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A DUALITY APPROXIMATION OF SOME NONLINEAR PDE's*

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Abstract

We discuss a discretization approach for the p - Laplacian equation and a variational inequality associated to fourth order elliptic operators, via a meshless approach based on duality theory.

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1 Introduction

We consider the well known p - Laplacian boundary value problem, expressed in the variational form (minimization of energy) via the Dirichlet principle:

$$\underset{y\in W_0^{1,p}(\Omega)}{\mathrm{Min}} \left\{ \frac{1}{p}\int_\Omega [|\nabla y|^p + |y|^p]dx - \int_\Omega fydx \right\}, \tag{1}$$

where $\Omega \subset R^d$ is a bounded domain and $p > d \geq 2$ is given, $f \in L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

The space $W_0^{1,p}(\Omega)$ is the usual Sobolev space.

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The existence of a unique weak solution $y \in W_0^{1,p}(\Omega)$ is standard due to the coercivity and strict convexity of the functional (1).

It may be interpreted as solving the nonlinear boundary value problem

$$-\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(|\nabla y|^{p-1} \frac{\partial y}{\partial x_i} \right) + |y|^{p-2} y = f \text{ in } \Omega, \tag{2}$$

$$y = 0 \qquad \text{on } \partial\Omega, \tag{3}$$

in a weak sense in $W_0^{1,p}(\Omega)$.

The basic idea of our approach is to interprete the boundary condition in (2), (3) as a constraint in the minimization problem (1), as it is in fact. These infinitely many pointwise constraints make sense due to the Sobolev embedding theorem $W_0^{1,p}(\Omega) \subset C(\overline{\Omega})$, if $p > d$.

The method that we introduce in this note, discretizes (3) without using a discretization of Ω or $\partial\Omega$, it is a meshless method.

Moreover, we are not applying the (rather complex) bases (like RBF for instance) that are usual in meshless methods [2], [10].

The idea goes back to [6], [11], [9], Ch. 6, where the Kirchhoff - Love model for arches is discussed. Since this model involves just a finite number of boundary conditions, then the explicit solution is obtained [6]. As a comparison, underlying the efficiency of our approach, in [3], [4] finite element discretizations and the locking problem are discussed for the Kirchhoff - Love arches.

In Section 2, we introduce the approximating problem and prove some convergence properties. Its solving is discussed in Section 3. In the recent papers [7], [8] similar ideas were used for the distributed obstacle problem in elliptic equations, including numerical experiments.

In Section 4, we briefly analyze an elliptic variational inequality for fourth order operators, via the same technique.

2 Approximation

Let $\{x_i\}_{i \in N}$ be a sequence of points dense in $\partial\Omega$. No "elements" are used, no uniformity or regularity assumptions are made for the geometry and its approximation.

The approximation of (1) is given by

$$\operatorname*{Min}_{\substack{y \in W^{1,p}(\Omega) \\ y(x_i)=0,\ i=\overline{1,n}}} \left\{ \frac{1}{p} \int_\Omega [|\nabla y|^p + |y|^p] dx - \int_\Omega f y dx \right\}, \tag{4}$$

for any $n \in N$. We denote by (P_n) the problem (4).

Proposition 1. *The problem (P_n) has a unique solution $y_n \in W^{1,p}(\Omega)$. Moreover, we have $y_n \to y$, the solution of (1), strongly in $W^{1,p}(\Omega)$.*

Proof. The existence and uniqueness of $y_n \in W^{1,p}(\Omega)$ is again a consequence of the coercivity and strict convexity of the functional (4). We have the following inequality

$$\frac{1}{p}\int_\Omega [|\nabla y_n|^p dx + |y_n|^p] - \int_\Omega f y_n dx \leq \frac{1}{p}\int_\Omega [|\nabla y|^p dx + |y|^p] - \int_\Omega f y dx, \quad (5)$$

since y is admissible for (P_n), any $n \in N$.

In particular, (5) shows that $\{y_n\}$ is bounded in $W^{1,p}(\Omega)$ due to the coercivity of (4). On a subsequence, denoted again by n, we may assume $y_n \to \widetilde{y}$ weakly in $W^{1,p}(\Omega)$.

Since the functional (4) is weakly lower semicontinuous, using (5) as well, we infer

$$\frac{1}{p}\int_\Omega [|\nabla \widetilde{y}|^p dx + |\widetilde{y}|^p] - \int_\Omega f \widetilde{y} dx \leq \frac{1}{p}\int_\Omega [|\nabla y|^p dx + |y|^p] - \int_\Omega f y dx. \quad (6)$$

Notice that, for any $i \in N$, we have $\widetilde{y}(x_i) = 0$. This is satisfied by y_n, $n > i$ and remains valid by passing to the limit in $W^{1,p}(\Omega)$ weak (and implicitly in $C(\overline{\Omega})$).

As $\widetilde{y} \in C(\overline{\Omega})$ and $\{x_i\}$ is dense in $\partial\Omega$, it yields $\widetilde{y} \in W_0^{1,p}(\Omega)$. Then, by (6) and the uniqueness of y, we obtain $\widetilde{y} = y$.

The inequalities (5), (6) also show that

$$\frac{1}{p}\int_\Omega [|\nabla y_n|^p dx + |y_n|^p] - \int_\Omega f y_n dx \to \frac{1}{p}\int_\Omega [|\nabla y|^p dx + |y|^p] - \int_\Omega f y dx, \quad (7)$$

for $n \to \infty$. Since clearly $\int_\Omega f y_n \to \int_\Omega f y$ due to the weak convergence of $\{y_n\}$, we obtain from (7) the convergence of the norms of the solutions and the strong convergence of y_n, via a strong convergence criterion in uniformly convex Banach spaces.

The convergence is valid on the initial sequence since the limit is unique.

This ends the proof.

□

Remark 1. *This approximation argument can be easily extended to many elliptic boundary value problems, linear or nonlinear. The key point is to ensure the continuity of the solutions. To circumvent this argument, by supplementary approximation techniques or other procedures, would be of interest.*

3 The duality argument

We investigate now the solution of (4). It has a finite number of constraints and we shall compute the dual problem, using the Fenchel theory. The dual problem is finite dimensional and easier to solve.

Define $g_n : W^{1,p}(\Omega) \to]-\infty, +\infty]$ as

$$g_n(y) = \begin{cases} 0 & y(x_i) = 0, \; i = \overline{1, n}, \\ +\infty & \text{otherwise.} \end{cases}$$

And denote by $h : W^{1,p}(\Omega) \to R$ the continuous, concave mapping

$$h(y) = -\frac{1}{p} \int_\Omega \left[|\nabla y|^p + |y|^p \right] dx + \int_\Omega f y dx. \tag{8}$$

Clearly, the problem (P_n) can be expressed as

$$\min_{y \in W^{1,p}(\Omega)} \{ g_n(y) - h(y) \}, \tag{9}$$

via (8), (8). Since g_n is convex, lower semicontinuous and finite in certain points of $W^{1,p}(\Omega)$, the Fenchel theorem [1], allows to rewrite (9) as

$$\max_{z \in W^{1,p}(\Omega)^*} \{ h^*(z) - g_n^*(z) \}, \tag{10}$$

where h^*, g_n^* denote concave/convex conjugates defined on the dual space $W^{1,p}(\Omega)^*$.

We compute now the conjugates from (10).

Lemma 1. *We have*

$$g_n^*(z) = \begin{cases} 0, & z = \sum_{i=1}^{n} \alpha_i \delta_{x_i}, \alpha_i \in R, \\ +\infty, & \text{otherwise,} \end{cases} \tag{11}$$

where δ_{x_i} is the Dirac measure concentrated in x_i.

Proof.

$$\begin{aligned} g_n^*(z) &= \sup_{w\in W^{1,p}(\Omega)} \left\{(z,w)_{W^{1,p}(\Omega)^*\times W^{1,p}(\Omega)} - g_n(w)\right\} = \\ &= \sup_{\substack{w\in W^{1,p}(\Omega)\\ w(x_i)=0,\ i=\overline{1,n}}} \left\{(z,w)_{W^{1,p}(\Omega)^*\times W^{1,p}(\Omega)}\right\} \geq 0. \end{aligned} \tag{12}$$

If in $\widehat{z} \in W^{1,p}(\Omega)^*$, we have $g_n^*(\widehat{z}) > 0$, then we may change in (12) $w \to \lambda w$ with $\lambda \to \infty$ and infer that $g_n^*(\widehat{z}) = +\infty$. Consequently, g_n^* takes either the value 0 or $+\infty$.

If δ_{x_i} is the Dirac measure concentrated in x_i, then $\delta_{x_i} \in W^{1,p}(\Omega)^*$ since $p > d$ and $W^{1,p}(\Omega) \subset C(\overline{\Omega})$.

By (12), we have $g_n^*(\delta_{x_i}) = 0$, for $i = \overline{1,n}$.

This remains valid for any $z^* = \sum\limits_{i=1}^{n} \alpha_i \delta_{x_i}$ with $\alpha_i \in R$ arbitrary scalars. Therefore, g_n^* is null on this finite dimensional subspace in $W^{1,p}(\Omega)^*$.

Notice that:

$$\mathrm{dom}\ g_n^* = \left\{z \in W^{1,p}(\Omega)^*;\ (z,w)_{W^{1,p}(\Omega)^*\times W^{1,p}(\Omega)} = 0,\ \forall\ w \in A_n\right\}, \tag{13}$$

where $A_n = \left\{y \in W^{1,p}(\Omega);\ y(x_i) = 0,\ i = \overline{1,n}\right\}$ is a closed linear subspace in $W^{1,p}(\Omega)$.

Denote $B_n = \left\{\sum\limits_{i=1}^{n} \alpha_i \delta_{x_i},\ \alpha_i \in R\right\} \subset W^{1,p}(\Omega)^*$, a linear closed subspace. The polar space of B_n is

$$\begin{aligned} B_n^\circ &= \left\{y \in W^{1,p}(\Omega);\ (y,z)_{W^{1,p}(\Omega)\times W^{1,p}(\Omega)^*} = 0;\ \forall\ z \in B_n\right\} = \\ &= \left\{y \in W^{1,p}(\Omega);\ y(x_i) = 0;\ i = \overline{1,n}\right\} = A_n. \end{aligned}$$

Since by (13), dom g_n^* is the polar of A_n, the bipolar theorem [1], p. 88, shows that dom $g_n^* = B_n$. This ends the proof. □

Proposition 2. *The dual problem (10) is given by:*

$$\min\left\{\frac{1}{q}|f - z|^q_{W^{1,p}(\Omega)^*};\ z = \sum_{i=1}^{n} \alpha_i \delta_{x_i},\ \alpha_i \in R\right\}. \tag{14}$$

It is a finite dimensional optimization problem.

Proof.

$$h^*(z) = \inf_{y \in W^{1,p}(\Omega)} \left\{ (y, z)_{W^{1,p}(\Omega) \times W^{1,p}(\Omega)^*} - h(y) \right\} =$$

$$= \inf_{y \in W^{1,p}(\Omega)} \left\{ (y, z - f)_{W^{1,p}(\Omega) \times W^{1,p}(\Omega)^*} + \frac{1}{p} \int_\Omega \left[|\nabla y|^p + |y|^p \right] \right\} \geq$$

$$\geq -\frac{1}{q} |z - f|^q_{W^{1,p}(\Omega)^*},$$

due to the elementary inequality $ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q$, with equality for $a = b^{\frac{1}{p-1}}$.

If we choose $y \in W^{1,p}(\Omega)$, $y = -J(z - f)$, then we get equality in (15) and, consequently, $h^*(z) = -\frac{1}{q} |z - f|^q_{W^{1,p}(\Omega)^*}$.

Above: $J : W^{1,p}(\Omega)^* \to W^{1,p}(\Omega)$ is the duality mapping with weight $e(\alpha) = \alpha^{\frac{1}{p-1}}$, $\alpha \in R$. That is, it satisfies (see [5], p. 38):

$$(J(z), z)_{W^{1,p}(\Omega) \times W^{1,p}(\Omega)^*} = |J(z)|_{W^{1,p}(\Omega)} |z|_{W^{1,p}(\Omega)^*},$$

$$|J(z)|_{W^{1,p}(\Omega)} = e(|z|_{W^{1,p}(\Omega)^*}).$$

Here, we also use that $W^{1,p}(\Omega)$ is a reflexive Banach space and J has nonvoid values on $W^{1,p}(\Omega)^*$.

Together with Lemma 1, this ends the proof. □

Remark 2. *If we denote by z_n the unique solution of the dual problem (14), then the solution y_n of the approximating problem (9) may be obtained from the relation $z_n - f \in \partial h(y_n)$, where ∂h is the subdifferential of (8). See [1], p. 188.*

Since z_n is a linear combination of Dirac measures, the solution of the above relation is much simplified. If $p = 2$, it becomes linear and one can also get continuity in space by taking in problem (2) the right hand side f in $L^s(\Omega)$, s big enough, depending on dimension.

Remark 3. *A formal interpretation of the approximating problem (4) is as a mixed boundary value problem: pointwise Dirichlet conditions in x_i, $i = \overline{1, n}$ and Neumann conditions in the remaining of $\partial\Omega$.*

4 A fourth order variational inequality

We formulate the variational problem

$$\operatorname*{Min}_{y\in K}\left\{\frac{1}{2}\int_\Omega |\Delta y|^2 dx - \int_\Omega hy dx\right\}, \tag{15}$$

$$K = \left\{z \in H^2(\Omega) \cap H_0^1(\Omega),\ \int_\Omega hz dx \geq -1\right\}, \tag{16}$$

where $h \in L^2(\Omega)$ is given and Ω is a bounded domain.

The problem (15), (16) can be interpreted as a simply supported plate and the unilateral condition in (16) is related to the mechanical work associated to the force h. It has a unique solution in K due to the coercivity and strict convexity of the functional (15).

We approximate (15), (16) by an optimization problem with a finite number of constraints:

$$\operatorname*{Inf}_{y\in K_n}\left\{\frac{1}{2}\int_\Omega |\Delta y|^2 dx - \int_\Omega hy dx\right\}, \tag{17}$$

$$K_n = \left\{z \in H^2(\Omega);\ z(x_i) = 0,\ i = \overline{1,n},\ \int_\Omega hz dx \geq -1\right\}, \tag{18}$$

where $\{x_i\}_{i\in N} \subset \partial\Omega$ is a dense subset as in the previous sections and K_n is well defined if z is continuous, for instance if $\Omega \subset R^3$, by the Sobolev theorem.

Notice that the existence question in (17), (18) is not easy to be settled, due to the possible lack of coercivity. Under conditions in (18), the seminorm in (17) may not be a norm.

However, the dual approximating problem (which is finite dimensional) has solutions and the Fenchel theorem may be applied. We fix:

$$g : L^2(\Omega) \to R,\ g(z) = -\frac{1}{2}\int_\Omega |z|^2 dx, \tag{19}$$

which is a concave continuous mapping,

$$D : H^2(\Omega) \to L^2(\Omega),\ Dy = \Delta y, \tag{20}$$

which is a linear continuous operator,

$$f_n : H^2(\Omega) \to]-\infty, +\infty], \tag{21}$$

$$f_n(z) = \begin{cases} -\int_\Omega hz & ,\ z \in K_n, \\ +\infty & \text{otherwise}, \end{cases}$$

a convex lower semicontinuous proper function.

By (19) - (21), the problem (17), (18) may be rewritten as

$$\operatorname*{Inf}_{z\in H^2(\Omega)} \{f_n(z) - g(Dz)\} \tag{22}$$

and the hypotheses of the Fenchel theorem are fulfilled (for $z = 0$, for instance).

The conjugate $g^*(z) = -\dfrac{1}{2}\displaystyle\int_\Omega |z|^2 dx$ is clear and, for f_n^*, we state

Lemma 2. *We have*

$$f_n^*(y) = \begin{cases} 0 & , \ y \in K_n^0 + h, \\ +\infty & \quad \text{otherwise}, \end{cases}$$

where $K_n^0 \subset H^2(\Omega)$ is the polar of the convex K_n.

Proof.

$$f_n^*(y) = \sup\left\{\langle z, y + h\rangle_{H^2(\Omega)\times H^2(\Omega)^*};\ z \in K_n\right\} \geq 0. \tag{23}$$

If y has the form $y = -h + \sum_{i=1}^{n} \alpha_i \delta_{x_i}$, $\alpha_i \in R$ and δ_{x_i} are Dirac measures from $H^2(\Omega)^*$ concentrated in x_i, then $f_n^*(y) = 0$.

Fix $z_0 \in K_n$, such that $\int_\Omega z_0 h dx = -1$, which clearly exists. Then, $K_n - z_0$ is a cone and the unilateral condition can be equivalently expressed:

$$\int_\Omega h(z - z_0)dx \geq 0.$$

If $y \in H^2(\Omega)^*$ is such that $f_n^*(y) > 0$, we take $z_0 + \lambda(z - z_0) \in K_n$, $\lambda > 0$ and compute

$$\begin{aligned} \langle z_0 + \lambda(z - z_0), y + h\rangle_{H^2(\Omega)\times H^2(\Omega)^*} &= -1 + \langle z_0, y\rangle_{H^2(\Omega)\times H^2(\Omega)^*} + \\ &+ \lambda\langle z - z_0, y + h\rangle \to \infty \quad \text{as} \quad \lambda \to \infty. \end{aligned} \tag{24}$$

The value $-\infty$ is impossible in (24), due to (23).

It yields that $f_n^*(y)$ takes either 0, or $+\infty$ values, due to (23), (24).

We also remark

$$f_n^*(y - h) = \sup\left\{\langle z, y\rangle_{H^2(\Omega)\times H^2(\Omega)^*};\ z \in K_n\right\} \leq 1$$

for $y \in K_n^0$, i.e. $f_n^*(y-h) = 0$ for $y - h \in K_n^0$.

Outside $K_n^0 + h$, we get greater than 1 values, i.e. $f_n^*(y-h) = +\infty$, $y \notin K_n^0$.

This ends the proof. $\square$

Remark 4. *Denote by* $A_n = \left\{ y = -h + \sum_{i=1}^{n} \alpha_i \delta_{x_i},\ \forall\, \alpha_i \in R \right\} \subset H^2(\Omega)^*$. *Then*

$$A_n^0 = \left\{ x \in H^2(\Omega);\ \langle x, z \rangle_{H^2(\Omega)\times H^2(\Omega)^*} \leq 1,\ \forall\, z \in A_n \right\} \subset$$
$$\subset \left\{ x \in H^2(\Omega);\ y(x_i) = 0,\ i = \overline{1,n} \right\},$$

since, otherwise, we may take $\alpha_i \to \pm\infty$ *conveniently, and obtain a contradiction. Using again the definition of the dual and this property, we get* $A_n^0 = \left\{ y \in H^2(\Omega);\ \langle y, -h \rangle_{H^2(\Omega)\times H^2(\Omega)^*} \leq 1,\ y(x_i) = 0,\ i = \overline{1,n} \right\} = K_n$.

It yields that $K_n^0 = A_n^{00} = \overline{\text{conv}}\,(\{0\} \cup A_n)$, by the bipolar theorem, [1]. This further clarifies Lemma 2. By Fenchel Theorem, we infer:

Proposition 3. *We have*

$$\operatorname*{Inf}_{z \in H^2(\Omega)} \{ f_n(z) - g(Dz) \} = \operatorname*{Max}_{y \in L^2(\Omega)} \{ g^*(y) - f_n^*(D^* y) \},$$

where $D^* : L^2 \to H^2(\Omega)^*$ *is the adjoint operator of* D.

Remark 5. *The dual problem reduces to the maximization of* g^* *under constraint* $D^* y \in K_n^0 + h$, *a finite dimensional optimization problem.*

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