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Eigenvalues of $-(\Delta_p + \Delta_q)$ under a Robin-like boundary condition*

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Dedicated to Professor Viorel Barbu on his 75^{th} anniversary

Abstract

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded open set with smooth boundary. Consider in Ω the equation $-\Delta_p u - \Delta_q u = \lambda |u|^{p-2}u$ subject to a Robin-like boundary condition involving a positive constant α, where $p, q \in (1, \infty)$, $p \neq q$, and $\lambda \in \mathbb{R}$. We show that there is no eigenvalue λ of the above problem in the interval $(-\infty, \lambda_R]$, where $\lambda_R := \inf\{\int_\Omega |\nabla v|^p\,dx + \alpha \int_{\partial\Omega} |v|^p\,ds;\ v \in W^{1,\max\{p,q\}}(\Omega),\ \int_\Omega |v|^p\,dx = 1\}$, while any $\lambda \in (\lambda_R, \lambda^*)$ is an eigenvalue of this problem, where $\lambda^* := \alpha m_{N-1}(\partial\Omega)/m_N(\Omega)$. Note that the case $p \neq q$ investigated here is complementary to the homogeneous case $p = q$ for which the set of eigenvalues is completely known only if $p = q = 2$.

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1 Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with smooth boundary $\partial\Omega$. Consider the eigenvalue problem

$$\begin{cases} Au := -\Delta_p u - \Delta_q u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu_A} + \alpha |u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $p, q \in (1, \infty)$, $p \neq q$, $\alpha > 0$, $\lambda \in \mathbb{R}$, and

$$\frac{\partial u}{\partial \nu_A} := \left(|\nabla u|^{p-2} + |\nabla u|^{q-2}\right) \frac{\partial u}{\partial \nu},$$

with $\nu = \nu(x)$ being the unit outward normal to $\partial\Omega$ at $x \in \partial\Omega$. The above PDE (as well as problem (1)) is called nonhomogeneous since $p \neq q$.

The solutions u of problem (1) will be sought in a weak sense, in the Sobolev space $W := W^{1,\max\{p,q\}}(\Omega)$, so that the above PDE is satisfied in the distribution sense, and the generalized normal derivative $\frac{\partial u}{\partial \nu_A}$ (associated with operator A) exists in a trace sense (see [4]). Using a Green formula (see [4, Corollary 2, p. 71]) one can define the eigenvalues of our problem in terms of weak solutions $u \in W$ as follows:

Definition 1. $\lambda \in \mathbb{R}$ *is an eigenvalue of problem (1) if there exists* $u \in W \setminus \{0\}$ *such that*

$$\int_\Omega (|\nabla u|^{p-2} + |\nabla u|^{q-2}) \nabla u \cdot \nabla v \, dx + \alpha \int_{\partial\Omega} |u|^{p-2}uv \, ds$$
$$= \lambda \int_\Omega |u|^{p-2}uv \, dx \quad \forall\, v \in W. \tag{2}$$

Indeed, it is easily seen that if $u \in W$ is a weak solution of (1) then u satisfies (2). Note that all the terms of (2) are well defined. Conversely, by virtue of the same Green formula it follows that if $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1) then any eigenfunction $u \in W \setminus \{0\}$ corresponding to it satisfies problem (1) in the distribution sense.

Our goal is to determine the set of eigenvalues of the Robin problem (1). We cannot achieve completely this goal, but we are able to show that there is no eigenvalue λ of the problem (1) in the interval $(-\infty, \lambda_R]$, where λ_R is defined by (3) below, while any $\lambda \in (\lambda_R, \lambda^*)$ is an eigenvalue of this problem, where λ^* is defined by (4). The case $\lambda \geq \lambda^*$ remains open.

Note that the homogeneous case $q = p > 1$ has been very much discussed in the literature, and a complete description of the corresponding eigenvalue set is available only if $p = q = 2$ (when problem (1) reduces to the classic Robin problem whose eigenvalue set is represented by a sequence of positive eigenvalues). If $p = q \in (1, \infty) \setminus \{2\}$ it is only known that, as a consequence of the Ljusternik-Schnirelman theory, there exists a sequence of positive eigenvalues of the corresponding operator, i.e., of the negative p-Laplacian (see [7, Theorem 3.4, p. 1068]), but this sequence may not constitute the whole eigenvalue set.

In order to state our results let us define

$$\begin{aligned} \lambda_R &= \inf\{\int_\Omega |\nabla v|^p \, dx + \alpha \int_{\partial\Omega} |v|^p \, ds;\; v \in W,\; \int_\Omega |v|^p \, dx = 1\} \\ &= \inf_{v \in W \setminus \{0\}} \frac{\int_\Omega |\nabla v|^p \, dx + \alpha \int_{\partial\Omega} |v|^p \, ds}{\int_\Omega |v|^p \, dx}\,, \end{aligned} \tag{3}$$

where $W = W^{1,\max\{p,q\}}(\Omega)$.

Next, we introduce the following constant

$$\lambda^* := \alpha \frac{m_{N-1}(\partial\Omega)}{m_N(\Omega)}\,, \tag{4}$$

where $m_{N-1}(\partial\Omega)$ and $m_N(\Omega)$ denote the corresponding $N-1$ and N dimensional Lebesgue measures of the boundary $\partial\Omega$ and the set Ω, respectively.

Remark 1. *If $q < p$ then $W = W^{1,p}(\Omega)$ so $\lambda_R = \lambda_1^R$, where λ_1^R denotes the first eigenvalue of the Robin problem*

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} + \alpha |u|^{p-2} u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{5}$$

If $p < q$ then $W = W^{1,q}(\Omega)$ which is a proper subset of $W^{1,p}(\Omega)$ so in this case $\lambda_R \geq \lambda_1^R$. In both cases λ_R is a positive number since λ_1^R is so. For information on the Robin problem (5) we refer the reader to [7]. See also [8].

Throughout in what follows we assume

(H) $\quad \Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded open set with smooth boundary $\partial\Omega$, and α is a positive real number.

Let us state first a nonexistence result:

Theorem 1. *Assume (H) is satisfied and $p, q \in (1, \infty)$, $p \neq q$. Then there exists no eigenvalue of problem (1) in the interval $(-\infty, \lambda_R]$.*

Theorem 2. *Assume* (H) *is satisfied and* $1 < p < q$. *Then any* $\lambda \in (\lambda_R, \lambda^*)$ *is an eigenvalue of problem (1).*

Theorem 3. *Assume* (H) *is satisfied and* $1 < q < p$. *Then, any* $\lambda \in (\lambda_1^R, \lambda^*)$ *is an eigenvalue of problem (1), where* λ_1^R *is the first eigenvalue of problem (5).*

Remark 2. *If in the definition of* λ_R, *with* $p \in (1, \infty)$, *we choose* $v(x) = 1 + t\phi(x)$, *where* $t > 0$ *is small and* $\phi \in C_0^\infty(\Omega)$, $\phi \geq 0$, ϕ *not identically zero, we obtain*

$$\frac{\int_\Omega |\nabla v|^p \, dx + \alpha \int_{\partial\Omega} |v|^p \, ds}{\int_\Omega |v|^p \, dx} = \frac{t^p \int_\Omega |\nabla \phi|^p \, dx + \alpha \, m_{N-1}(\partial\Omega)}{\int_\Omega (1 + t\phi)^p \, dx} =: f(t) \, .$$

Note that $f(0) = \lambda^*$. *Since* $t \longmapsto f(t)$ *is strictly decreasing on an interval* $[0, \delta]$, *we have* $\lambda_R \leq f(\delta) < f(0)$ *and hence*

$$\lambda_R < \lambda^* \, .$$

Thus the statements of Theorems 2 and 3 make sense.

Recall that the spectrum of $A = -\Delta_p - \Delta_q$ under the Neumann boundary condition (i.e., the case $\alpha = 0$ in (1)) has been discussed in [10], with p and q satisfying $p \in [2, \infty)$, $q \in (1, \infty)$, $p \neq q$ (thus extending the results obtained in [5] and [9] for the particular case $p = 2$). In this case the eigenvalue set is $\{0\} \cup (\lambda_N, \infty)$, where

$$\begin{aligned} \lambda_N \;\; = \;\; & \inf \{ \int_\Omega |\nabla v|^p \, dx; \; v \in W^{1,\max\{p,q\}}(\Omega), \\ & \qquad \int_\Omega |v|^p \, dx = 1, \int_\Omega |v|^{p-2} v \, dx = 0 \} \, . \end{aligned}$$

If the Robin condition in (1) is replaced by the Dirichlet condition, i.e., $u = 0$ on $\partial\Omega$, then the corresponding set of eigenvalues is equal to (λ_D, ∞), where

$$\lambda_D = \inf \{ \int_\Omega |\nabla v|^p \, dx; \; v \in W_0^{1,\max\{p,q\}}(\Omega), \; \int_\Omega |v|^p \, dx = 1 \} \, , \tag{6}$$

provided that $p, q \in (1, \infty)$, $p \neq q$. For details see [1] and [2]. We only point out that their definition of λ_D (with infimum over $C_0^\infty(\Omega)$ instead of $W_0^{1,\max\{p,q\}}(\Omega)$) is in fact equivalent with (6), as the reader can easily check by using the density of $C_0^\infty(\Omega)$ in $W_0^{1,\max\{p,q\}}(\Omega)$.

It is worth pointing out that in all cases (Dirichlet, Neumann or Robin) one can use essentially the same strategy to derive the corresponding eigenvalue sets. However there are significant differences so separate analysis is required in each case. In particular, the Robin eigenvalue problem (1) is the most difficult one and we cannot provide for the moment a complete description of the eigenvalue set in this case. For the convenience of the reader we shall provide complete proofs of our results above (Theorems 1, 2, 3).

2 Proofs

We start with the

Proof of Theorem 1.

Assume (H) is satisfied, and $p, q \in (1, \infty),\ p \neq q$. The proof is divided into four steps.

Step 1: *there is no negative eigenvalue of problem (1).*

Indeed, if $(\lambda,\, u) \in \mathbb{R} \times (W \setminus \{0, \})$ is an eigenpair of problem (1), then choosing $v = u$ in (2) we obtain

$$\int_\Omega (|\nabla u|^p + |\nabla u|^q)\, dx + \alpha \int_{\partial\Omega} |u|^p\, ds = \lambda \int_\Omega |u|^p \quad \forall\, v \in W\,, \tag{7}$$

which clearly shows that $\lambda \geq 0$.

Step 2: $\lambda = 0$ *is not an eigenvalue for problem (1).*

Assume by contradiction that $\lambda = 0$ is an eigenvalue of problem (1), and let $u \in W \setminus \{0\}$ be a corresponding eigenfunction. It follows from (7) that $\nabla u = 0$ as an element of W. By Weyl's regularity lemma (see, e.g., [12]) $u \in C^\infty(\Omega)$, hence u is a constant function. Since $\int_{\partial\Omega} |u|^p\, ds = 0$ it follows that u is the null function, contradiction.

Step 3: *there is no eigenvalue of problem (1) in* $(0, \lambda_R)$.

Assume the contrary, that there exists an eigenvalue $\lambda \in (0, \lambda_R)$. Let

$u \in W\setminus\{0\}$ be an eigenfunction of problem (1) corresponding to λ. Then

$$\begin{aligned} 0<(\lambda_R-\lambda)\int_\Omega |u|^p\,dx &\le \int_\Omega |\nabla u|^p\,dx+\alpha\int_{\partial\Omega}|u|^p\,ds-\lambda\int_\Omega |u|^p\,dx \\ &\le \int_\Omega |\nabla u|^p\,dx+\alpha\int_{\partial\Omega}|u|^p\,ds-\lambda\int_\Omega |u|^p\,dx \\ &\qquad +\int_\Omega |\nabla u|^q\,dx=0 \end{aligned}$$

which implies the impossible inequality $0<0$.

Step 4: $\lambda=\lambda_R$ *is not an eigenvalue for problem (1).*

Assume by contradiction that λ_R is a eigenvalue of problem (1), and let $u_R\in W\setminus\{0\}$ be an eigenfunction corresponding to λ_R. Choosing $\lambda=\lambda_R$ and $v=u=u_R$ in (2) yields

$$\int_\Omega \left(|\nabla u_R|^p+|\nabla u_R|^q\right)dx+\alpha\int_{\partial\Omega}|u_R|^p\,ds=\lambda_R\int_\Omega |u_R|^p\,. \tag{8}$$

From (8) and the definition of λ_R we derive

$$\begin{aligned} \int_\Omega |\nabla u_R|^q\,dx+\lambda_R\int_\Omega |u_R|^p\,dx &\le \int_\Omega |\nabla u_R|^q+\int_\Omega |\nabla u_R|^p\,dx \\ &\quad +\alpha\int_{\partial\Omega}|u_R|^p\,ds=\lambda_R\int_\Omega |u_R|^p\,dx\,, \end{aligned}$$

hence

$$\int_\Omega |\nabla u_R|^q\,dx=0\,.$$

This implies that u_R is a constant function, and as u_R is a solution of the first equation in (1) with $\lambda=\lambda_R>0$, it follows that u_R is the null function, a contradiction.

Proof of Theorem 2.

We assume that (H) is satisfied and $1<p<q$. Then $W=W^{1,q}(\Omega)$.

We choose a $\lambda\in(\lambda_R,\lambda^*)$ which will remain fixed throughout the proof of Theorem 2. Define $J:W\to\mathbb{R}$ by

$$J(v):=\frac{1}{p}\int_\Omega |\nabla v|^p\,dx+\frac{1}{q}\int_\Omega |\nabla v|^q\,dx+\frac{\alpha}{p}\int_{\partial\Omega}|v|^p\,ds-\frac{\lambda}{p}\int_\Omega |v|^p\,dx\,.$$

It is easily seen that $J \in C^1(W, \mathbb{R})$ with the derivative given by

$$\begin{aligned}\langle J'(v), \phi\rangle &= \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\phi \, dx + \int_\Omega |\nabla v|^{q-2}\nabla v \cdot \nabla \phi \, dx \\ &\quad + \alpha \int_{\partial\Omega} |v|^{p-2} v\phi \, ds - \lambda \int_\Omega |v|^{q-2} v \phi \, dx \,,\end{aligned}$$

for all v, $\phi \in W$. It is worth pointing out that λ is an eigenvalue of problem (1) if and only if there exists $u \in W \setminus \{0\}$ such that $J^{'}(u) = 0$. In the following we shall prove the existence of a nontrivial critical point of J by using a classical result in Calculus of Variations, see [13, Theorem 1.2]. Obviously, $W = W^{1,q}(\Omega)$ is a reflexive Banach space. We have

Claim 1: *functional J is coercive, i.e., $J(v) \to \infty$ as $\|v\|_{W^{1,q}(\Omega)} \to \infty$*

Assume by contradiction that functional J is not coercive. Then there exist a constant $M > 0$ and a sequence $(u_n)_{n\geq 1}$ in $W^{1,q}(\Omega)$ such that $\|u_n\|_{W^{1,q}(\Omega)} \to +\infty$ and $J(u_n) \leq M$, for $n = 1, 2, \ldots$ We have

$$\begin{aligned} 0 \leq \frac{p}{q}\int_\Omega |\nabla u_n|^q \, dx &< \lambda \int_\Omega |u_n|^p \, dx - \alpha \int_{\partial\Omega} |u_n|^p \, ds - \int_\Omega |\nabla u_n|^p \, dx + pM \\ &\leq \lambda \int_\Omega |u_n|^p \, dx - \alpha \int_{\partial\Omega} |u_n|^p \, ds + pM \qquad (9) \\ &\leq \lambda \|u_n\|^p_{L^p(\Omega)} + pM. \qquad (10)\end{aligned}$$

Taking into account the fact that the norm $\|\nabla u\|_{L^q(\Omega)} + \|u\|_{L^p(\Omega)}$ is equivalent to the usual norm $\|u\|_{W^{1,q}(\Omega)}$ in $W^{1,q}(\Omega)$ (see [3, Remark 15, p. 286]) we conclude from (10) that $\|u_n\|_p := \|u_n\|_{L^p(\Omega)} \to +\infty$. Set $v_n := u_n/\|u_n\|_p$ and divide (10) by $\|u_n\|_p^q$. Then $\|\nabla v_n\|_{L^q(\Omega)} \to 0$ implying that (v_n) converges (on a subsequence) strongly in $W^{1,q}(\Omega)$ to some constant function. Indeed, (v_n) is bounded in $W^{1,q}(\Omega)$, hence there exists $v_\infty \in W^{1,q}(\Omega)$ such that (on a subsequence) $v_n \to v_\infty$ weakly in $W^{1,q}(\Omega)$ and strongly in $L^p(\Omega)$ and $L^p(\partial\Omega)$. Next, we have

$$\|\nabla v_\infty\|_{L^q(\Omega)} \leq \lim_{n\to\infty} \|\nabla v_n\|_{L^q(\Omega)} = 0 \,,$$

so v_∞ is a constant function, say $v_\infty \equiv C$. In addition,

$$\|v_\infty\|_p = \lim_{n\to\infty} \|v_n\|_p = 1 \,,$$

thus $C \neq 0$. On the other hand, if we divide (9) by $\|u_n\|_p^p$ and take the limit as $n \to \infty$, then we obtain

$$0 \leq \lambda \int_\Omega |v_\infty|^p \, dx - \alpha \int_{\partial\Omega} |v_\infty|^p \, ds = |C|^p(\lambda - \lambda^*) m_N(\Omega) < 0 \,,$$

a contradiction.

Thus functional J is indeed coercive.

Claim 2: *for any sequence $v_k \to v$ weakly in $W^{1,q}(\Omega)$ we have* $J(v) \le \liminf J(v_k)$.

Since the canonical injections from $W^{1,q}(\Omega)$ to $L^p(\Omega)$ and $L^p(\partial\Omega)$ are both compact, we have

$$\lim_{k\to\infty} \int_\Omega |v_k|^p\,dx = \int_\Omega |v|^p\,dx, \quad \lim_{k\to\infty} \int_{\partial\Omega} |v_k|^p\,ds = \int_{\partial\Omega} |v|^p\,ds\,. \tag{11}$$

From (11) and the weak lower semicontinuity of the norms in $L^p(\Omega)$ and $L^q(\Omega)$, we see that $J(v) \le \liminf J(v_k)$, as claimed.

Now, taking into account Claim 1 and Claim 2, we infer by [13, Theorem 1.2] that J is bounded from below and has a global minimizer, say $u \in W^{1,q}(\Omega)$, i.e., $J(u) = \min_{v\in W^{1,q}(\Omega)} J(v)$. Hence u is a critical point of J: $J'(u) = 0$. One can show that $J(u) < 0$. To this purpose let us first observe that

$$\lambda_R = \inf_{v\in W\setminus\{0\}} \frac{\frac{1}{p}\int_\Omega |\nabla v|^p\,dx + \frac{1}{q}\int_\Omega |\nabla v|^q\,dx + \frac{\alpha}{p}\int_{\partial\Omega} |v|^p\,ds}{\frac{1}{p}\int_\Omega |v|^p\,dx}\,. \tag{12}$$

Indeed, the inequality $\le$ in (12) is obvious, while the converse inequality follows easily by replacing $v \in W\setminus\{0\}$ in the right hand side by tv, $t > 0$,

$$\begin{aligned}&\frac{\frac{1}{p}\int_\Omega |\nabla(tv)|^p\,dx + \frac{1}{q}\int_\Omega |\nabla(tv)|^q\,dx + \frac{\alpha}{p}\int_{\partial\Omega} |tv|^p\,ds}{\frac{1}{p}\int_\Omega |tv|^p\,dx}\\ &\qquad = \frac{\int_\Omega |\nabla v|^p\,dx + \alpha\int_{\partial\Omega} |v|^p\,ds}{\int_\Omega |v|^p\,dx} + \frac{pt^{q-p}}{q}\,\frac{\int_\Omega |\nabla v|^q\,dx}{\int_\Omega |v|^q\,dx}\,,\end{aligned}$$

and observing that $t^{q-p} \to 0$ as $t \to 0^+$. (Note that in fact (12) is also valid if $q < p$ since $t^{q-p} \to 0$ as $t \to \infty$). As $\lambda_R < \lambda$ it follows from (12) that there exists a $v^* \in W^{1,q}(\Omega)\setminus\{0\}$ such that $J(v^*) < 0$. Hence $J(u) \le J(v^*) < 0$, showing that $u \ne 0$.

Consequently, as $J'(u) = 0$, λ is an eigenvalue of problem (1).

Proof of Theorem 3.

We assume that $1 < q < p$. It follows that $W = W^{1,p}(\Omega)$ and $\lambda_R = \lambda_1^R$.

We again fix a $\lambda \in (\lambda_R, \lambda^*)$. Under the present conditions we cannot derive coercivity of J on $W = W^{1,p}(\Omega)$ so have to use another method. Note that in this case $J \in C^1(W^{1,p}(\Omega), \mathbb{R})$. Note also that any eigenfunction u corresponding to λ satisfies equation (2) so testing with $v = 1$ and with $v = u$ we obtain

$$\alpha \int_{\partial\Omega} |u|^{p-2}u \, ds = \lambda \int_\Omega |u|^{p-2}u \, dx,$$
$$\int_\Omega (|\nabla u|^p + |\nabla u|^q) \, dx + \alpha \int_{\partial\Omega} |u|^p \, ds = \lambda \int_\Omega |u|^p \, dx \, .$$

We plan to show that J has a nonzero critical point (which will be an eigenfunction corresponding to λ) so it is natural to investigate the restriction of functional J to the Nehari type manifold (see [14])

$$\begin{aligned} M &= \{ v \in W \setminus \{0\};\, \alpha \int_{\partial\Omega} |v|^{p-2}v \, ds = \lambda \int_\Omega |v|^{p-2}v \, dx,\, \langle J'(v), v\rangle = 0 \} \\ &= \{ v \in W \setminus \{0\};\, \alpha \int_{\partial\Omega} |v|^{p-2}v \, ds = \lambda \int_\Omega |v|^{p-2}v \, dx, \\ &\qquad \int_\Omega (|\nabla v|^p + |\nabla v|^q) \, dx + \alpha \int_{\partial\Omega} |v|^p \, ds = \lambda \int_\Omega |v|^p \, dx \}. \end{aligned}$$

We shall prove that J attains its infimum $m := \inf_{v\in M} J(v)$ at some point $u \in M$ and $J'(u) = 0$. The proof is based on several claims as follows:

Claim (a): $M \neq \emptyset$.

We will first show that there exists a function w which belongs to the set

$$S = \{ v \in W^{1,p}(\Omega);\, \alpha \int_{\partial\Omega} |v|^{p-2}v \, ds = \lambda \int_\Omega |v|^{p-2}v \, dx \},$$

such that

$$\int_\Omega |\nabla w|^p \, dx + \alpha \int_{\partial\Omega} |w|^p \, ds < \lambda \int_\Omega |w|^p \, dx \, . \tag{13}$$

Recall that there exists a positive eigenfunction $u_R \in W^{1,p}(\Omega) \setminus \{0\}$ corresponding to the first eigenvalue λ_1^R of the problem (5) (see, e.g., [7] and [8]). In particular,

$$\alpha \int_{\partial\Omega} |u_R|^{p-2}u_R \, ds = \lambda_1^R \int_\Omega |u_R|^{p-2}u_R \, dx < \lambda \int_\Omega |u_R|^{p-2}u_R \, dx, \tag{14}$$

$$\int_\Omega |\nabla u_R|^p \, dx + \alpha \int_{\partial\Omega} |u_R|^p \, ds = \lambda_1^R \int_\Omega |u_R|^p \, dx \; < \lambda \int_\Omega |u_R|^p \, dx \, . \tag{15}$$

Denote

$$\gamma(v) := \alpha \int_{\partial\Omega} |v|^{p-2}v ds - \lambda \int_\Omega |v|^{p-2}v dx. \tag{16}$$

One can easily check that

$$\lim_{\xi\to\pm\infty}\frac{\gamma(v+\xi)}{|\xi|^{p-2}\xi}=\alpha m_{N-1}(\partial\Omega)-\lambda m_N(\Omega)>0. \tag{17}$$

We take w in the form $w=u_R+\xi^*$, where ξ^* is the least positive constant such that

$$\alpha\int_{\partial\Omega}|u_R+\xi^*|^{p-2}(u_R+\xi^*)\,ds=\lambda\int_{\Omega}|u_R+\xi^*|^{p-2}(u_R+\xi^*)\,dx.$$

Indeed, such constant exists since the function $\xi\mapsto\gamma(u_R+\xi)$ is continuous, negative at $\xi=0$ (due to inequality (14)) and $\gamma(u_R+\xi)\to+\infty$ as $\xi\to+\infty$ (according to (17)). In addition, we have

$$\alpha\int_{\partial\Omega}|w|^p\,ds-\lambda\int_{\Omega}|w|^p\,dx<\alpha\int_{\partial\Omega}|u_R|^p\,ds-\lambda\int_{\Omega}|u_R|^p\,dx$$

since

$$\frac{d}{d\xi}\left(\alpha\int_{\partial\Omega}|u_R+\xi|^p\,ds-\lambda\int_{\Omega}|u_R+\xi|^p\,dx\right)=p\gamma(u_R+\xi)<0$$

for any $\xi\in[0,\xi^*)$. Hence the inequality (13) holds for $w=u_R+\xi^*$ due to (15). Then there exists a $t>0$ such that $tw\in M$, i.e.,

$$t^p\int_{\Omega}|\nabla w|^p\,dx+t^q\int_{\Omega}|\nabla w|^q\,dx+\alpha t^p\int_{\partial\Omega}|w|^p\,ds=\lambda t^p\int_{\Omega}|w|^p\,dx\,, \tag{18}$$

noting that $tw\in S$. Indeed, equation (18) can be solved for t:

$$t=\Big(\frac{\int_{\Omega}|\nabla w|^q\,dx}{\lambda\int_{\Omega}|w|^p\,dx-\alpha\int_{\partial\Omega}|w|^p\,ds-\int_{\Omega}|\nabla w|^p\,dx}\Big)^{\frac{1}{p-q}}\,, \tag{19}$$

which is a positive number (since $\int_{\Omega}|\nabla w|^q\,dx=0\Leftrightarrow w=const.$, which contradicts $w\in S\setminus\{0\}$) and for this t we have $tw\in M$. Therefore $M\neq\emptyset$ as claimed.

Claim (b): $m\geq 0$.

Indeed, for all $v\in M$, we have

$$\begin{aligned}
J(v) &= \frac{1}{q}\int_{\Omega}|\nabla v|^q\,dx+\frac{1}{p}\int_{\Omega}|\nabla v|^p\,dx+\frac{\alpha}{p}\int_{\partial\Omega}|v|^p\,ds-\frac{\lambda}{p}\int_{\Omega}|v|^p\,dx\\
&= \frac{1}{q}\int_{\Omega}|\nabla v|^q\,dx-\frac{1}{p}\int_{\Omega}|\nabla v|^q\,dx\\
&= \frac{p-q}{pq}\int_{\Omega}|\nabla v|^q\,dx\\
&\geq 0\,.
\end{aligned}$$

Claim (c): *every minimizing sequence for J on the set M is bounded in $W^{1,p}(\Omega)$.*

Let (u_n) be a minimizing sequence in M, i.e.,

$$\begin{aligned} 0 &\leq \lambda \int_\Omega |u_n|^p \, dx - \int_\Omega |\nabla u_n|^p \, dx - \alpha \int_{\partial\Omega} |u_n|^p \, ds \\ &= \int_\Omega |\nabla u_n|^q \, dx \to \frac{pq}{p-q} m \quad \text{as } n \to \infty \, . \end{aligned} \tag{20}$$

Assume by contradiction that (u_n) is unbounded in $W^{1,p}(\Omega)$, i.e.,

$$\int_\Omega |\nabla u_n|^p \, dx + \int_\Omega |u_n|^p \, dx$$

is unbounded. According to (20), $\|u_n\|_p := \|u_n\|_{L^p(\Omega)}$ is unbounded too, i.e., for a subsequence of (u_n), still denoted (u_n), we have $\|u_n\|_p \to \infty$. Set $v_n := u_n/\|u_n\|_p$. We divide (20) by $\|u_n\|_p$ and derive $\int_\Omega |\nabla v_n|^p \, dx \leq \lambda$, i.e., (v_n) is bounded in $W^{1,p}(\Omega)$. Therefore there exists $v_\infty \in W^{1,p}(\Omega)$ such that (on a subsequence)

$$v_n \to v_\infty \text{ weakly in } W^{1,p}(\Omega), \text{ and strongly in both } L^p(\Omega) \text{ and } L^p(\partial\Omega) \, , \tag{21}$$

since $W^{1,p}(\Omega)$ is reflexive and the canonical injections from this space to $L^p(\Omega)$ and $L^p(\partial\Omega)$ are compact. Then, as $v_n \in S$ for all n, it follows by Lebesgue's Dominated Convergence Theorem and [3, Theorem 4.9, p. 94] that $v_\infty \in S$. Now, since $\int_\Omega |\nabla u_n|^q \, dx$ is bounded (cf. (20) we have

$$\int_\Omega |\nabla v_n|^q \, dx \to 0 \, ,$$

so

$$\int_\Omega |\nabla v_\infty|^q \, dx \leq \liminf \int_\Omega |\nabla v_n|^q \, dx = 0 \, ,$$

since v_n converges to v_∞ weakly in $W^{1,p}(\Omega)$, hence in $W^{1,q}(\Omega)$ as well. Therefore v_∞ is a constant function. This fact combined with $v_\infty \in S$ implies $v_\infty \equiv 0$. On the other hand, from $\|v_n\|_p = 1$ and $\|v_n - v_\infty\|_p \to 0$, we derive $\|v_\infty\|_p = 1$ contradicting $v_\infty \equiv 0$. Therefore Claim (c) holds true.

Claim (d): $m = \inf_{v \in M} J(v)$ *is positive*: $m > 0$.

Assume the contrary, i.e., in view of Claim (b), $m = 0$. Let (u_n) be a minimizing sequence, i.e., $u_n \in M$ for all n and $J(u_n) \to 0$. So we can write (see (20))

$$\begin{aligned} 0 &\le \lambda \int_\Omega |u_n|^p \, dx - \int_\Omega |\nabla u_n|^p \, dx - \alpha \int_{\partial\Omega} |u_n|^p \, ds \\ &= \int_\Omega |\nabla u_n|^q \, dx \to 0 \quad \text{as } \ n \to \infty \, . \end{aligned} \tag{22}$$

By Claim (c) we know that (u_n) is bounded in $W^{1,p}(\Omega)$. It follows that there exists $u_\infty \in W^{1,p}(\Omega)$ such that (on a subsequence) u_n converges to u_∞ weakly in $W^{1,p}(\Omega)$ (hence also in $W^{1,q}(\Omega)$) and u_n converges strongly to u_∞ in both $L^p(\Omega)$ and $L^p(\partial\Omega)$. Therefore, $u_\infty \in S$ and

$$\int_\Omega |\nabla u_\infty|^q \, dx \le \liminf \int_\Omega |\nabla u_n|^q \, dx = 0 \, ,$$

and consequently u_∞ is the null function. Summarizing, we see that u_n converges to 0 weakly in $W^{1,p}(\Omega)$.

Now, set $v_n := u_n / \|u_n\|_p$. From (22) we deduce that

$$\int_\Omega |\nabla u_n|^p \, dx + \alpha \int_{\partial\Omega} |u_n|^p \, ds \le \lambda \int_\Omega |u_n|^p \, dx \quad \forall n \, ,$$

which implies

$$\int_\Omega |\nabla v_n|^p \, dx + \alpha \int_{\partial\Omega} |v_n|^p \, ds \le \lambda \quad \forall n \, .$$

Therefore, (v_n) is bounded in $W^{1,p}(\Omega)$. It follows that there exists $v_\infty \in W^{1,p}(\Omega)$ such that (on a subsequence) v_n converges to v_∞ weakly in $W^{1,p}(\Omega)$ and strongly in both $L^p(\Omega)$ and $L^p(\partial\Omega)$. Moreover, as $v_n \in S$ for all n, we also have $v_\infty \in S$.

Now, dividing (22) by $\|u_n\|_p^q$ we obtain

$$\int_\Omega |\nabla v_n|^q \, dx = \|u_n\|_p^{p-q} \left[\lambda - \int_\Omega |\nabla v_n|^p \, dx - \alpha \int_{\partial\Omega} |v_n|^p \, ds \right] \to 0 \, .$$

Next, since v_n converges (on a subsequence) to v_∞ weakly in $W^{1,p}(\Omega)$ and hence in $W^{1,q}(\Omega)$, we have

$$\int_\Omega |\nabla v_\infty|^q \, dx \le \liminf \int_\Omega |\nabla v_n|^q \, dx = 0 \, ,$$

and consequently v_∞ is a constant function. In fact $v_\infty = 0$ since $v_\infty \in S$. Thus v_n converges strongly to 0 in $L^p(\Omega)$, which contradicts the fact that $\|v_n\|_p = 1$ for all n.

This contradiction shows that $m > 0$ as asserted.

Claim (e): *there exists* $u \in M$ *such that* $J(u) = m$, *i.e.,* $m = \inf_M J$ *is attained.*

Let (u_n) be a sequence in M such that $J(u_n) \to m$. By Claim (c) (u_n) is bounded in $W^{1,p}(\Omega)$. Thus, on a subsequence, u_n converges to some u weakly in $W^{1,p}(\Omega)$ and strongly in both $L^p(\Omega)$ and $L^p(\partial\Omega)$. Therefore,

$$J(u) \le \liminf J(u_n) = m\,. \tag{23}$$

As $u_n \in M$ for all n we have

$$\int_\Omega (|\nabla u_n|^p + |\nabla u_n|^q)\,dx + \alpha \int_{\partial\Omega} |u_n|^p\,ds = \lambda \int_\Omega |u_n|^p\,dx \quad \forall n\,, \tag{24}$$

and

$$\alpha \int_{\partial\Omega} |u_n|^{p-2} u_n\,ds = \lambda \int_\Omega |u_n|^{p-2} u_n\,dx \quad \forall n\,. \tag{25}$$

Assuming $u = 0$ we can infer from (24) that u_n converges to 0 strongly in $W^{1,p}(\Omega)$. Arguing as in the proof of Claim (d) we reach a contradiction. Hence $u \ne 0$. Passing to the limit in (25) we find that $u \in S$. Now, letting $n \to 0$ in (24) we get

$$\int_\Omega |\nabla u|^p\,dx + \int_\Omega |\nabla u|^q\,dx + \alpha \int_{\partial\Omega} |u|^p\,ds \le \lambda \int_\Omega |u|^p\,. \tag{26}$$

If we have equality in (26) then $u \in M$ and the proof is complete since by (23) $J(u) = m$. In what follows we show that the strict inequality

$$\int_\Omega |\nabla u|^p\,dx + \int_\Omega |\nabla u|^q\,dx + \alpha \int_{\partial\Omega} |u|^p\,ds < \lambda \int_\Omega |u|^p\,. \tag{27}$$

is impossible. Let us assume by contradiction that (27) holds true. Then we can find a $t > 0$ such that $tu \in M$. This t is given by (19) where $w = u$. Note that $\int_\Omega |\nabla u|^q\,dx \ne 0$ because otherwise u is a constant function which contradicts the fact that $u \in S \setminus \{0\}$. By (27) we have $t \in (0,1)$. Since $tu \in M$ we can write

$$J(tu) = \frac{p-q}{pq} \int_\Omega |\nabla (tu)|^q\,dx = \frac{(p-q)t^q}{pq} \int_\Omega |\nabla u|^q\,dx\,.$$

We also have

$$J(u_n) = \frac{p-q}{pq} \int_\Omega |\nabla u_n|^q\,dx \;\Rightarrow\; m = \lim_{n\to\infty} J(u_n) \ge \frac{p-q}{pq} \int_\Omega |\nabla u|^q\,dx\,.$$

Therefore,

$$\begin{aligned} 0 < m \le J(tu) &= \frac{(p-q)t^q}{pq} \int_\Omega |\nabla u|^q \, dx \\ &\le t^q \lim_{n\to\infty} J(u_n) \\ &= t^q m \\ &< m\,, \end{aligned}$$

which is impossible, hence inequality (27) is so, as claimed. So $u \in M$ and $J(u) = m$.

Claim (f): *if* $u \in M$ *is the minimizer found before, i.e.,* $J(u) = m = \inf_M J$ *then* $J'(u) = 0$.

We will first prove that in fact u minimizes the functional J on the larger set

$$N := \{\, v \in W^{1,p}(\Omega) \setminus \{0\};\ \langle J'(v), v\rangle = 0\,\}.$$

Obviously, M is a proper subset of N.

Take an arbitrary element $v \in N$. Denote again

$$\gamma(v) := \alpha \int_{\partial\Omega} |v|^{p-2} v ds - \lambda \int_\Omega |v|^{p-2} v dx.$$

Obviously, if $\gamma(v) = 0$, then $v \in M$ and $J(v) \ge J(u)$. So, we have to investigate the case when $\gamma(v) \ne 0$.

Let us first consider the case when $\gamma(v)$ is positive. Therefore, there exists $\xi^* < 0$ such that $\gamma(v + \xi^*) = 0$, i.e., $v + \xi^* \in S$ since $\gamma(v+\xi) \to -\infty$ as $\xi \to -\infty$ (see (17)). In addition, let ξ^* be the greatest possible, i.e.,

$$\xi^* := \sup\{\xi < 0 : \gamma(v + \xi^*) = 0\} < 0\,.$$

Then $\gamma(v+\xi) > 0$ for any $\xi \in (\xi^*, 0]$, which means that the function

$$\xi \mapsto \alpha \int_{\partial\Omega} |v+\xi|^p ds - \lambda \int_\Omega |v+\xi|^p dx$$

is strictly increasing with respect to ξ on the interval $[\xi^*, 0]$. In particular,

$$\alpha \int_{\partial\Omega} |v+\xi^*|^p ds - \lambda \int_\Omega |v+\xi^*|^p dx < \alpha \int_{\partial\Omega} |v|^p ds - \lambda \int_\Omega |v|^p dx. \tag{28}$$

Let t be defined as in (19) with $w = v + \xi^*$, i.e.,

$$t := \left(\frac{\int_\Omega |\nabla v|^q \, dx}{\lambda \int_\Omega |v+\xi^*|^p \, dx - \alpha \int_{\partial\Omega} |v+\xi^*|^p \, ds - \int_\Omega |\nabla v|^p \, dx} \right)^{\frac{1}{p-q}} < 1. \tag{29}$$

The last inequality is a consequence of (28) and the fact that $v \in N$, i.e.,

$$\begin{aligned}\int_\Omega |\nabla v|^q\,dx &= \lambda \int_\Omega |v|^p\,dx - \alpha \int_{\partial\Omega} |v|^p\,ds - \int_\Omega |\nabla v|^p\,dx \\ &< \lambda \int_\Omega |v+\xi^*|^p\,dx - \alpha \int_{\partial\Omega} |v+\xi^*|^p\,ds - \int_\Omega |\nabla v|^p\,dx.\end{aligned}$$

Hence, $t(v+\xi^*) \in M$ and then

$$\begin{aligned}J(v) = \frac{p-q}{pq}\int_\Omega |\nabla v|^q\,dx &> \frac{p-q}{pq}t^q \int_\Omega |\nabla(v+\xi^*)|^q\,dx \\ &= J(t(v+\xi^*)) \geq J(u).\end{aligned} \tag{30}$$

Similarly, (17) implies that if $v \in N$ and $\gamma(v) < 0$, then we can choose $\xi^* := \inf\{\xi > 0 : \gamma(v+\xi^*) = 0\} > 0$ and as before, we again deduce the inequality (30). Hence, u is a minimizer of the functional J on the set N.

In fact u is a solution of the minimization problem, denoted (P),

$$\min_{v \in W^{1,p}(\Omega)\setminus\{0\}} J(v)\,,$$

under the following constraint

$$h(v) := \int_\Omega |\nabla v|^p\,dx + \int_\Omega |\nabla v|^q\,dx + \alpha \int_{\partial\Omega} |v|^p\,ds - \lambda \int_\Omega |v|^p\,dx = 0\,. \tag{31}$$

For such a problem we can use the well known Lagrange multiplier rule (see [15, Theorem 3.3.3, p.179] or [11, Theorem 2.2.10, p. 76]):

Lemma 1. *Let X, Y be real Banach spaces and let $f : D \to \mathbb{R}$, $h : D \to Y$ be C^1 functions on the open set $D \subset X$. If y is a local solution of the minimization problem*

$$\min f(x), \qquad h(x) = 0\,,$$

and $h'(y)$ is a surjective operator, then there exists $y^* \in Y^\star$ such that

$$f'(y) + y^* \circ h'(y) = 0\,, \tag{32}$$

where $Y^\star$ stands for the dual of Y.

We choose $X = W^{1,p}(\Omega)$, $Y = \mathbb{R}^2$, $D = W^{1,p}(\Omega) \setminus \{0\}$, $f = J$, and let h be the function defined by (31). Obviously, $Y^\star$ can be identified with

$Y = \mathbb{R}$. Note that all the conditions of Lemma 1 are satisfied in our case, including the surjectivity of $h'(u)$ which means

$$\forall \xi_1 \in \mathbb{R},\ \exists w \in W^{1,p}(\Omega) \ \text{ such that } \langle h'(u), w\rangle = \xi_1,$$

i.e.,

$$p\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w\,dx + q\int_\Omega |\nabla u|^{q-2}\nabla u\cdot\nabla w\,dx \\ + p\alpha\int_{\partial\Omega}|u|^{p-2}uw\,ds - p\lambda\int_\Omega |u|^{p-2}uw\,dx = \xi_1\,. \quad (33)$$

We try to determine w of the form $w = au$, where $a \in \mathbb{R}$. Replacing this w in (33) and having in mind that $u \in N$, we obtain

$$a\underbrace{\left[p\int_\Omega |\nabla u|^p\,dx + q\int_\Omega |\nabla u|^q\,dx + p\alpha\int_{\partial\Omega}|u|^p\,ds - p\lambda\int_\Omega |u|^p\,dx\right]}_{=(q-p)\int_\Omega |\nabla u|^q\,dx} = \xi_1,$$

i.e.,

$$a(q-p)\underbrace{\int_\Omega |\nabla u|^q\,dx}_{\neq 0} = \xi_1\,.$$

So a can be uniquely determined, and this shows that $h'(u)$ is indeed surjective. Therefore Lemma 1 is applicable to our minimization problem (P). Specifically, there exist $c \in \mathbb{R}$ such that (see (32))

$$\langle J'(u), \phi\rangle + c\langle h'(u), \phi\rangle = 0\,, \quad \forall\, \phi \in W^{1,p}(\Omega). \quad (34)$$

Testing with $\phi = u$ in (34) we derive (having in mind that $u \in M$)

$$c(q-p)\underbrace{\int_\Omega |\nabla u|^q\;dx}_{\neq 0} = 0\,,$$

which implies $c = 0$.

Consequently,

$$\langle J'(u), \phi\rangle = 0\,,$$

for all $\phi \in W^{1,p}(\Omega)$, i.e., $J'(u) = 0$, as claimed.

Therefore λ is an eigenvalue of problem (1).

3 Final Comments

1. In fact all our results hold true for any bounded open set $\emptyset \neq \Omega \subset \mathbb{R}^N$, $N \geq 2$, which is of class C^1. For the definition of this class see [3, p. 272].

2. We ask ourselves whether there are any eigenvalues of problem (1) in the interval $[\lambda^*, +\infty)$, where λ^* is the constant defined by (4). This is an open problem. Probably the first step in its investigation would be to find necessary and sufficient conditions under which the set M (see the proof of Theorem 3) is nonempty since any eigenfunction belongs to it. There is an evidence that when $\lambda^* < \lambda < \lambda_1^N$ and $p = 2$ the set M is empty. Here λ_1^N denotes the first positive eigenvalue of the classical Laplace operator under the Neumann boundary condition. Indeed, let $\lambda^* < \lambda < \lambda_1^N$ and u be such that $\lambda \int_\Omega u dx = \alpha \int_{\partial\Omega} u ds$. According to the variational characterization of λ_1^N we have

$$\lambda_1^N = \inf \left\{ \int_\Omega |\nabla v|^2 \, dx : \int_\Omega v^2 dx = 1, \int_\Omega v dx = 0 \right\},$$

hence

$$\int_\Omega |\nabla u|^2 \, dx \geq \lambda_1^N \int_\Omega \left(u - \frac{\int_\Omega u dx}{m_N(\Omega)} \right)^2 dx = \lambda_1^N \left(\int_\Omega u^2 dx - \frac{\left(\int_\Omega u dx\right)^2}{m_N(\Omega)} \right).$$

On the other hand,

$$\begin{aligned}
&\lambda_1^N \left(\int_\Omega u^2 dx - \frac{\left(\int_\Omega u dx\right)^2}{m_N(\Omega)} \right) - \lambda \int_\Omega u^2 dx + \alpha \int_{\partial\Omega} u^2 dx \\
&= \left(\lambda_1^N - \lambda\right) \underbrace{\left(\int_\Omega u^2 dx - \frac{\left(\int_\Omega u dx\right)^2}{m_N(\Omega)} \right)}_{\geq 0} + \alpha \int_{\partial\Omega} u^2 dx - \lambda \frac{\left(\int_\Omega u dx\right)^2}{m_N(\Omega)} \\
&\geq \alpha \int_{\partial\Omega} u^2 dx - \frac{\alpha^2}{\lambda} \frac{\left(\int_{\partial\Omega} u dx\right)^2}{m_N(\Omega)} = \alpha \left(\int_{\partial\Omega} u^2 dx - \frac{\lambda^*}{\lambda} \frac{\left(\int_{\partial\Omega} u dx\right)^2}{m_{N-1}(\partial\Omega)} \right) \\
&\geq 0,
\end{aligned}$$

where we have used the Cauchy–Schwarz inequality. Then

$$\int_\Omega |\nabla u|^2 \, dx - \lambda \int_\Omega u^2 dx + \alpha \int_{\partial\Omega} u^2 dx \geq 0,$$

i.e.,

$$\int_\Omega |\nabla u|^q \, dx + \int_\Omega |\nabla u|^2 \, dx - \lambda \int_\Omega u^2 dx + \alpha \int_{\partial\Omega} u^2 dx \geq \int_\Omega |\nabla u|^q \, dx > 0,$$

since u is a non-constant function. Thus the set M is empty when $\lambda \in (\lambda^*, \lambda_1^N)$. So, generally speaking, it seems $\lambda = \lambda^*$ is a bifurcation point.

3. Combinations of different conditions on $\partial\Omega$ (Dirichlet, Neumann and Robin) can also be investigated by using a similar method.

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