau\alpha LM\rangle = \tau(\tau+3)|\tau\alpha LM\rangle. \tag{3.6}$$

Consequently, for the γ-unstable regime, the separation constant is

$$W_{\gamma u} = \tau(\tau+3). \tag{3.7}$$

The other extreme limit, called γ-stable, corresponds to a $w(\gamma)$ potential with a very sharp minimum in $\gamma = 0^\circ$, which is suitable for highly stabilized prolate shapes [25]. In the small γ limit, a further separation of the γ variable from the Euler angles is

possible. It's starting point is the following approximation of the rotational term [26]

$$\sum_{k=1}^{3} \frac{Q_k^2}{\sin^2\left(\gamma - \frac{2}{3}\pi k\right)} \approx \frac{4}{3}\mathbf{Q}^2 + Q_3^2\left(\frac{1}{\sin^2\gamma} - \frac{4}{3}\right). \tag{3.8}$$

A harmonic-like γ potential then leads to a separation constant

$$W_{\gamma s} = \frac{L(L+1)}{3}, \tag{3.9}$$

defined up to a constant for the $K^\pi = 0^+$ states. This result can be achieved in two alternative ways. One is the somewhat old fashion adiabatic separation of the β and γ fluctuations [26], while the other follows a suitable adjustment of a zero point energy of the γ excitations corresponding to the separated $w(\gamma)$ potential [11].

Imposing a fixed value for γ, changes the classical under-structure of the Bohr Model, leading after quantization to distinct Hamiltonian operators. This is the so called γ-rigid case of the collective motion. For example, a fixed value of γ in the interval $(0, \pi/3)$ will end up generating the Davydov-Chaban Hamiltonian [27]:

$$H = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^3}\frac{\partial}{\partial\beta}\beta^3\frac{\partial}{\partial\beta} - \sum_{k=1}^{3}\frac{Q_k^2}{4\sin^2\left(\gamma - \frac{2}{3}\pi k\right)}\right] + U(\beta). \tag{3.10}$$

Similarly, for an axially symmetric γ-rigid nuclear surface, the number of independent degrees of freedom is even further reduced, because the rotational motion can now be described by only two rotation angles instead of three. Consequently the quantum Hamiltonian will be [28]:

$$H = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^2}\frac{\partial}{\partial\beta}\beta^2\frac{\partial}{\partial\beta} + \frac{\Delta_{\theta,\varphi}}{3\beta^2}\right] + U(\beta), \tag{3.11}$$

where

$$\Delta_{\theta,\varphi} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \tag{3.12}$$

The resulting reduced β equation for all presented cases, that is γ-unstable, stable, triaxial rigid and prolate rigid, can be written as

$$-\frac{\partial^2}{\partial\beta^2} + \frac{1}{\beta^2}\left[\frac{(d-1)(d-3)}{4} + W\right] + u(\beta), \tag{3.13}$$

where d is the number of dimensions (active variables). This one-dimensional-like equation is obtained by changing the original β wave-function as $R(\beta) = \beta^{-\frac{d-1}{2}} f(\beta)$.

The separation constant W for the Davydov-Chaban Hamiltonian is actually the eigenvalue of a triaxial rigid rotor Hamiltonian with hydrodynamical moments

of inertia. The hydrodynamic nature of the moments of inertia facilitates a situation when two of them are equal when the triaixiality is $\gamma = 30^\circ$. The separation constant then can be analytically expressed through the simple formula [29]:

$$W_{\gamma t} = L(L+1) - \frac{4}{3}R^2, \tag{3.14}$$

where R is the angular momentum projection on the body fixed first principal axis. For ground and β excited bands $R = L$, while for the γ band $R = L - 2$ for even L and $R = L - 1$ for odd L.

For the γ-rigid prolate case, the separation constant is readily obtained by averaging Eq.(3.11) with angular momentum spherical harmonics:

$$W_{\gamma p} = \frac{L(L+1)}{3}. \tag{3.15}$$

It is up to a constant term the same as for the γ-stable case. It is not surprising, because there are obvious similarities between the two collective conditions. One must however remember that their associated β variable kinetic operator act in different space dimensions.

4 Bohr Hamiltonian solutions for energy-dependent potentials

In this section, one will discuss the solution of the effective β equation (3.13) in connection to different collective conditions and for two families of energy-dependent potentials, namely Kratzer [13, 30] and Davidson [14]. Both potentials have a centrifugal term and therefore can in general account for possible contribution from the γ-rotational degrees of freedom. The distinctive term in the Kratzer potential corresponds to a Coulomb-like interaction and is therefore singular, whereas the non-centrifugal term of the Davidson potential is a confining harmonic oscillator potential.

4.1 Kratzer potential

Let us first solve Eq.(3.13) for the Kratzer potential with a coupling constant of the hyperbolic (Coulomb-like) term depending linearly on the energy of the system:

$$u(\beta) = \frac{a_1}{\beta^2} - \frac{1 + a_2\epsilon}{\beta}. \tag{4.1}$$

This is done by following up to a certain point the procedure for a state-independent Coulomb potential. First of all, Eq.(3.13) is rewritten into a Whittaker differential equation [31]:

$$f''(x) + \left[\frac{k}{x} - \frac{1}{4} + \frac{\left(\frac{1}{4} - \mu^2\right)}{x^2}\right] f(x) = 0, \tag{4.2}$$

by making the change of variable $x = 2\sqrt{\varepsilon}\beta$ and where

$$\varepsilon = -\epsilon, \quad k = \frac{1 - a_2\varepsilon}{2\sqrt{\varepsilon}}, \quad \mu = \sqrt{\frac{(d-2)^2}{4} + W_\lambda + a_1}. \tag{4.3}$$

Here, λ denotes the set of quantum numbers describing the γ-angular excitations associated to different collective conditions, including those presented in the previous section. A regular solution in both origin and the asymptotic limit is possible if:

$$\mu + \frac{1}{2} - k = -n, \tag{4.4}$$

where n is a positive integer. This condition leads to a quadratic equation for ϵ

$$(1 + a_2\epsilon)^2 + 4\epsilon\left(n + \frac{1}{2} + \sqrt{\frac{(d-2)^2}{4} + W + a_1}\right)^2 = 0, \tag{4.5}$$

and the reduction of the wave-function in terms of Laguerre polynomials [32]. The physical solution of the quadratic equation for the energy is

$$\begin{aligned}\epsilon_{n\lambda} = \frac{1}{a_2^2}\Bigg[2\left(n + \frac{1}{2} + \sqrt{\frac{(d-2)^2}{4} + W_\lambda + a_1}\right) \\ \sqrt{\left(n + \frac{1}{2} + \sqrt{\frac{(d-2)^2}{4} + W_\lambda + a_1}\right)^2 + a_2} \\ -2\left(n + \frac{1}{2} + \sqrt{\frac{(d-2)^2}{4} + W_\lambda + a_1}\right)^2 - a_2\Bigg].\end{aligned} \tag{4.6}$$

From this, one can deduce that the integer number n plays the role of the β excitation quantum number.

Making the following notation:

$$\eta_{n\lambda} \quad = \quad \sqrt{-\epsilon} = \frac{1 + a_2\epsilon_{n\lambda}}{2\left(n + \frac{1}{2} + \sqrt{\frac{(d-2)^2}{4} + W_\lambda + a_1}\right)}, \tag{4.7}$$

the total β wave-function can then be written in the following analytical closed form [9]:

$$R_{n\lambda}(\beta) = N_{n\lambda}\beta^{\mu_\lambda + 1 - \frac{d}{2}} e^{-\eta_{n\lambda}\beta} L_n^{2\mu_\lambda}(2\eta_{n\lambda}\beta). \tag{4.8}$$

The normalization constant is determined from the condition

$$\int_0^\infty \left[R_{n\lambda}(\beta)\right]^2 \beta^{d-1}\left(1 + \frac{a_2}{\beta}\right) d\beta = 1. \tag{4.9}$$

Using the integration properties of the associated Laguerre polynomials [32], the normalization constant is readily obtained as:

$$N_{n\lambda} = \sqrt{\frac{n!}{\Gamma(n+2\mu_\lambda+1)\,(2a_2\eta_{n\lambda}+2n+2\mu_\lambda+1)}}\,(2\eta_{n\lambda})^{\mu_\lambda+1}\,. \tag{4.10}$$

The dependence of the energy function (4.6) on parameter a_2 allows for a special case in its asymptotic limit. For very large values of a_2 the energy can be approximated by:

$$\epsilon_{n\lambda}^{(asymp)} = -\frac{1}{a_2}+\frac{2}{a_2^{3/2}}\left(n+\frac{1}{2}+\mu_\lambda\right). \tag{4.11}$$

The peculiarity of this case resides in the fact that a_2 becomes a simple scaling factor for the excitation energies. The scaling property of a_2 is also reflected on the wave-function, which becomes energy-independent

$$R_{n,\lambda}^{(asymp)}(\beta) = \mathcal{N}_{n\lambda}^{(asymp)}\beta^{\mu_\lambda+1-\frac{d}{2}}e^{-\frac{\beta}{\sqrt{a_2}}}L_n^{2\mu_\lambda}\left(2\beta/\sqrt{a_2}\right), \tag{4.12}$$

because

$$\eta_{n\lambda}^{(asymp)} = \frac{1}{\sqrt{a_2}}. \tag{4.13}$$

The norm of the asymptotic function is obtained either from making the large a_2 limit of (4.10), or by integrating the square of the above wave-function within the the integration metric $(a_2/\beta)d\beta$. Either way, the result is:

$$\mathcal{N}_{n\lambda}^{(asymp)} = \left(\frac{2}{\sqrt{a_2}}\right)^{\mu_\lambda+\frac{1}{2}}\sqrt{\frac{n!}{a_2\Gamma(n+2\mu_\lambda+1)}}. \tag{4.14}$$

It is worth to mention here that if μ_λ is given only in terms of γ-rotational quantum numbers, then in the asymptotic limit of energy dependence of the Kratzer potential (4.1) the total excitation spectrum is also parameter-free. Such a solution was for example given in Ref.[9]. A parameter-free energy spectrum imply the existence of a symmetry in the system. Constructing an effective Hamiltonian (2.19) for the asymptotic regime of the linear energy dependence, one found in Ref.[33] that the $SU(1;1)$ is its spectrum generating algebra. Although the same dynamical symmetry is associated also to the bound spectrum of the state-independent Coulomb potential, in the present case the same algebra closes on the actual wave-functions and not Sturmian contractions.

4.2 Davidson potential

Considering a Davidson (pseudo-harmonic) potential [14] with a coupling constant depending linearly on the energy:

$$u(\beta) = \frac{a_1}{\beta^2} + (1 + a_2\epsilon)\beta^2, \tag{4.15}$$

one arrives to the following equation for determining the energy:

$$\epsilon = 2\sqrt{1 + a_2\epsilon}\,(2n + p_\lambda + 1)\,, \tag{4.16}$$

where

$$p_\lambda = \sqrt{\frac{(d-2)^2}{4} + W_\lambda + a_1}. \tag{4.17}$$

The physical solution of this quadratic equation is

$$\epsilon_{n\lambda} = 2a_2\,(2n + p_\lambda + 1)^2 + 2\,(2n + p_\lambda + 1)\,\sqrt{1 + a_2^2\,(2n + p_\lambda + 1)^2}, \tag{4.18}$$

while the corresponding wave-function reads

$$R_{n\lambda}(\beta) = N_{n\lambda}\beta^{p_\lambda + 1 - \frac{d}{2}} e^{-\frac{(1+a_2\epsilon_{n\lambda})\beta^2}{2}} L_n^{p_\lambda}\left[(1 + a_2\epsilon_{n\lambda})\beta^2\right], \tag{4.19}$$

where $N_{n\lambda}$ is a normalization factor which can be analytically determined from the properties of integrals involving associated Laguerre polynomials [32]:

$$(N_{n\lambda})^{-2} = \frac{\Gamma(n + p_\lambda + 1)}{2n!(1 + a_2\epsilon_{n\lambda})^{p+1}}\left[1 - \frac{a_2(2n + p + 1)}{1 + a_2\epsilon_{n\lambda}}\right]. \tag{4.20}$$

The above formula was obtained by considering the modified integration metric

$$\beta^{d-1}d\beta \;\longrightarrow\; \left(1 - \frac{\partial u(\beta)}{\partial\epsilon}\right)\beta^{d-1}d\beta = (1 - a_2\beta^2)\beta^{d-1}d\beta. \tag{4.21}$$

In order to have a fully coherent quantum theory, the deformation probability distribution associated to this integration metric

$$\rho_{n\lambda}(\beta) = \left|R_{Ln_\beta}(\beta)\right|^2\left(1 - a_2\beta^2\right)\beta^{d-1}, \tag{4.22}$$

must be positive definite. This condition is exactly realized only for $a_2 \leq 0$. Checking Eq.(4.18), one can see that the $a_2 > 0$ case has the practical value of providing monotonously increasing excitation energies with quantum numbers. As it happens, all norms (4.20) are positive definite for any a_2, p_λ and n. The same is true for averages for small powers of β, up to certain critical quantum numbers which are usually very high and not relevant for the purpose of studying collective excitations

in nuclei. It is then possible to have a working quantum theory in a limited but exhaustive framework of the model for $a_2 > 0$. Moreover, the mathematical restriction of $a_2 \leq 0$ can be safely disregarded because the negative contribution to the total probability distribution is easily negligible for any $a_2 > 0$. Indeed, the negative apport of ρ for the ground state was found [10] to be under 1% and decreasing with the state.

Another argument for the extension of the energy dependence to the $a_2 > 0$ case, is the fact that despite its controversial interpretation, the negative probability is considered a solid mathematical concept in quantum theory [34] when dealing with intermediate quantities and not physical observables. Then the origin of negative probabilities lies in the fact that the quantum hypotheses and initial conditions are not always directly relatable with measurable physical quantities. This generates an acceptable interpretation for negative probability, which is ascribed to states whose assumed conditions of preparation or verification are experimentally unattainable [35]. As a matter of fact, negative probabilities were a constant issue even from the foundation days of quantum mechanics [36, 37]. The Wigner quasi-probability function [38] is a famous example of negative probability in quantum mechanics. The probability distribution (4.22) is also an intermediate quantity which is not directly observed or measured, and therefore is not strictly limited only to positive values.

Similarly to the energy-dependent Kratzer potential, the asymptotic regime of parameter a_2 have special properties. More precisely, the energy (4.18) becomes scalable with parameter a_2 in its large value limit [8]:

$$\varepsilon_{n\lambda}^{(asymp)} = 2a_2(2n + 2p_\lambda + 1)^2. \tag{4.23}$$

The saturation of the energy spectrum is quite rapid with the increase of the a_2 parameter. Thus, comparing to the Kratzer potential case, the energy becomes scalable with a_2 at relatively low values of a_2. The scaling of the wave-function is demonstrated numerically [8]. An analytical confirmation is problematic due to the fact that the wave-function is mostly confined to small values of the variable, where the correction term of metric (4.21) becomes indeterminate for high values of a_2. The numerical analysis also revealed that the negative probability distribution have contribution to the norm tending to zero in the asymptotic limit of a_2 [8].

The model with a negative a_2 can also have some practical value. For example it can hinder the rapid increase of energy with rotational quantum numbers, keeping however a monotonically increasing energy function. From phenomenological point of view, the $a_2 > 0$ and $a_2 < 0$ model realizations correspond to a stiffening and respectively a softening of the β vibrations.

The model was employed in Refs.[10, 11] for the description of many transitional nuclei, with a different normalization convention for the eigenfunctions. The expressions for the energy are related to the present formalism through a factor 2 for the reduced energy and parameter a_2.

4.3 Bohr model with deformation-dependent mass term versus energy-dependent potentials

The present formalism of energy-dependent potentials (EDP) have some common aspects with the Bohr Hamiltonian solutions involving a deformation dependent mass (DDM) term [39, 40]. In order to show the distinction between the two approaches, let us start by presenting a general form for a kinetic operator with a position dependent mass $m(x) = m_0/[g(x)]^2$ [41]:

$$T_{kin} = -\frac{\hbar^2}{2m_0}\left[g^\delta(x)\nabla g^k(x)\nabla g^\sigma(x) + g^\sigma(x)\nabla g^k(x)\nabla g^\delta(x)\right], \tag{4.24}$$

where the powers δ, σ and k are real and must satisfy $\delta + \sigma + k = 2$. The final scale-free eigenvalues were found to be independent of the choice for δ, σ and k.

Implementing now this generalization into the five-dimensional Bohr Hamiltonian for a deformation-dependent mass $B = B_0/\xi^2(\beta)$, one obtains

$$\begin{aligned} H_{DDM} &= -\frac{\sqrt{\xi(\beta)}}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\xi(\beta)\frac{\partial}{\partial\beta}\sqrt{\xi(\beta)} - \xi^2(\beta)\frac{W_\lambda}{\beta^2} + v(\beta) \\ &\quad +\frac{1}{2}(1-\delta-\sigma)\xi(\beta)\xi''(\beta) + \left(\frac{1}{2}-\delta\right)\left(\frac{1}{2}-\sigma\right)\left[\xi'(\beta)\right]^2. \end{aligned} \tag{4.25}$$

This differential operator must be now compared with the modified Hamiltonian obtained as in Eq.(2.19):

$$H_{EDP} = \tilde{H} = \zeta(\beta)H\zeta(\beta) = -\frac{\zeta(\beta)}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta}\zeta(\beta) - \zeta^2(\beta)\frac{\hat{\Lambda}_{\gamma\Omega}}{\beta^2} + v(\beta), \tag{4.26}$$

by a transformation involving the function $\zeta(\beta) = 1/\sqrt{1-av(\beta)}$. H denotes here the five-dimensional Bohr Hamiltonian with a linear energy dependence of the potential. In both Hamiltonians, the constant mass factor B_0 or B is dropped and W_λ denotes the γ-angular part of the kinetic operator including a possible separated γ interaction. An effective comparison between the two Hamiltonians can be made by considering $\zeta(\beta) \equiv \xi(\beta)$ and $\delta = \sigma = 1$, and making the derivatives in both equations. Doing so, one observes that the two differential operators are identical except a free term $-2\zeta(\beta)\zeta'(\beta)/\beta$. Thus, even if there are some similarities between the two concepts, they are not equivalent.

4.4 Electromagnetic transitions

Along with energy spectra, the associated quadrupole transition probabilities are another important observables for the description of the collective states in nuclei. Having the analytical structure of the total wave-function, the $E2$ transition probabilities can be then calculated as

$$B(E2, n\lambda \to n'\lambda') = \langle \Psi_{n'\lambda'}||T(E2)||\Psi_{n\lambda}\rangle^2, \tag{4.27}$$

with the help of the quadrupole transition operator:

$$T_\mu(E2) = t\beta \left[D^2_{\mu 0} \cos\gamma + \frac{1}{\sqrt{2}} \left(D^2_{\mu 2} + D^2_{\mu -2} \right) \sin\gamma \right], \tag{4.28}$$

where t is a scaling factor. Exploiting the separation of the β variable from the γ-angular ones, the reduced matrix element involved in (4.27) acquires a factorized form. One factor is an integral over the β variable with a modified integration metric. The remaining quantity deals with γ-rotational part of the wave-functions. It has closed analytical forms which depend on the specifics of the collective conditions. For the γ unstable case, the γ-rotational wave-function is $SO(5)$ spherical harmonic [22], while for the γ-stable case it is a product of rotational Wigner functions with γ-vibrational wave-functions [26]. Similarly, in the γ-rigid case, the rotational part is given by usual spherical harmonics for the prolate case [28], and Wigner rotation functions for the rigid triaxial deformation [27].

5 Numerical applications

In what follows, one will make an overview analysis of the model characteristics related to the energy levels. For simplicity, one will not mention the energy-dependence of the discussed potentials. Only the state-independent solutions will be identified explicitly when discussed.

5.1 Model characteristics

The focus here is on the energy spectra. Due to the scaling properties of the Bohr Hamiltonian solutions, the energy spectra are usually described in terms of excitation energies normalized to the excitation energy of the first excited 2^+_g state. In this way, the spectra are scale-free and depend only on linearly independent parameters and quantum numbers. Additionally, such a representation offers some spectral observables with major insight on the nuclear shape and the behavior of its collective excitations. These are the ratios

$$R_{4/2} = \frac{E(4^+_g)}{E(2^+_g)}, \;\; R_{0/2} = \frac{E(0^+_\beta)}{E(2^+_g)}, \;\; R_{2/2} = \frac{E(2^+_\gamma)}{E(2^+_g)}, \tag{5.1}$$

where $E(i) = (\epsilon_i - \epsilon_{g.s.})\hbar^2/2B$ are excitation energies in respect to the ground state. The first one describes the rotational properties of the system, ranging from a vibrational value of 2, to the axially symmetric limit of 3.33. The other two denote the β and γ band-head energy positions relative to the rotational excitation. The evolution of the first two observables with parameters a_1 and a_2 in the γ-unstable, γ-stable, and prolate γ-rigid cases is depicted in Fig. 1 for the Kratzer potential and in Fig. 2 for the Davidson potential. The γ band states in the γ-unstable case are degenerated with the ground band states, while in the axially

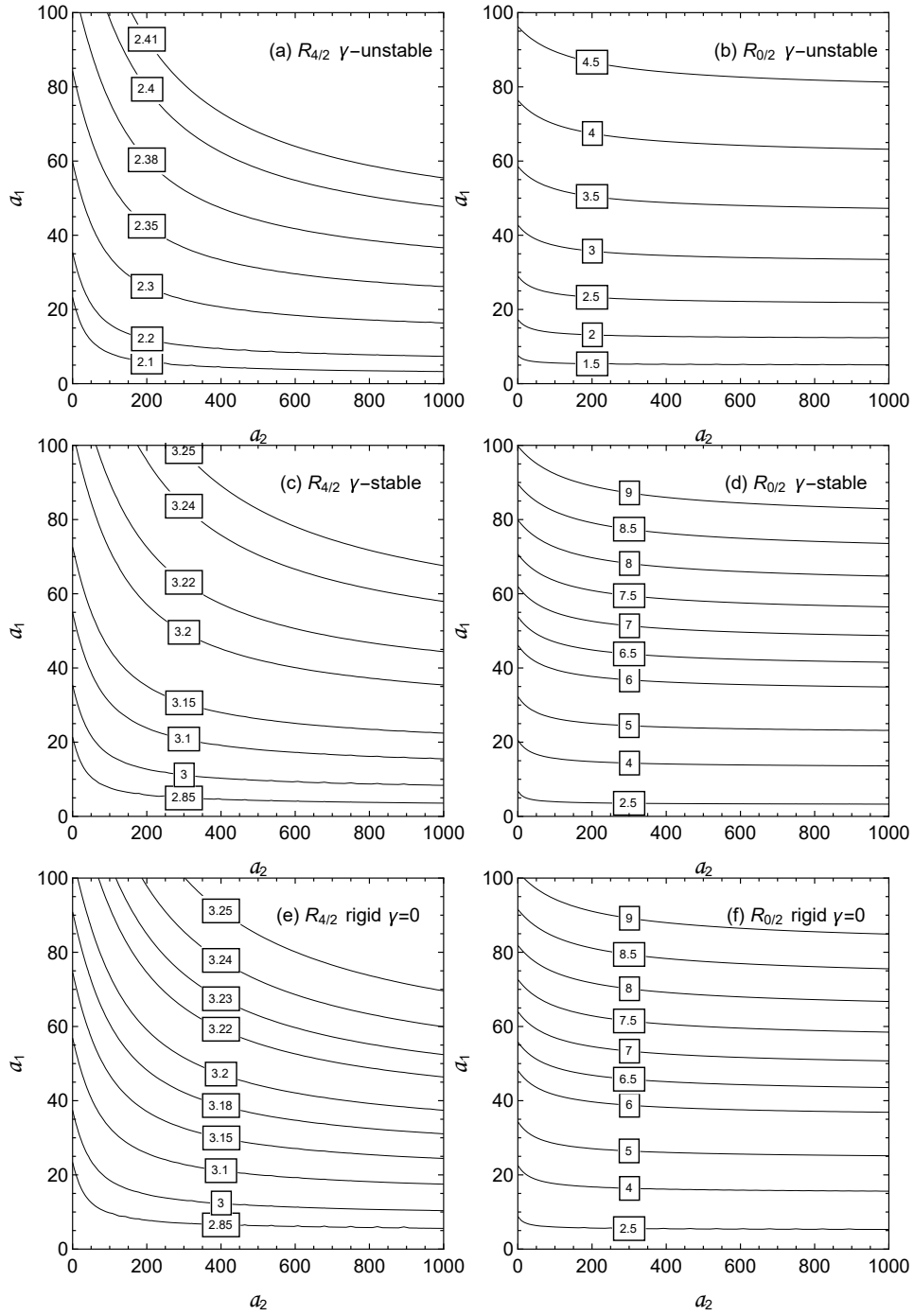

Figure 1: Constant lines of spectral observables $R_{4/2}$ and $R_{0/2}$ as a function of parameters a_1 and a_2 defining an energy-dependent Kratzer potential for γ-unstable (a,b), γ-stable (c,d) and prolate γ-rigid (e,f) cases.

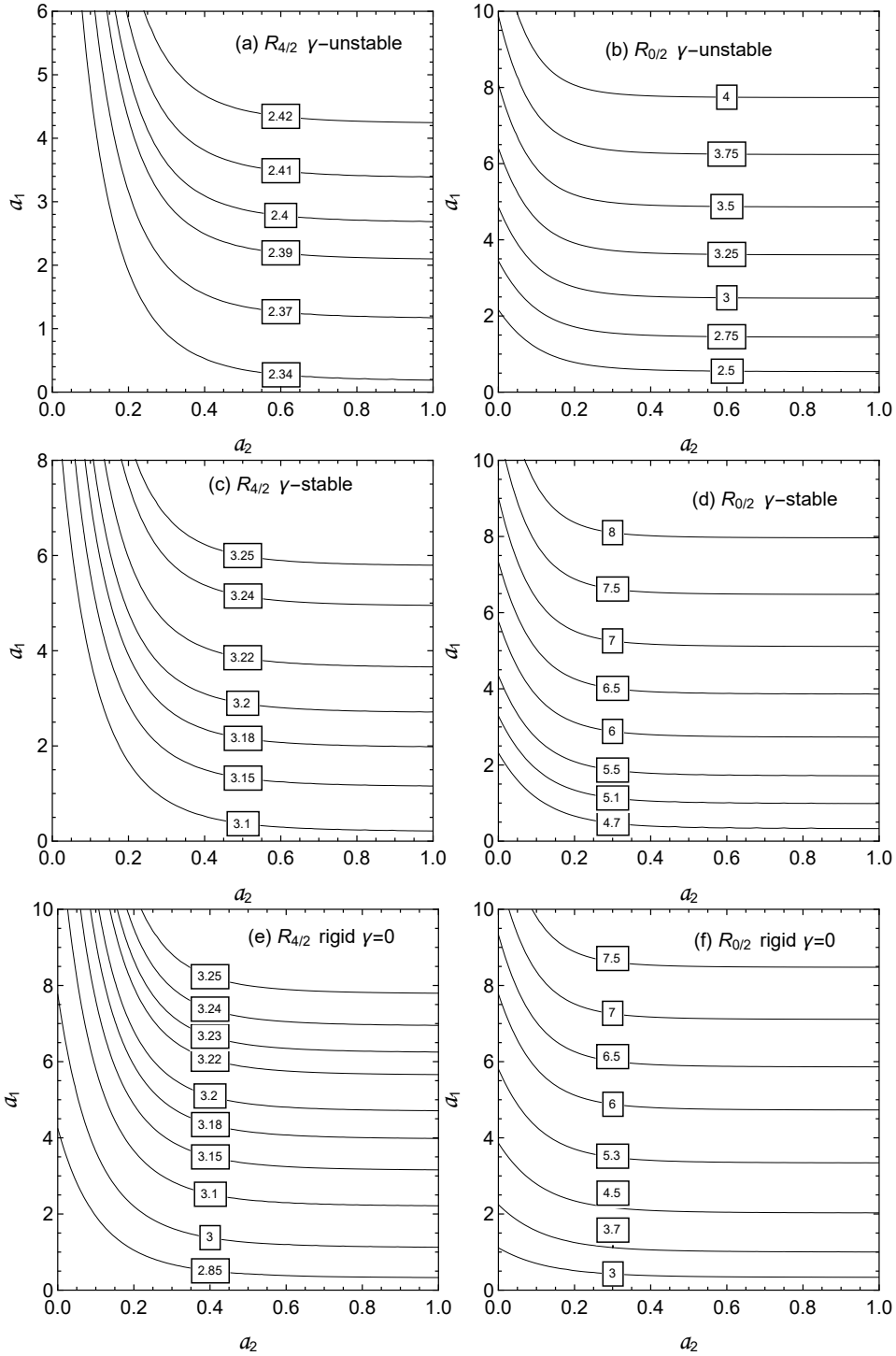

Figure 2: Constant lines of spectral observables $R_{4/2}$ and $R_{0/2}$ as a function of parameters a_1 and a_2 defining an energy-dependent Davidson potential for γ-unstable (a,b), γ-stable (c,d) and prolate γ-rigid (e,f) cases.

symmetric γ-stable situation, it requires an additional parameter and are therefore not considered in this analysis. In the prolate γ-rigid case, there is no γ excited states whatsoever. As can been seen, there is a predictable similarity between prolate γ-stable and γ-rigid spectra. The first striking difference between the Kratzer and Davidson energy-dependent solutions, is the variation rates of the spectra with the two parameters. In the Kratzer case, the domain of values for both a_1 and a_2, which show a similar evolution of spectral observables as in the Davidson case, are few orders of magnitude higher. Another distinguishing aspect related to this disparity in the scale of parameters, is the fact that the spectra generated by the Davidson potential are more sensitive in the small regime of a_2. This fact is reflected in more abrupt curves of Fig.2. Comparing to the Davidson potential results, the constant energy curves of the Kratzer potential are more equidistant when a_1 is varied. The spectral ratios corresponding to Davidson potential have a more restricted domain of values, than in the case of the Kratzer potential. Both potentials are found to offer $R_{0/2}$ ratios with a slow variation as a function of a_2 parameter. This invariance is strengthened in the large value regime of both parameters a_1 and a_2, where the corresponding β potentials have very sharp and deep minima consistent with β-rigidity.

The same conclusions can be drawn for the spectral ratios (5.1) in the triaxial γ-rigid case with $\gamma = \pi/6$, whose dependence on a_1 and a_2 is shown in Fig.3 for the Kratzer potential and in Fig.4 for the Davidson potential. This is valid aslo for the γ band head ratio $R_{2/2}$ which in this particular case does not need an additional parameter. The γ band evolution with spin can be better traced through the measure of the γ band staggering pattern proposed in Ref.[42] and defined as:

$$S(L) = \frac{E(L) - 2E(L-1) + E(L-2)}{E(2_g^+)}. \tag{5.2}$$

This is an important quantity especially when dealing with triaxiality, be it rigid or dynamical as in the case of the γ-unstable conditions. For the low-lying states, the $S(4)$ value is a good indicative for the effect of triaxiality on the rotationally excited states. This quantity is represented in the same manner in Figs. 3 and 4, along the spectral observables (5.1). In case of the Kratzer potential, the range of values for $S(4)$ is more extended in the lower part, having the same upper bound as the results obtained for the Davidson potential. The highest value of $S(4)$ in both cases is close to the value provided by the rigid trixial rotor model of Davydov and Filipov [43].

When $a_1 = 0$, Davidson potential transforms to a simple harmonic oscillator potential, while the Kratzer potential retains just the hyperbolic term, similar to the attractive Coulomb interaction. These simpler models represent a perfect opportunity to show the saturation of the energy levels at high values of parameter a_2. This is done for all considered collective conditions in Fig.5 for the hyperbolic potential and in Fig.6 for the harmonic oscillator potential. As was mentioned previously, the saturation of the whole spectra is achieved earlier for the harmonic oscillator

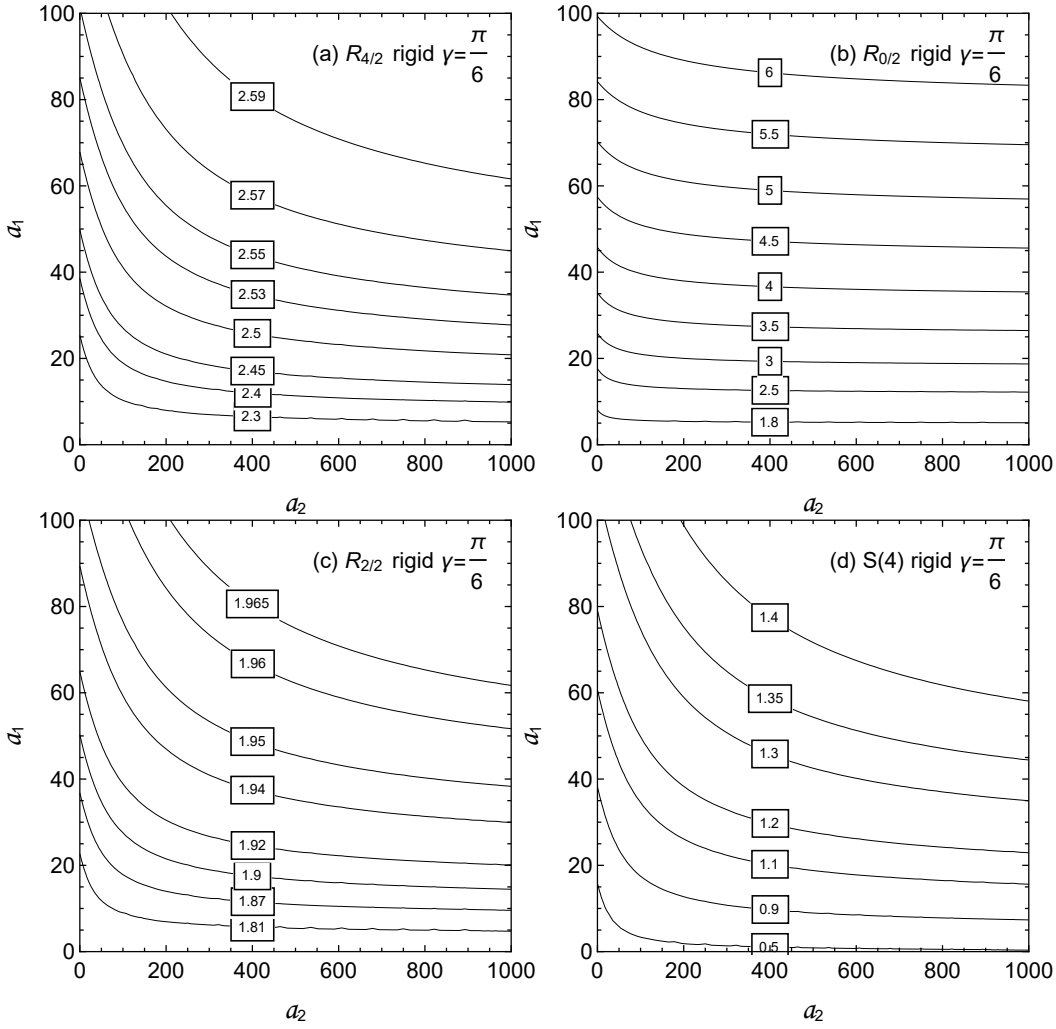

Figure 3: Constant lines of spectral observables $R_{4/2}$ (a), $R_{0/2}$ (b), $R_{2/2}$ (c) and $S(4)$ (d), as a function of parameters a_1 and a_2 defining an energy-dependent Kratzer potential in triaxial γ-rigid conditions.

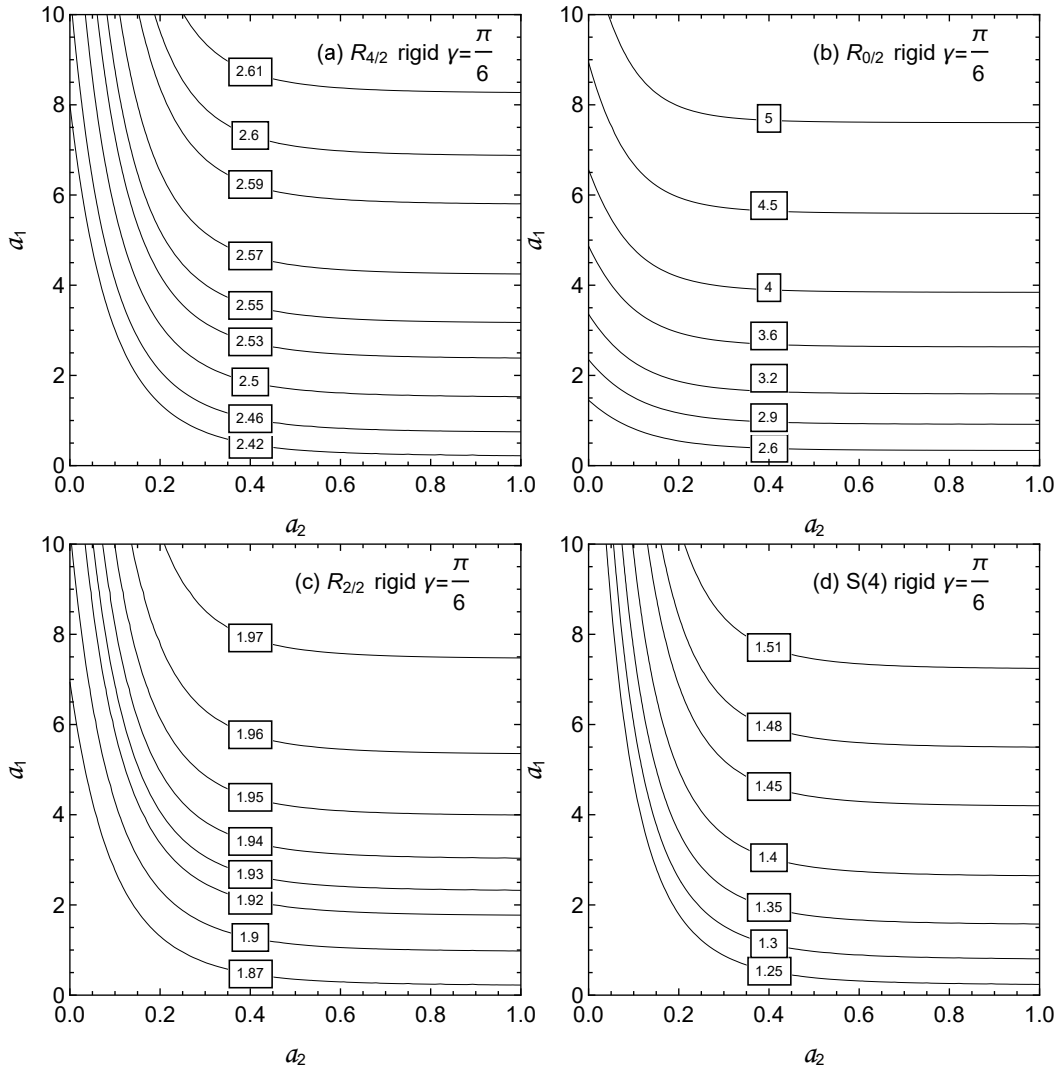

Figure 4: Constant lines of spectral observables $R_{4/2}$ (a), $R_{0/2}$ (b), $R_{2/2}$ (c) and $S(4)$ (d), as a function of parameters a_1 and a_2 defining an energy-dependent Davidson potential in triaxial γ-rigid conditions.

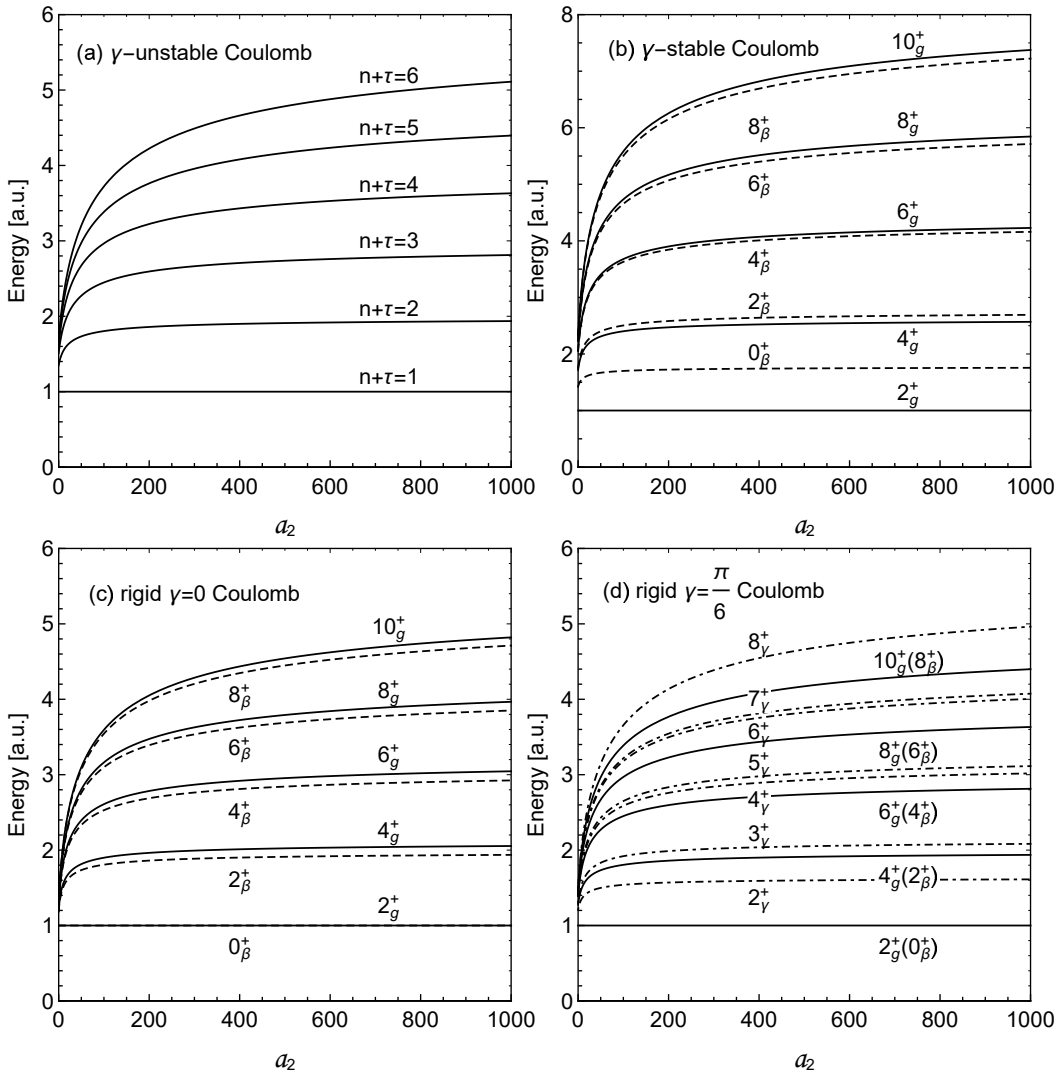

Figure 5: The energy spectrum in the (a) γ-unstable, (b) γ-stable, (c) prolate γ-rigid and (d) triaxial γ-rigid cases of energy-dependent hyperbolic potential ($a_1 = 0$) shown as a function of parameter a_2. The excitation energies are in respect to the ground state and are normalized to the energy of the first excited state.

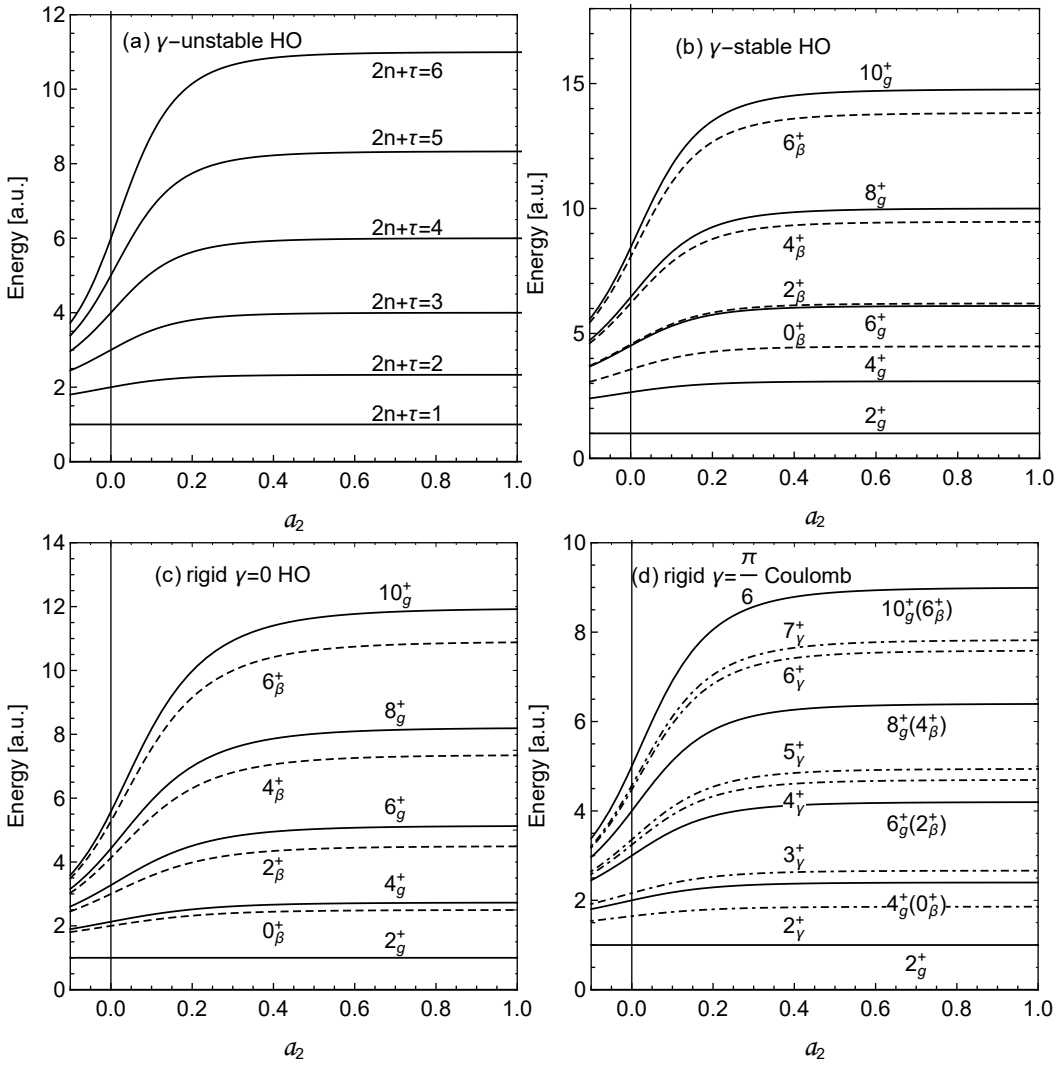

Figure 6: The energy spectrum in the (a) γ-unstable, (b) γ-stable, (c) prolate γ-rigid and (d) triaxial γ-rigid cases of energy-dependent harmonic oscillator potential ($a_1 = 0$) shown as a function of parameter a_2. The excitation energies are in respect to the ground state and are normalized to the energy of the first excited state.

at quite low values of a_2, whereas for the hyperbolic potential, values of a_2 in the order of thousands assures the convergence just for the few low-lying states. This property is obviously state-dependent. Therefore, to have an extended spectrum of the hyperbolic potential which is independent of the a_2 variation, its value must be considered much higher.

For the Davidson potential curves of Fig.6, one represented also a small interval of negative a_2 values. The collective excitations are usually associated with a ratio $R_{4/2} > 2$. In the usual Bohr Model with simple potentials, the limiting value $R_{4/2} = 2$ corresponds to the five-dimensional γ-unstable spherical vibrator model. This value is recovered by the asymptotic hyperbolic potential, but can also be obtained in the $a_2 < 0$ regime of the prolate conditions. The negative value of a_2 suppresses the excitation spectrum, bunching the states together at a finite upper threshold. If the spectrum for the energy-independent case is sufficiently rapidly increasing, as in the case of prolate γ-stable and γ-rigid cases of Davidson potential, small negative a_2 values still provide spectra with monotonically increasing excitation energies. Decreasing excitation energy is not a disqualifying rule for nuclear spectra. There are plenty examples of such spectra, whose interpretation is however regarded as non-collective, mostly based on the incompatibility with the simple cornerstone solutions of the collective models.

The actual asymptotic spectra are not evident from Figs. 3 and 4, whose purpose is just to exemplify the tendency of the spectra to saturate. The actual numerical values of the energy levels for the asymptotic models in the γ-unstable regime are shown in Fig.7 for the hyperbolic potential, and in Fig.8 for the Harmonic oscillator potential. These spectra are also complemented by the relevant quadrupole transition probabilities. The other collective conditions associated to the asymptotic models are listed in Table 1 for the prolate γ-stable and γ-rigid cases and in Table 2 for the γ-rigid triaxial shape with $\gamma = \pi/6$. The numbers in these tables were obtained for a_2 parameter values which assures that all considered energy ratios do not change its second decimal when a_2 is further increased. Having thus a common rule for establishing this minimal value of a_2, one finds, that in both hyperbolic and oscillator cases, this limiting value decreases from the γ-stable results, to the γ-rigid ones, with the triaxial γ-rigid case having the smallest a_2 asymptotic values.

There is a peculiar similarity of the energy level sequence for the asymptotic hyperbolic model with the spectrum associated to the energy-independent harmonic oscillator in the γ-unstable and the γ-rigid ($\gamma = \pi/6$) cases. The only difference is for the β excited states. These are shifted down in the asymptotic energy-dependent model, but still follow the same level sequence as the ground band states. This happens because a particular relation between the vibrational quantum number n and the rotational quantum numbers is realized for these models. Checking the analytical formulas, one finds that in the asymptotic regime, the γ-unstable and triaxial γ-rigid energy levels are given by $n + \tau$ and respectively $n + L/2$, while the corresponding state-independent harmonic oscillator energies are defined by $2n + \tau$ and $2n + L/2$.

Table 1: Asymptotic spectra corresponding to the γ-stable and prolate γ-rigid realizations of the hyperbolic (Hyper) and harmonic oscillator (HO) potentials. Energies are given in terms of the first excited state energy. The last row denotes the approximate value of the a_2 parameter for which the sufficiently precise realization of the asymptotic spectrum is achieved.

L_i^+	γ-stable		γ-rigid	
	Hyper	HO	Hyper	HO
4_g^+	2.65	3.09	2.13	2.73
6_g^+	4.51	6.10	3.27	5.14
8_g^+	6.45	10.00	4.42	8.21
10_g^+	8.44	14.77	5.58	11.95
12_g^+	10.44	20.41	6.73	16.36
14_g^+	12.46	26.90	7.88	21.44
16_g^+	14.49	34.24	9.03	27.18
0_β^+	1.78	4.48	1.00	2.50
2_β^+	2.78	6.20	2.00	4.50
4_β^+	4.43	9.47	3.13	7.36
6_β^+	6.29	13.83	4.27	10.91
8_β^+	8.23	19.13	5.42	15.13
10_β^+	10.22	25.33	6.58	20.03
12_β^+	12.23	32.40	7.73	25.59
a_2	10^{10}	3	10^9	3

The particular combination of the quantum numbers in the energy-dependent case induces a different degeneracy of the energy levels which consequently leads to a higher density of energy degenerated states. The distinct role played by the rotational quantum numbers and n have a more profound effect on the wave-functions. The most important aspect is related to the quadrupole transition probabilities with $\Delta n = 1$, which are essentially forbidden due to the restriction imposed on the selection rules for the rotational quantum numbers. As a result, consecutive β excited bands are fully decoupled.

From the phenomenological point of view, it is quite obvious that the energy-dependent pseudo-harmonic (Davidson) potential with $a_2 > 0$ simulates a stiffening of the nuclear surface against axially-symmetric vibrations. The picture is not so clear with the Kratzer potential. The energy dependence, deepens the minimum of the Kratzer potential, but also spreads its outer wall. Having the exact energy

Table 2: Asymptotic spectra corresponding to the triaxial γ-rigid realizations of the hyperbolic and harmonic oscillator potentials for $\gamma = \pi/6$. Energies are given in terms of the first excited state energy. The asymptotic spectra in the listed numerical format, are obtained for the minimal values $a_2 = 10^9$ for the hyperbolic potential (Hyper) and $a_2 = 2$ for the harmonic oscillator (HO) potential.

ground band			β-band			γ-band		
L_i^+	Hyper	HO	L_i^+	Hyper	HO	L_i^+	Hyper	HO
4_g^+	2.00	2.40	0_β^+	1.00	2.40	2_γ^+	1.65	1.86
6_g^+	3.00	4.20	2_β^+	2.00	4.20	3_γ^+	2.16	2.66
8_g^+	4.00	6.40	4_β^+	3.00	6.40	4_γ^+	3.24	4.70
10_g^+	5.00	9.00	6_β^+	4.00	9.00	5_γ^+	3.36	4.94
12_g^+	6.00	12.00	8_β^+	5.00	12.00	6_γ^+	4.57	7.83
14_g^+	7.00	15.40	10_β^+	6.00	15.40	7_γ^+	4.48	7.59
16_g^+	8.00	19.20	12_β^+	7.00	19.20	8_γ^+	5.78	11.31

spectrum of a Schrödinger equation with an energy-dependent potential, one can in general construct an equivalent local (energy-independent) potential associated to an energy-dependent one [44]. Specifically the energy-dependent Coulomb potential, was found to be very well approximated by a state-independent Wood-Saxon form. The final clues are provided by the associated wave-functions, which suggests a softening of the β excitations with energy.

The energy-dependence in both cases of Davidson and Kratzer potentials, diminishes the difference between the equilibrium deformations of the ground state and the β excited state. In the state-independent situation, the average deformation in the β excited states is higher than the ground state expectation value. By increasing the slope of the energy-dependence, the β excited state equilibrium deformation decreases, becoming at some point equal and even smaller than the ground state value.

As a final conclusion, one must emphases the versatility of the presented models represented by the wide range of $R_{4/2}$ values combined with even a wider range of $R_{0/2}$ values for the β excited band-head state, including even very low ones. The utility of this characteristic will be shown in the next section, where some uncommon collective spectra are successfully described.

5.2 Experimental realization

The novel spectral characteristics induced by the energy-dependence of the potential extend the applicability of the phenomenological Bohr model to nuclei never

described before in this manner. Additionally, the effect of the energy dependence improves the agreement with experiment in comparison with traditional algebraic solutions of the Bohr Hamiltonian, especially for nuclei where the later is deficient. In what follows, one presents examples of both these model contributions to the description of collective excitations in even-even nuclei.

Model calculations for actual nuclei and comparison to experimental data were performed in Refs.[9, 12] with hyperbolic and Kratzer potentials, and in Refs.[8, 10, 11] for the Davidson potential. Here one will briefly discuss these applications with few additional examples.

The simplicity and the interesting properties of the asymptotic hyperbolic γ-unstable case were used to describe nuclei with a vibrational-like ground band spectrum, but for which the 2_g^+ and 0_β^+ states are nearly degenerated. A low-lying excited 0^+ state is one of the telltales associated with shape coexistence. Consequently, the search was directed to regions in the nuclide chart where the shape coexistence phenomenon is encountered, that is in the vicinity of mass numbers $A \sim 70$ and $A \sim 100$. A good experimental realization of the model was found for the ^{72}Se, ^{74}Se, ^{72}Kr, ^{98}Mo and ^{100}Mo nuclei. In Fig.7 one compared the theoretical spectrum with the experimental data for the most promising candidates. Although the energy levels of ^{72}Se and ^{72}Kr are sparse and deviate at higher spins from the vibrational sequence, there is a very good agreement in what concerns the quadrupole transition probabilities. Indeed, the asymptotic hyperbolic model predicts very large $E2$ rates which are consistent with experimental data for these nuclei. In contradistinction, the ^{100}Mo nucleus exhibit very suppressed transitions far from the theoretical predictions, but has an impressive agreement with theory when comparing the distribution and position of the energy levels.

Alternatively, another modification of the spherical vibrator model is brought by the γ-unstable energy-dependent harmonic oscillator solution in its asymptotic limit. Here the distribution and degeneracy of the states are the same as in the spherical vibrator model, but the spectrum is overall expanded, while the quadrupole transition probabilities are smaller. A very good candidate for this asymptotic model was found to be the ^{116}Cd nucleus [8]. Other possible candidates include the ^{104,106}Pd nuclei. As can be seen from Fig.8, where one compared the theoretical predictions with experimental data, the energy states are very well reproduced, while the transition probabilities are substantially underestimated.

Full Davidson potential calculations were performed in the γ-stable regime, which is a more common situation for deformed nuclei. Applications made on Pt and Os isotopes [10] showed that an increased dependence on energy of the potential is expected for transitional nuclei near the critical point of the shape phase transition between deformed and near spherical deformation. This feature was later supported by calculations made in Ref.[11] on $N = 90$ rare-earth nuclei ^{148}Ce, ^{150}Nd, ^{152}Sm, ^{154}Gd and ^{156}Dy, which represent the best examples of such critical behaviour, in comparison to their neighbouring isotopes ^{146}Ce, ^{148}Nd, ^{154}Sm, ^{156}Gd and ^{158}Dy.

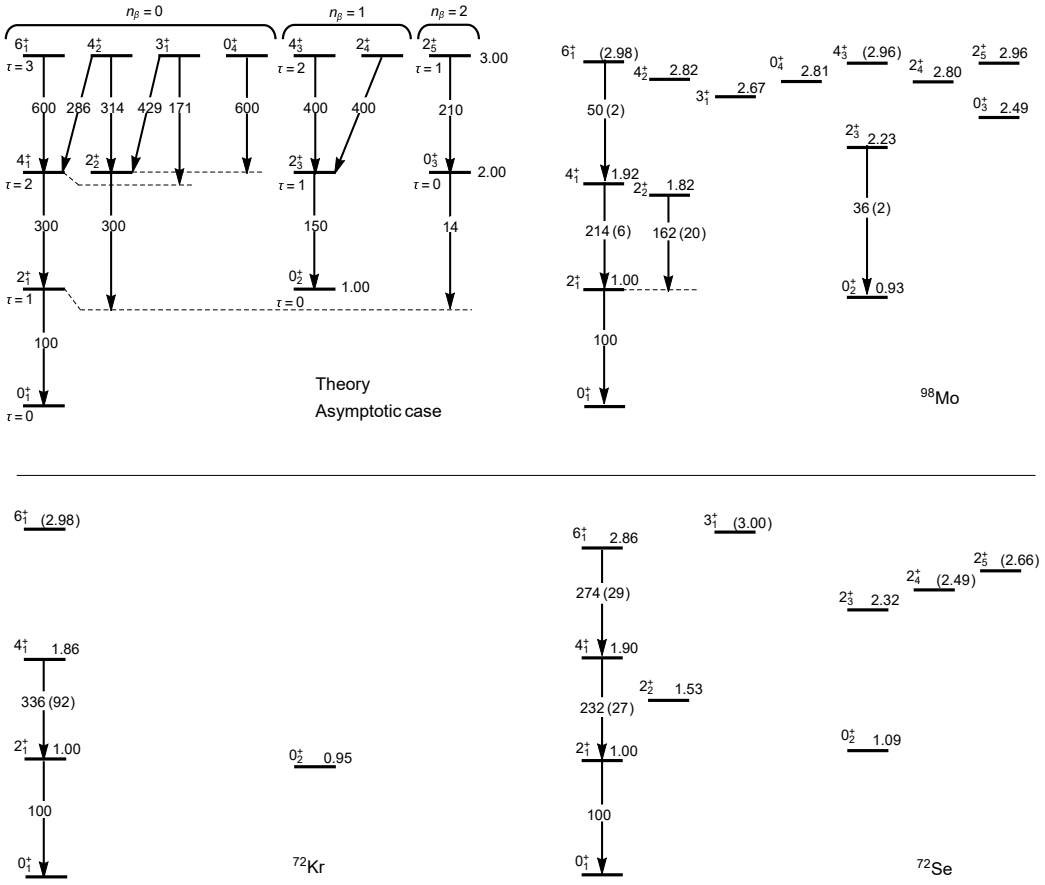

Figure 7: Predictions of the γ-unstable hyperbolic potential with an asymptotically increasing coupling for the lowest portion of the spectrum are compared to the experimental level scheme of ^{98}Mo [45], ^{72}Kr [46], and ^{72}Se [46]. Energies are given in terms of the first excited state energy, while the corresponding $B(E2)$ transition probabilities are normalized to $B(E2, 2_g^+ \rightarrow 0_g^+) = 100$ in order to lose the corresponding scaling constants.

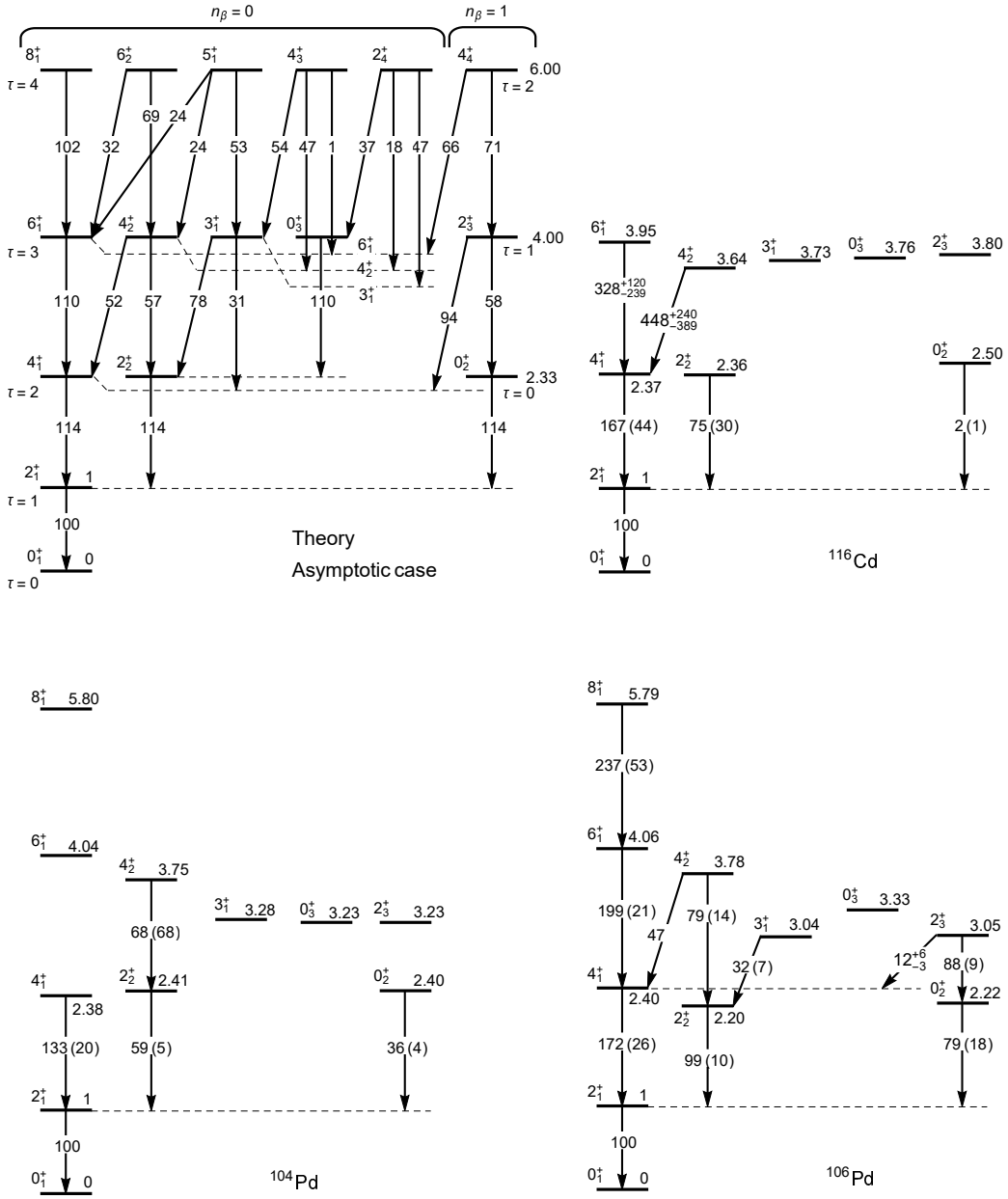

Figure 8: Predictions of the five-dimensional spherical vibrator model with an asymptotically increasing frequency for the lowest portion of the spectrum are compared to the experimental level scheme of ^{116}Cd [47], ^{104}Pd [48] and ^{106}Pd [49]. The same scaling as in Fig.7 is applied.

High slopes of energy dependence was also determined for two nuclei which are considered as endpoints of a similar shape phase transition but in the $N = 60$ region, namely ^{102}Zr, ^{106}Mo.

The energy-dependent Kratzer potential is suitable for describing nuclear collective spectra with a relatively low β band-head. Such a potential was recently used to describe the evolution of the deformation properties for an extended set of Xe isotopes [12]. This was done by high quality model fits on energy levels for $^{118-128}$Xe nuclei. Once again, the highest energy dependence was found to occur near the neutron mid-shell $N = 66$, where the collective behavior suffers a change.

A special feature of the model descriptions with Davidson and Kratzer energy-dependent potentials is the superior reproduction of the β band states in comparison to traditional Bohr model solutions with local potentials.

6 Summary

Energy-dependent potentials are an accessible tool for the generalization of the stationary Schrödinger equation. However, the induced non-linear effects have important consequences on the fundamental quantum structure of the model. The issues were discussed in detail for a general energy dependence, pointing out that only for a linear energy dependence, a coherent quantum theory can be realized. The formalism is applied to exactly solvable Kratzer and Davidson potentials in the frame of the Bohr-Mottelson model subjected to particular conditions which are associated with specific deformation properties of the nuclear shape. The selected collective conditions assure a separation of the equation for the β shape variable from the γ-angular degrees of freedom. As a consequence of this separation, a sizable part of the energy spectrum is determined by just two parameters defining the potential: one factorizing the centrifugal-like term and the other one giving the slope of the energy dependence for the coupling constant of the remaining term. The eigenvalue problem for the β part of the Bohr Hamiltonian leads then to a quadratic equation for the system's energy. The physically meaning energy solution is chosen by imposing a good origin and asymptotic behavior of the associated wave-functions. For all considered cases, energy-related spectral observables are calculated as a function of the two parameters for the purpose of determining the model's domain of applicability to nuclear spectra. Special cases of the hyperbolic and harmonic oscillator potentials are used to ascertain the effect of the slope parameter on the model characteristics. It is observed that the energy spectrum is in general expanded in reference to the energy-independent case. This analysis is especially insightful in what concerns the saturation property of the model at large values of the slope parameter. In this asymptotic regime, the slope parameter acquires just a scaling role, leading thus to parameter-free collective solutions. This feature has an interesting consequence for the hyperbolic potential. Its corresponding energy spectrum is no longer bounded by the energy threshold corresponding to an infinite quantum number as in the un-

perturbed problem. Moreover, in the γ-unstable and triaxial γ-rigid situations, the energy levels in the asymptotic regime of the hyperbolic potential are arranged in a sequence specific to the states described by a corresponding energy-independent model with harmonic oscillator potential. Due to the changes to the quantum theory, the similarity between the two results stops at the energy level scheme.

The success of the formalism in describing nuclei with uncommon spectral characteristics is shown through selected numerical applications. Parameter-free models of the hyperbolic and harmonic oscillator potentials are especially interesting, and an effort was made for finding an experimental realization of them. The energy levels of the most suitable candidate nuclei are very well described, while the theoretical selection rules for the electromagnetic transitions are rarely in a good agreement.

Model calculations with full Kratzer and Davidson potentials were also briefly discussed in connection to high fidelity description of collective excitations in many transitional nuclei. Analyzing the effect of the slope parameter from the phenomenological point of view, a specific behaviour was identified for critical point nuclei. More specifically, nuclei near the critical point of a shape phase transition exhibit a heightened energy dependence. This can be understood by remembering that such nuclei are usually in the vicinity of a mid-shell, such that their valence space is larger than their neighboring nuclei. This in turn allows more substantial microscopic changes between collective states as a function of energy excitation.

Despite being restricted only to a linear energy dependence, Bohr model solutions are greatly improved by the introduction of the energy-dependent potentials. The exact solvability of a local potential is transposed also to the energy-dependent case. This leads to fully analytical Bohr model solutions with a wide range of novel spectral characteristics used to describe collective excitations for even-even nuclei.

References

[1] A. G. Ushveridze, *Quasi-Exactly Solvable Models in Quantum Mechanics* (Institute of Physics Publishing, Bristol, 1994).

[2] G. P. Lepage, Phys. Rev. A **16**, 863 (1977).

[3] H. Sazdjian, J. Math. Phys. **29**, 1620 (1988).

[4] J. Formanek, R. J. Lombard, and J. Mares, Czech. J. Phys. **54**, 289 (2004).

[5] A. Bohr, Mat. Fys. Medd. K. Dan. Vidensk. Selsk. **26**, 14 (1952).

[6] A. Bohr and B. R. Mottelson, Mat. Fys. Medd. Dan. Vidensk. Selsk. **27**, 16 (1953).

[7] R. Budaca, P. Buganu, M. Chabab, A. Lahbas, and M. Oulne, Ann. Phys. (NY) **375**, 65 (2016).

[8] R. Budaca, Phys. Lett. B **751**, 39 (2015).

[9] R. Budaca, Eur. Phys. J. A **52**, 314 (2016).

[10] A. I. Budaca and R. Budaca, Phys. Scr. **92**, 084001 (2017).

[11] A. I. Budaca and R. Budaca, Eur. Phys. J. Plus **134**, 145 (2019).

[12] A. I. Budaca and R. Budaca, (2020) under review.

[13] A. Kratzer, Z. Phys. **3**, 289 (1920).

[14] P. M. Davidson, Proc. R. Soc. A **135**, 459 (1932).

[15] R. Yekken, M. Lassaut, and R. J. Lombard, Ann. Phys. (NY) **338**, 195 (2013).

[16] G. Darboux, C. R. Acad. Sci. **94**, 1456 (1882).

[17] Ji Lin, Yi-Shen Li, and Xian-Min Qian, Phys. Lett. A **362**, 212 (2007).

[18] L. Wilets and M. Jean, Phys. Rev. **102**, 788 (1956).

[19] L. Fortunato, Phys. Rev. C **70**, 011302 (2004).

[20] D. J. Rowe, T. A. Welsh, and M. A. Caprio, Phys. Rev. C **79**, 054304 (2009).

[21] D. J. Rowe and J. L. Wood, *Fundamentals of nuclear models: Foundational Models* (World Scientific, Singapore, 2010).

[22] D. J. Rowe, P. S. Turner, and J. Repka, J. Math. Phys. **45**, 2761 (2004).

[23] G. Rakavy, Nucl. Phys. **4**, 289 (1957).

[24] D. R. Bes, Nucl. Phys. **10**, 373 (1959).

[25] M. A. Caprio, Phys. Rev. C **72**, 054323 (2005).

[26] F. Iachello, Phys. Rev. Lett. **87**, 052502 (2001).

[27] A. S. Davydov and A. A. Chaban, Nucl. Phys. **20**, 499 (1960).

[28] D. Bonatsos, D. Lenis, D. Petrellis, P. A. Terziev and I. Yigitoglu, Phys. Lett. B **632**, 238 (2006).

[29] J. Meyer-ter-Vehn, Nucl. Phys. A **249**, 111 (1975).

[30] L. Fortunato and A. Vitturi, J. Phys. G: Nucl. Part. Phys. **29**, 1341 (2003).

[31] E. T. Whittaker, Bull. Am. Math. Soc. **10**, 125 (1904).

[32] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Dover, New York, 1972).

[33] R. Budaca, Ann. Acad. Rom. Sci. Ser. Math. Appl. **9**, 205 (2017).

[34] W. Mückenheim *et al.*, Phys. Rep. **133**, 337 (1986).

[35] R. P. Feynman, *Negative Probability in Quantum Implications: Essays in Honour of David Bohm* (London: Routledge and Kegan Paul, 1987), pages 235248.

[36] P. A. M. Dirac, Proc. Cambridge Philos. Soc. **26**, 376 (1930).

[37] W. Heisenberg, Physik. Z. **32**, 737 (1931).

[38] E. P. Wigner, Phys. Rev. **40**, 749 (1932).

[39] D. Bonatsos, P. E. Georgoudis, D. Lenis, N. Minkov, and C. Quesne, Phys. Rev. C **83**, 044321 (2011).

[40] D. Bonatsos, P. E. Georgoudis, N. Minkov, D. Petrellis, and C. Quesne, Phys. Rev. C **88**, 034316 (2013).

[41] O. Von Roos, Phys. Rev. B **27**, 7547 (1983).

[42] N. V. Zamfir and R. F. Casten, Phys. Lett. B **260**, 265 (1991).

[43] A. S. Davydov and G. F. Filippov, Nucl. Phys. **8**, 237 (1958).

[44] R. Yekken and R. J. Lombard, J. Phys. A: Math. Theor. **43**, 125301 (2010).

[45] B. Singh and Z. Hu, Nucl. Data Sheets **98**, 335 (2003).

[46] D. Abriola and A. A. Sonzogni, Nucl. Data Sheets **111**, 1 (2010).

[47] J. Blachot, Nucl. Data Sheets **111**, 717 (2010).

[48] J. Blachot, Nucl. Data Sheets **108**, 2035 (2007).

[49] D. D. Frenne and A. Negret, Nucl. Data Sheets **109**, 943 (2008).