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The Integral Version of Popoviciu's Inequality on the Real Line*

Constantin P. Niculescu† Gabriel T. Prăjitură‡

Dedicated to the memory of Tiberiu Popoviciu, on the occasion of his 110th birthday anniversary

Abstract

T. Popoviciu has proved in 1965 an interesting characterization of the convex functions of one real variable, relating the arithmetic mean of its values and the values taken at the barycenters of certain subfamilies of the given family of points. The aim of our paper is to prove an integral analogue in the framework of absolutely continuous probability measures on the real line.

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1 Introduction

Fifty one years ago Tiberiu Popoviciu [11] published a striking result concerning the averaging properties of convex functions. Its essence is as follows:

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†cpniculescu@gmail.com Academy of Romanian Scientists, Bucharest 050094, Romania;

‡State University of New York, College at Brockport, Brockport NY 14420, USA, gprajitu@brockport.edu

Theorem 1. *If f is a convex function on an interval I, then*

$$\frac{f(x)+f(y)+f(z)}{3}+f\left(\frac{x+y+z}{3}\right) \tag{P}$$
$$\geq \frac{2}{3}\left(f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right)$$

for all $x, y, z \in I$. The inequality is strict when f is strictly convex and the points x, y, z are different from each other.

The book of Niculescu and Persson [6] offers three different proofs; see Theorem 1.1.8, p. 12, Remark 1.5.5, p. 33 and the discussion after Theorem 1.5.7 at p. 35. Many useful comments can be found in the monographs of Mitrinović [4] and Pečarić, Proschan and Tong [9]. Some refinements of (P) appeared in [7], while in [1] is outlined a higher dimensional analogue of Popoviciu's inequality.

A Riemann integral analogue of Theorem 1 concerning convex functions defined on compact intervals is presented in [5]. An important step in deriving that analogue is the following remark:

Lemma 1. *The inequality*

$$\frac{1}{b-a}\int_a^b f(x)dx + f\left(\frac{a+b}{2}\right) \geq \frac{4}{(b-a)^2}\int_a^b\int_a^x f\left(\frac{x+t}{2}\right)dtdx \tag{R-IP}$$

holds for all convex functions $f : [a, b] \to \mathbb{R}$ whose restrictions to the interval $\left[\frac{5a+3b}{8}, \frac{3a+5b}{8}\right]$ are affine functions.

Lemma 1 warns that the Riemann integral analogue of Popoviciu's inequality is not just $(R\text{-}IP)$ (as it might appear at a first glance). Some extra terms should be added to work for all convex functions. See [5] for details.

The aim of this paper is to discuss the continuous analogue of (P) within the framework of absolutely continuous probability measures $\mu = \varphi dx$ on $\mathbb{R}$ having the property that

$$\int_{\mathbb{R}} |x|\, \varphi(x)dx < \infty. \tag{B}$$

The condition (B) assures that μ has a barycenter, precisely the point

$$b_\mu = \int_{\mathbb{R}} x\varphi(x)d\mu(x).$$

This framework encompasses a large variety of measures. First, the probability measures φdx supported by a compact interval $[a, b]$ (and having the barycenters $\int_a^b x\varphi(x)dx$). Some other interesting examples on noncompact intervals are $(-\log x)\,\chi_{(0,1)}dx$, $(e^{-x})\,\chi_{(0,\infty)}dx$ and $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$; their barycenters are respectively 1/4, 1 and 0.

In the next section we will describe an algorithm leading to an extension of Lemma 1 in this framework (and generating integral analogues of Popoviciu's inequality). This algorithm is illustrated in two special cases, those of the probability measures $\frac{1}{2}(\sin x)\,\chi_{[0,\pi]}dx$ and $(e^{-x})\,\chi_{(0,\infty)}dx$, in Sections 3 and 4 respectively. The paper ends with conclusions and some open problems.

2 The integral extension

We start searching for an extension of Lemma 1.

Let $\mu = \varphi dx$ be a probability measure supported by an interval I that verifies the condition $\int_I |x|\,\varphi(x)dx < \infty$. For which convex functions $f : I \to \mathbb{R}$ belonging to $L^1(\varphi(x)dx)$ does the inequality

$$\int_I f(x)\varphi(x)dx + f\left(\int_I x\varphi(x)dx\right)$$
$$\geq \frac{2}{(\mu\otimes\mu)\left(\{(t,x)\in I^2 : t<x\}\right)}\iint_{\{(t,x)\in I^2:\ t<x\}} f\left(\frac{x+t}{2}\right)\varphi(x)\varphi(t)dxdt$$

hold?

Since

$$(\mu\otimes\mu)\left(\left\{(t,x)\in I^2 : t<x\right\}\right) = \int_I\int_{I\cap(-\infty,x]}\varphi(x)\varphi(t)dtdx$$
$$= \int_I\left[\int_{I\cap(-\infty,x]}\varphi(t)dt\right]\varphi(x)dx$$
$$= \frac{1}{2}\int_I\frac{d}{dx}\left[\int_{I\cap(-\infty,x]}\varphi(t)dt\right]^2 dx = \frac{1}{2}\left[\int_I\varphi(t)dt\right]^2 = \frac{1}{2},$$

the problem raised above is equivalent to the following one:

$$\int_I f(x)\varphi(x)dx + f\left(\int_I x\varphi(x)dx\right)$$
$$\geq 4 \iint_{\{(t,x)\in I^2:\ t<x\}} f\left(\frac{x+t}{2}\right)\varphi(x)\varphi(t)dxdt. \quad (IP)$$

The next Lemma collects some useful remarks simplifying the analysis of the inequality (IP).

Lemma 2. (i) *The set of all convex functions $f \in L^1(\varphi(x)dx)$ which verify the inequality (IP) is a convex cone.*

(ii) *The inequality (IP) works (as an equality) for every affine function.*

(iii) *One can reduce the analysis of (IP) to the case of continuous convex functions, by modifying the values at the finite endpoints (if necessary).*

Proof. The assertions (i) and (iii) are clear. This reduces the proof of (ii) to the case of the identity function, $f(x) = x$. Then,

$$\int_I f(x)\varphi(x)dx + f\left(\int_I x\varphi(x)dx\right) = 2\int_I x\varphi(x)dx.$$

On the other hand, denoting $I_*^x = I \cap (-\infty, x]$, we have

$$\begin{aligned}
4\int_I\int_{I_*^x}&\left(\frac{x+t}{2}\right)\varphi(x)\varphi(t)dtdx = 2\int_I\left[\int_{I_*^x}(x+t)\varphi(t)dt\right]\varphi(x)dx\\
&= 2\int_I\left\{\left[\int_{I_*^x}\varphi(t)dt\right]x\varphi(x) + \varphi(x)\int_{I_*^x}t\varphi(t)dt\right\}dx\\
&= 2\int_I\left\{\left[\int_{I_*^x}\varphi(t)dt\right]\frac{d}{dx}\left[\int_{I_*^x}t\varphi(t)dt\right] + \frac{d}{dx}\left[\int_{I_*^x}\varphi(t)dt\right]\int_{I_*^x}t\varphi(t)dt\right\}dx\\
&= 2\int_I\frac{d}{dx}\left(\left[\int_{I_*^x}\varphi(t)dt\right]\left[\int_{I_*^x}t\varphi(t)dt\right]\right)dx\\
&= 2\left[\int_I\varphi(t)dt\right]\left[\int_I t\varphi(t)dt\right] = 2\int_I t\varphi(t)dt,
\end{aligned}$$

and the proof of (ii) is done. □

Lemma 3 and Lemma 4 below provide a density argument that reduces the proof of inequality (IP) to the case of piecewise linear convex functions.

Lemma 3. *Suppose that the function f verifies the assumptions accompanying (IP). Then for every $\varepsilon > 0$ there exists a piecewise linear convex function f_ε such that*

$$\int_{\mathbb{R}} |f(x) - f_\varepsilon(x)| \, \varphi(x)dx < \varepsilon.$$

Lemma 4. (*T. Popoviciu* [10]; *see also* [6], *p.* 34). *Let $f\colon [a,b] \to \mathbb{R}$ be a piecewise linear convex function. Then f is the sum of an affine function and a linear combination, with positive coefficients, of translates of the positive part function. In other words, f is of the form*

$$f(x) = Ax + B + \sum_{k=1}^{n} \lambda_k \left(x - c_k\right)^+$$

for suitable $A, B, c_1, ..., c_n \in \mathbb{R}$ and suitable nonnegative coefficients $\lambda_1, \ldots, \lambda_n$.

As a consequence of the last three lemmas the proof of inequality (IP) reduces to the case of convex functions of the form

$$f(x) = (x - c)^+,$$

where c is a real parameter. In other words, the critical case is that of inequalities of the form

$$\begin{aligned}
&\int_I (x-c)^+ \varphi(x)dx + \left(\int_I x\varphi(x)dx - c\right)^+ \\
&\geq 4 \iint_{\{(t,x)\in I^2\colon\, t<x\}} \left(\frac{x+t}{2} - c\right)^+ \varphi(x)\varphi(t)dxdt,
\end{aligned}$$

equivalently,

$$\begin{aligned}
&\int_I (x-c)^+ \varphi(x)dx + \left(\int_I x\varphi(x)dx - c\right)^+ \\
&\qquad \geq 2 \int_I \left[\int_{I\cap(-\infty,x]} (t - (2c-x))^+ \varphi(t)dt\right] \varphi(x)dx \\
&\qquad = 2 \int_{I\cap[c,\infty]} \left[\int_{I\cap(2c-x,x]} (t - (2c-x))\, \varphi(t)dt\right] \varphi(x)dx. \quad (IPS)
\end{aligned}$$

The analysis of this simplified form of the inequality (IP) gives in principle the intervals where f must be affine. Unfortunately, at this level of generality the continuation will be too intricate and the reader will not see the forest for the trees. Thus, for the sake of clarity, we will detail in the next two sections the particular cases of the probability measures $\frac{1}{2}(\sin x)\,\chi_{[0,\pi]}dx$ and $(e^{-x})\,\chi_{(0,\infty)}dx$ respectively.

3 The case of $\mu = \frac{1}{2}(\sin x)\,\chi_{[0,\pi]}dx$

The probability measure $\mu = \frac{1}{2}(\sin x)\,\chi_{[0,\pi]}dx$ has the barycenter $b_\mu = \frac{\pi}{2}$ and the inequality (IPS) is equivalent to the fact that

$$E = \frac{1}{2}\int_0^\pi (x-c)^+ \sin x dx + \left(\frac{\pi}{2} - c\right)^+ - \int_0^\pi \left[\int_0^x (t+x-2c)^+ \sin t dt\right] \sin x dx$$

is nonnegative.

If $c \leq 0$, we have

$$\begin{aligned} E &= \frac{1}{2}\int_0^\pi (x-c)\sin x dx + \left(\frac{\pi}{2} - c\right) - \int_0^\pi \left[\int_0^x (t+x-2c)\sin t dt\right] \sin x dx \\ &= 2\left(\frac{\pi}{2} - c\right) - \int_0^\pi (x - 2c + \sin x + 2c\cos x - 2x\cos x)\sin x dx \\ &= \pi - 2c - (2\pi - 4c) = 2c - \pi < 0. \end{aligned}$$

If $c \in [0, \pi/2]$, the expression E becomes

$$\begin{aligned} E &= \frac{1}{2}\int_c^\pi (x-c)\sin x dx + \left(\frac{\pi}{2} - c\right) \\ &- \int_c^{2c} \left[\int_{2c-x}^x (t+x-2c)\sin t dt\right] \sin x dx \\ &- \int_{2c}^\pi \left[\int_0^x (t+x-2c)\sin t dt\right] \sin x dx \\ &= \frac{1}{2}\pi - \frac{1}{2}c - \frac{1}{2}\sin c + \left(\frac{\pi}{2} - c\right) \\ &- \int_c^{2c} (\sin x - \sin(2c-x) + 2c\cos x - 2x\cos x)\sin x dx \\ &- \int_{2c}^\pi (x - 2c + \sin x + 2c\cos x - 2x\cos x)\sin x dx \end{aligned}$$

$$= \pi - \frac{3}{2}c - \frac{1}{2}\sin c + \left(\frac{1}{2}\sin 4c - \frac{1}{4}\sin 2c - \frac{1}{2}c - \frac{1}{2}c\cos 2c - \frac{1}{2}c\cos 4c\right)$$
$$+ \left(\frac{7}{2}c - 2\pi + \sin 2c - \frac{1}{2}\sin 4c + \frac{1}{2}c\cos 4c\right)$$
$$= \frac{3}{2}c - \pi - \frac{1}{2}\sin c + \frac{3}{4}\sin 2c - \frac{1}{2}c\cos 2c.$$

Using elementary calculus one can easily show that $E < 0$. See Fig. 1 for the graph of E in this case.

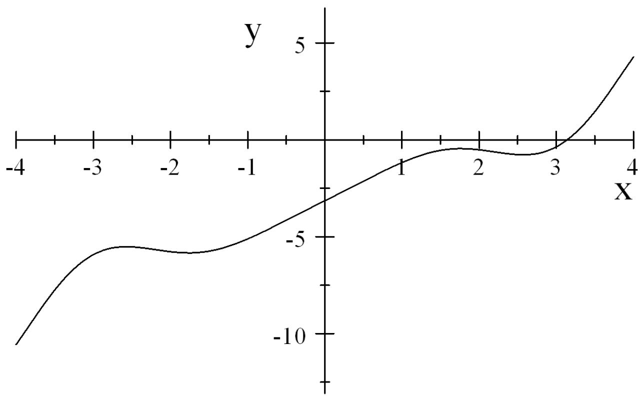

Figure 1: The graph of the function $\frac{3}{2}x - \pi - \frac{1}{2}\sin x + \frac{3}{4}\sin 2x - \frac{1}{2}x\cos 2x$

When $c \in [\pi/2, \pi]$,

$$E = \frac{1}{2}\int_c^\pi (x-c)\sin x dx - \int_c^{2c}\left[\int_{2c-x}^x (t+x-2c)\sin t dt\right]\sin x dx$$
$$= -\int_c^{2c} (\sin x - \sin(2c-x) + 2c\cos x - 2x\cos x)\sin x dx$$
$$= \frac{1}{2}\pi - \frac{1}{2}c - \frac{1}{2}\sin c + \left(\frac{1}{2}\sin 4c - \frac{1}{4}\sin 2c - \frac{1}{2}c - \frac{1}{2}c\cos 2c - \frac{1}{2}c\cos 4c\right)$$
$$= \frac{1}{2}\pi - c - \frac{1}{2}\sin c - \frac{1}{4}\sin 2c + \frac{1}{2}\sin 4c - \frac{1}{2}c\cos 2c - \frac{1}{2}c\cos 4c$$

and $E \geq 0$ if and only if $c \in [u^*, v^*]$, where

$$u^* = 1.782\,848\,790 \ldots \text{ and } v^* = 2.412\ 885\ 603 \ldots$$

are the two roots of the equation

$$\frac{1}{2}\pi - x - \frac{1}{2}\sin x - \frac{1}{4}\sin 2x + \frac{1}{2}\sin 4x - \frac{1}{2}x\cos 2x - \frac{1}{2}x\cos 4x = 0$$

in the interval $[\pi/2, \pi]$. See also Fig. 2.

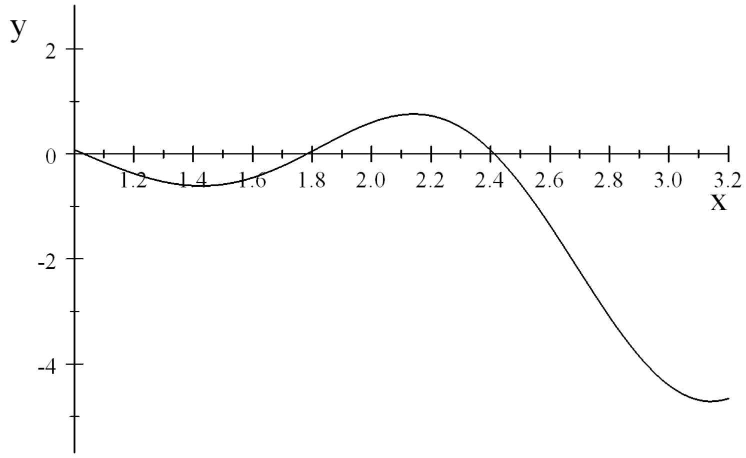

Figure 2: The graph of the function E on the interval $[\pi/2, \pi]$.

Consequently, the analogue of Lemma 1 in the case of the probability measure $\frac{1}{2}(\sin x)\,\chi_{[0,\pi]}dx$ is as follows:

Lemma 5. *In the case of the probability measure* $\mu = \frac{1}{2}(\sin x)\,\chi_{[0,\pi]}dx$, *the inequality*

$$\frac{1}{2}\int_0^{\pi} f(x)\sin x dx + f\left(\frac{\pi}{2}\right) \geq 4\int_0^{\pi}\left[\int_0^{x} f\left(\frac{t+x}{2}\right)\sin t dt\right]\sin x dx$$

works for all continuous convex functions $f : [0,\pi] \to \mathbb{R}$ *whose restrictions to the intervals* $[0, u^*]$ *and* $[v^*, \pi/2]$ *are affine functions.*

Proof. In fact, the above formula works precisely for all convex functions $f : [0,\pi] \to \mathbb{R}$ in the closed convex cone generated by the affine functions and the functions of the form $(x-c)^+$ with $c \in [u^*, v^*]$. See Lemma 2. On the intervals $[0, u^*]$ and $[v^*, \pi/2]$ these functions should be affine since the limit of a pointwise convergent sequence of affine functions is itself affine. $\square$

In the general case of an arbitrary continuous convex function F on $[0,\pi]$, the analogue of Popoviciu's inequality results form Lemma 5, by applying it to the function

$$f(x)=\begin{cases} F\left(0\right)+\frac{F(u^*)-F(0)}{u^*}x & \text{if } x\in[0,u^*] \\ F(x) & \text{if } x\in[u^*,v^*] \\ f\left(v^*\right)+\frac{f(\pi/2)-f(v^*)}{\pi/2-v^*}\left(x-v^*\right) & \text{if } x\in[v^*,\pi/2]. \end{cases}$$

4 The case of $\mu=\left(e^{-x}\right)\chi_{(0,\infty)}dx$

The barycenter of the probability measure $\left(e^{-x}\right)\chi_{(0,\infty)}dx$ is 1 and the inequality (IPS) is equivalent to the fact that

$$E=\int_0^\infty (x-c)^+e^{-x}dx+(1-c)-2\int_0^\infty\left[\int_0^x\left(t+x-2c\right)e^{-t}dt\right]e^{-x}dx$$

is nonnegative.

If $c\leq 0$, the inequality (IPS) fails because

$$\begin{aligned}E&=\int_0^\infty (x-c)e^{-x}dx+(1-c)-2\int_0^\infty\left[\int_0^x\left(t+x-2c\right)e^{-t}dt\right]e^{-x}dx\\&=2-2c-2\int_0^\infty\left[x-e^{-x}-xe^{-x}+2c\left(e^{-x}-1\right)+1\right]e^{-x}dx\\&=2-2c-2\left(\frac{5}{4}-c\right)=-\frac{1}{2}<0.\end{aligned}$$

If $c\geq 0$, the inequality (IPS) is equivalent to

$$\begin{aligned}E&=\int_c^\infty (x-c)e^{-x}dx+(1-c)^+\\&\quad-2\int_c^{2c}\left[\int_{2c-x}^x\left(t+x-2c\right)e^{-t}dt\right]e^{-x}dx\\&\quad-2\int_{2c}^\infty\left[\int_0^x\left(t+x-2c\right)e^{-t}dt\right]e^{-x}dx\\&\quad=e^{-c}+(1-c)^+-2\int_c^{2c}\left[e^{x-2c}-e^{-x}+2ce^{-x}-2xe^{-x}\right]e^{-x}dx\\&\quad-2\int_{2c}^\infty\left[x-2c-e^{-x}+2ce^{-x}-2xe^{-x}+1\right]e^{-x}dx\\&\quad=e^{-c}+(1-c)^+-2e^{-2c}\left(c+e^{-2c}+ce^{-2c}-1\right)\\&\quad+2e^{-2c}\left(e^{-2c}+ce^{-2c}-2\right)\\&\quad=e^{-c}+(1-c)^+-2e^{-2c}\left(c+1\right)\geq 0.\end{aligned}$$

Since

$$E=\begin{cases} e^{-c}+1-c-2e^{-2c}\,(c+1) & \text{if } c\in[0,1]\\ e^{-c}-2e^{-2c}\,(c+1) & \text{if } c\in[1,\infty),\end{cases}$$

we obtain that

$$E\geq 0 \text{ if and only if } c\in[0,c^{*}]\cup[d^{*},\infty),$$

where

$$c^{*}=0.520\,120\,114\,...$$

is the positive solution of the equation $e^{-x}+1-x-2e^{-2x}\,(x+1)=0$ and

$$d^{*}=1.\,678\,346\,990...$$

and $d^{*}=1.\,678\,346\,990...$ is the solution of the equation $e^{-x}-2e^{-2x}\,(x+1)=0.$

Combining this fact with Lemma 2 above we infer the following result:

Lemma 6. *In the case of the probability measure* $\mu=(e^{-x})\,\chi_{(0,\infty)}dx$, *the inequality*

$$\int_{0}^{\infty}f(x)e^{-x}dx+f\,(1)\geq 4\iint_{\{(t,x)\in(0,\infty)^{2}:\ t<x\}}f\left(\frac{x+t}{2}\right)e^{-x-t}dxdt$$

works for all continuous convex functions $f\in L^{1}(\mu)$ *whose restrictions to the interval* $[c^{*},d^{*}]$ *are affine functions.*

Proof. In fact, the above formula works precisely for all convex functions $f:(0,\infty)\rightarrow\mathbb{R}$ in the closed convex cone generated by the affine functions and the functions of the form $(x+c)^{+}$ with $c\in[0,c^{*}]\cup[d^{*},\infty)$. In the interval $[c^{*},d^{*})$ these functions should be affine since the limit of a pointwise convergent sequence of affine functions is itself affine. □

In the general case of an arbitrary continuous convex function F on $[0,\infty)$, the analogue of Popoviciu's inequality results form Lemma 6, by applying it to the convex function

$$f(x)=\begin{cases} F(x) & \text{if } x\in[0,c^{*}]\cup[d^{*},\infty)\\ F\left(c^{*}\right)+\frac{F(d^{*})-F(c^{*})}{d^{*}-c^{*}}\left(x-c^{*}\right) & \text{if } x\in[c^{*},d^{*}],\end{cases}$$

obtained from F by replacing the portion over $[c^{*},d^{*}]$ by the affine function joining the points $(c^{*},F(c^{*}))$ and $(d^{*},F(d^{*}))\,.$

5 Conclusions and some open problems

The two examples detailed above outline a big difference between the discrete Popoviciu's inequality and its continuous analogue. While the discrete case works for **all** convex functions, the continuous case imposes certain restrictions (that depend on the measure under attention).

An interesting phenomenon is the existence of probability measures on $\mathbb{R}$ for which their corresponding Popoviciu's inequalities work only for affine functions. Indeed, if $\mu_1 = \varphi_1 dx$ and $\mu_2 = \varphi_2 dx$ are absolutely continuous probability measures on $\mathbb{R}$ having the property that

$$\int_{\mathbb{R}} |x| \, \varphi_k(x) dx < \infty \quad \text{for } k = 1, 2,$$

then $\mu = (\varphi_1 + \varphi_2) dx$ also admits barycenter and the cone C of convex functions for which the integral Popoviciu's inequality works for μ equals the intersections of the cones C_1 and C_2 corresponding to μ_1 and μ_2 respectively.

We end our paper with some open problems that might be of interest.

The measure $\frac{1}{2} (\sin x) \, \chi_{[0,\pi]} dx$ is symmetric with respect to its barycenter $\pi/2$ but the interval $[u^*, v^*]$ that appears in Lemma 5 is not. How can this fact be explained?

Describe the integral analogue of Popoviciu's inequality in the case of the Gaussian measure $\frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.

Our algorithm seems not practical in this case because it leads to the evaluation of integrals containing the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2/2} dx.$$

Last, but not least, is the problem concerning the role and place of the new Popoviciu type inequalities. How do the various integral analogues of Popoviciu's inequality relate to other known inequalities?

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