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High Dimensional Applications of Implicit Parametrizations in Nonlinear Programming*

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Abstract

Several experiments are reported, related to the implicit parametrization method and its application in optimization, together with comparisons with the existing methodology.

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1 Introduction

In the recent papers [5], [3], [6] a new approach to the solution of general implicit systems has been introduced, based on the application of certain iterated ordinary differential systems. This method can be extended to the critical case via the use of generalized solutions [6]. Some numerical experiments are reported in [3] and [2]. It was noticed that one may apply such techniques to constrained nonlinear programming problems, by solving the equality constraints and reducing the dimension of the minimization problem [6]. In this work, we report on several large scale numerical experiments in constrained optimization and compare them with other approaches

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from the literature, [1], [4] or from MatLab. It is a continuation of [6], in the sense that the provided examples are related to the theoretical developments therein.

2 An example in $\mathbb{R}^6$

Let $f : \mathbb{R}^6 \to \mathbb{R}$ be a continuous function and $F_i : \mathbb{R}^6 \to \mathbb{R}$, $i = \overline{1,3}$ be of class C^1, such that:

$$F_i(x^0) = 0,\ i = \overline{1,3},$$

$x^0 \in \mathbb{R}^6$ given.

Consider the following constrained optimization problem:

$$\min f(x_1, x_2, x_3, x_4, x_5, x_6), \tag{1}$$

with the restrictions:

$$F_i(x_1, x_2, x_3, x_4, x_5, x_6) = 0, \quad i = 1, 2, 3. \tag{2}$$

We eliminate the constraints (2) using the implicit parametrization method from [6].

To solve the problem (1)-(2), we also assume:

$$\frac{D(F_1, F_2, F_3)}{D(x_1, x_2, x_3)} \neq 0, \ \ in\ x^0. \tag{3}$$

Let $A(x)$ be the corresponding 3×3 non-singular submatrix of the Jacobian (3).

We introduce the following linear system:

$$v_j(x) \cdot \nabla F_i(x) = 0,\ i = \overline{1,3},\ \ j = \overline{1,3}, \tag{4}$$

where $v_j(x)$ has the last three components given by the rows of $I_3 \cdot det(A(x))$, I_3 beeing the identity matrix in $\mathbb{R}^3$. Using the three solutions $v_j(x)$, $j = \overline{1,3}$, thus obtained from (4), we define the following three iterated differential systems, in $\mathbb{R}^6$ each:

$$
\begin{aligned}
&\frac{\partial y_1(t_1)}{\partial t_1} = v_1(y_1(t_1)),\\
&y_1(0) = x_0,\\
&\frac{\partial y_2(t_1,t_2)}{\partial t_2} = v_2(y_2(t_1,t_2)),\ t_2 \in I_2(t_1) \subset \mathbb{R},\\
&y_2(t_1,0) = y_1(t_1),\\
&\frac{\partial y_3(t_1,t_2,t_3)}{\partial t_3} = v_3(y_2(t_1,t_2,t_3)),\ t_3 \in I_3(t_1,t_2),\\
&y_3(t_1,t_2,0) = y_2(t_1,t_2).
\end{aligned} \tag{5}
$$

In [6], it is shown that the systems (5) solve the restrictions (2) around x^0.

We discuss now an example from [4].

Example 1. *Let $Z \subset \mathbb{R}^3$ and $P \subset \mathbb{R}^3$. Consider the objective function $f : Z \times P \to \mathbb{R}$:*

$$
\begin{aligned}
f(z,p) &= \sum_{j=1}^{3} \left([a_j(p_j - c_j)]^2 + \sum_{i \neq j} a_j(p_i - c_i) \right. \\
&- \left. 5\left((j-1)(j-2)(z_2 - z_1) + \sum_{i=1}^{3} (-1)^{i+1} z_j \right) \right)^2
\end{aligned} \tag{6}
$$

where a_i, c_i, $i = 1,2,3$ are fixed constants, given in Table 1 and the equality constraints are:

$$
\begin{aligned}
h_1(z,p) &= 10^{-9}\left(e^{38z_1} - 1\right) + p_1 z_1 - 1.6722 z_1 + 0.6689 z_3 - 8.0267 = 0,\\
h_2(z,p) &= 1.98 \cdot 10^{-9}\left(e^{38z_2} - 1\right) + 0.6622 z_1 + p_2 z_2 + 0.6622 z_3 + 4.0535 = 0,\\
h_3(z,p) &= 10^{-9}\left(e^{38z_3} - 1\right) + z_1 - z_2 + p_3 z_3 - 6 = 0.
\end{aligned} \tag{7}
$$

	$i=1$	$i=2$	$i=3$
a_i	$37,3692$	$18,5805$	6.25
c_i	0.602	1.211	3.6

Table 1: constants values

We choose two different initial conditions, that are indicated in [4] as local, respectively global solutions of (6), (7):

$$x_0^1 = [0.56, -3.3158, 0.5115, 0.602, 1.4685, 3.6563],$$

$$x_0^2 = [0.56, -3.3158, 0.5115, 0.7039, 1.4364, 3.6113].$$

To apply the implicit parametrization method, we solve the three differential systems (5) using the routine $ode15s$, for stiff problems, from MatLab.

In both cases, we use the parameter integration intervals: $[-0.2, 0.2]$, with different discretizations: for the first system in (5), we take the step 0.00001 and for the second and the third system 0.001. In this way we obtain a discretization around x^0 of the manifold generated by (7) in $\mathbb{R}^6$.

For the first initial condition, we obtain for the objective function (6) the minimal value $348, 1322$, corresponding to $z = [0.5580, -3.3172, 0.5136]$, $p = [0.4692, 1.4635, 3, 5890]$. The values for the restrictions are $h_1 = -0.2557$, $h_2 = -0.0021$, $h_3 = 0.0026$.

For the second initial condition, the minimum value for f is $384, 8258$. The point where the function touches its minimum is $z = [0.565, -3.3154, 0.5089]$, $p = [0.5047, 1.4365, 3.6777]$ and the constraints have the values: $h_1 = -0.2557$, $h_2 = -0.1083$, $h_3 = 0.0193$. In each experiment the necessary time was about three minutes on a medium size laptop.

The constraints are not very close to zero due to the approximations.

We now make a correction by keeping fixed, in each case, the last three coordinates of the minimum point, and we obtain the following results, by solving (7) in MatLab:

• for the first initial condition:
$f_{min} = 343.7695$ in $[0.5631, -3.3258, 0.5159, 0.4692, 1.4635, 3.5890]$ and the constraints values are $h_1 = 2.9387 \cdot 10^{-39}$, $h_2 = 0$, $h_3 = 0$.

• for the second initial condition:
$f_{min} = 383.7265$ in $[0.5616, -3.3154, 0.5089, 0.5047, 1.4365, 3.6777]$ and the constraints values are $h_1 = 2.9387 \cdot 10^{-39}$, $h_2 = -1.4693 \cdot 10^{-39}$, $h_3 = 0$.

While our solutions improve the numerical results in [4], this is not a contradiction, since the implicit parametrization method extends the search region. Clearly, it is possible, for instance, to further extend the parameter integration interval $[-0.2, 0.2]$, in a very simple way, if needed.

3 High dimensional examples

In this section, we investigate the possibility to apply the implicit parametrization method in complex nonlinear programming problems. We just discuss examples in $\mathbb{R}^{12}$ and $\mathbb{R}^{50}$, for simplicity of writing. Large scale examples, for instance, in dimension 1000, are possible by the same technique. In [1] such examples are discussed for unconstrained problems, with different methods.

Example 2. *In* $\mathbb{R}^{12}$ *we consider the problem:*

$$\min f(x), \tag{8}$$

where $f(x) = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12}$
or $f(x) = x_1^2 + 3x_2 + x_3 + x_4x_5 + x_6 + x_7^3 + x_8 + x_9 - x_{10}x_{11} + x_{12}$,
with ten equality constraints:

$$\begin{aligned}
F_1(x) &= x_1 - x_{11}x_{12}, \\
F_2(x) &= 2x_1^2 - x_2 - x_{11}^2 x_1 2, \\
F_3(x) &= x_1^2 - x_2 - x_3 + x_{11}x_{12}, \\
F_4(x) &= 2x_1 - x_2 - x_3 + x_4 + x_{11}^2 x_{12}, \\
F_5(x) &= x_1^3 + x_2 - x_3 + 3x_4 - x_5 - 3x_{11}x_{12}, \\
F_6(x) &= x_1^3 + x_2^2 + x_3 + x_5 - x_6 - x_{11}^2 - 2x_{12}, \\
F_7(x) &= x_1 - x_2^3 - x_4 + x_5 - x_7 + x_{11}x_{12}, \\
F_8(x) &= x_1 - x_3^3 - 2x_5 + x_7 - x_8 + x_{11}^2 + x_{12}^2, \\
F_9(x) &= -x_2 + x_3^3 - x_4 + 3x_6 - x_7 - x_9 - x_{11} + x_{12}^2, \\
F_{10}(x) &= x_1^3 + x_3^2 - x_4^2 + 2x_6 - x_7 - 3x_8 + x_9 - x_1 0 + x_{11}^3 - x_{12}^2,
\end{aligned} \tag{9}$$

We first choose the initial condition $x^0 = [1,1,1,1,1,1,1,1,1,1,1,1]$ which satisfies

$$F_i(x^0) = 0, \ i = \overline{1,10}$$

and

$$\frac{D(F_1, \cdots, F_{10})}{D(x_1, \cdots, x_{10})}(x^0) \neq 0. \tag{10}$$

Let $A(x)$ be the corresponding 10×10 nonsingular submatrix in (10), of the Jacobian. We solve the linear system:

$$v_j(x) \cdot \nabla F_i(x) = 0, \ i = \overline{1,10}, \ j = 1,2, \tag{11}$$

where the last two components of $v_j(x)$ are the rows of $I_2 \cdot det(A(x))$, I_2 being the identity matrix in $\mathbb{R}^2$, and we obtain $v_j(x) \in \mathbb{R}^{12}, \ j = 1,2$, x in a neighborhood of x^0.

By applying the method from [6], we solve the problem (8)-(9) via the iterated differential systems (12), (13), with right-hand given by $v_j \in \mathbb{R}^{12}$, $j = 1, 2$:

$$
\begin{aligned}
x_1'(t_1) &= -x_{12}(t_1), \\
x_2'(t_1) &= -(4x_1(t_1)x_{12}(t_1) - 2x_{11}(t_1)x_{12}(t_1)); \\
x_3'(t_1) &= -(x_{12}(t_1) - 2x_1(t_1)x_{12}(t_1) + 2x_{11}(t_1)x_{12}(t_1)), \\
x_4'(t_1) &= -(x_{12}(t_1) - 2x_1(t_1)x_{12}(t_1) + 2x_{11}(t_1)x_{12}(t_1)), \\
x_5'(t_1) &= -(3x_{12}(t_1)x_1^2(t_1) - x_{12}(t_1) + 2x_{11}(t_1)x_{12}(t_1)), \\
x_6'(t_1) &= -(4x_{11}(t_1)x_{12}(t_1) - 2x_1(t_1)x_{12}(t_1) - 2x_{11}(t_1) + 6x_1(t_1)^2x_{12}(t_1) \\
&\quad +8x_1(t_1)x_2(t_1)x_{12}(t_1) - 4x_2(t_1)x_{11}(t_1)x_{12}(t_1)) \\
x_7'(t_1) &= -(2x_1(t_1)x_{12}(t_1) - 3x_2^2(t_1)(4x_1(t_1)x_{12}(t_1) - 2x_{11}(t_1)x_{12}(t_1)) + \\
&\quad 3x_1(t_1)^2x_{12}(t_1)), \\
x_8'(t_1) &= -(2x_{11}(t_1) + 3x_{12}(t_1) + 2x_1(t_1)x_{12}(t_1) - 4x_{11}(t_1)x_{12}(t_1) - \\
&\quad 3x_1^2(t_1)x_{12}(t_1) - 3x_3(t_1)^2x_{12}(t_1) - 12x_1(t_1)x_2^2(t_1)x_{12}(t_1) + \\
&\quad 6x_1x_3(t_1)^2x_{12}(t_1) + 6x_2^2(t_1)x_{11}(t_1)x_{12}(t_1) - \\
&\quad 6x_3^2(t_1)x_{11}(t_1)x_{12}(t_1)) \qquad (12) \\
x_9'(t_1) &= -(12x_{11}(t_1)x_{12}(t_1) - x_{12}(t_1) - 10x_1x_{12}(t_1) - 6x_{11}(t_1) + \\
&\quad 15x_1^2(t_1)x_{12}(t_1) + 3x_3^2(t_1)x_{12}(t_1) + 12x_1(t_1)x_2^2(t_1)x_{12}(t_1) - \\
&\quad 6x_1(t_1)x_3^2(t_1)x_{12}(t_1) - 6x_2^2(t_1)x_{11}(t_1)x_{12}(t_1) + \\
&\quad 6x_3^2(t_1)x_{11}(t_1)x_{12}(t_1) + 24x_1(t_1)x_2(t_1)x_{12}(t_1) \\
&\quad -12x_2(t_1)x_{11}(t_1)x_{12}(t_1) - 1) \\
x_{10}'(t_1) &= -(2x_3(t_1)x_{12}(t_1) - 10x_{12}(t_1) - 24x_1(t_1)x_{12}(t_1) - 18x_{11}(t_1) - \\
&\quad 2x_4(t_1)x_{12}(t_1) + 36x_{11}(t_1)x_{12}(t_1) + 42x_1^2(t_1)x_{12}(t_1) + \\
&\quad 12x_3^2(t_1)x_{12}(t_1) + 3y(11)^2(t_1) + 60x_1(t_1)x_2^2(t_1)x_{12}(t_1) - \\
&\quad 24x_1(t_1)x_3^2(t_1)x_{12}(t_1) - 30x_2^2(t_1)x_{11}(t_1)x_{12}(t_1) + \\
&\quad 24x_3^2(t_1)x_{11}(t_1)x_{12}(t_1) + 48x_1(t_1)x_2(t_1)x_{12}(t_1) - \\
&\quad 4x_1(t_1)x_3(t_1)x_{12}(t_1) + 4x_1(t_1)x_4(t_1)x_{12}(t_1) - \\
&\quad 24x_2(t_1)x_{11}(t_1)x_{12}(t_1) + 4x_3(t_1)x_{11}(t_1)x_{12}(t_1) - \\
&\quad 4x_4(t_1)x(11)(t_1)x_{12}(t_1) - 1), \\
x_{11}'(t_1) &= -1, \\
x_{12}'(t_1) &= 0,
\end{aligned}
$$

$$\begin{aligned}
x_1(0) &= 1, x_2(0) = 1, x_3(0) = 1, x_4(0) = 1, x_5(0) = 1, x_6(0) = 1,\\
x_7(0) &= 1, x_8(0) = 1, x_9(0) = 1, x_{10}(0) = 1, x_{11}(0) = 1, x_{12}(0) = 1,
\end{aligned}$$

$$\begin{aligned}
\dot{y}_1(t_1,t_2) &= -y_{11}(t_1,t_2),\\
\dot{y}_2(t_1,t_2) &= -(-y_{11}(t_1,t_2)^2 + 4y_1(t_1,t_2)y_{11}(t_1,t_2))\\
\dot{y}_3(t_1,t_2) &= -(y_{11}(t_1,t_2) - 2y_1(t_1,t_2)y_{11}(t_1,t_2) + y_{11}(t_1,t_2)^2)\\
\dot{y}_4(t_1,t_2) &= -(y_{11}(t_1,t_2) - 2y_1(t_1,t_2)y_{11}(t_1,t_2) + y_{11}^2(t_1,t_2))\\
\dot{y}_5(t_1,t_2) &= -(3y_1^2(t_1,t_2)y_{11}(t_1,t_2) + y_{11}^2(t_1,t_2) - y_{11}(t_1,t_2))\\
\dot{y}_6(t_1,t_2) &= -(6y_1^2(t_1,t_2)y_{11}(t_1,t_2) - 2y_1(t_1,t_2)y_{11}(t_1,t_2) -\\
&\quad 2y_2(t_1,t_2)y_{11}^2(t_1,t_2) + 2y_{11}^2(t_1,t_2) +\\
&\quad 8y_1(t_1,t_2)y_2(t_1,t_2)y_{11}(t_1,t_2) - 2)\\
\dot{y}_7(t_1,t_2) &= -(2y_1(t_1,t_2)y_{11}(t_1,t_2) - 3y_2^2(-y_{11}^2(t_1,t_2) +\\
&\quad 4y_1(t_1,t_2)y_{11}(t_1,t_2)) + 3y_1^2(t_1,t_2)y_{11}(t_1,t_2))\\
\dot{y}_8(t_1,t_2) &= -(3y_{11}(t_1,t_2) + 2y_{12}(t_1,t_2) - 3y_2^2(-y_{11}(t_1,t_2)^2 +\\
&\quad 4y_1(t_1,t_2)y_{11}(t_1,t_2)) - 3y_3^2(y_{11} - 2y_1y_{11} + y_{11}^2) +\\
&\quad 2y_1(t_1,t_2)y_{11}(t_1,t_2) - 3y_1^2(t_1,t_2)y_{11}(t_1,t_2) - 2y_{11}^2(t_1,t_2))\\
\dot{y}_9(t_1,t_2) &= -(15y_1^2(t_1,t_2)y_{11}(t_1,t_2) +\\
&\quad 12y_1(t_1,t_2)y_2^2(t_1,t_2)y_{11}(t_1,t_2) +\\
&\quad 24y_1(t_1,t_2)y_2(t_1,t_2)y_{11}(t_1,t_2) - \qquad (13)\\
&\quad 6y_1(t_1,t_2)y_3^2(t_1,t_2)y_{11}(t_1,t_2) - 10y_1(t_1,t_2)y_{11}(t_1,t_2) -\\
&\quad 3y_2^2(t_1,t_2)y_{11}^2(t_1,t_2) - 6y_2(t_1,t_2)y_{11}^2(t_1,t_2) +\\
&\quad 3y_3^2(t_1,t_2)y_{11}^2(t_1,t_2) + 3y_3^2(t_1,t_2)y_{11}(t_1,t_2) + 6y_{11}^2(t_1,t_2) -\\
&\quad y_{11}(t_1,t_2) + 3y_{12}^2(t_1,t_2) - 6)\\
\dot{y}_{10}(t_1,t_2) &= -(12y_3^2(t_1,t_2)y_{11}^2(t_1,t_2) - 8y_{12}(t_1,t_2) - 15y_2^2(t_1,t_2)y_{11}^2(t_1,t_2) -\\
&\quad 10y_{11}(t_1,t_2) - 24y_1(t_1,t_2)y_{11}(t_1,t_2) + 2y_3(t_1,t_2)y_{11}(t_1,t_2) -\\
&\quad 2y_4(t_1,t_2)y_{11}(t_1,t_2) + 42y_1^2(t_1,t_2)y_{11}(t_1,t_2) -\\
&\quad 12y_2(t_1,t_2)y_{11}^2(t_1,t_2) + 2y_3(t_1,t_2)y_{11}^2(t_1,t_2) +\\
&\quad 12y_3^2(t_1,t_2)y_{11}(t_1,t_2) - 2y_4(t_1,t_2)y_{11}^2(t_1,t_2) + 18y_{11}^2(t_1,t_2) +\\
&\quad 3y_{12}^2(t_1,t_2) + 60y_1(t_1,t_2)y_2^2(t_1,t_2)y_{11}(t_1,t_2) -\\
&\quad 24y_1(t_1,t_2)y_3^2(t_1,t_2)y_{11}(t_1,t_2) +\\
&\quad 48y_1(t_1,t_2)y_2(t_1,t_2)y_{11}(t_1,t_2) -\\
&\quad 4y_1(t_1,t_2)y_3(t_1,t_2)y_{11}(t_1,t_2) +\\
&\quad 4y_1(t_1,t_2)y_4(t_1,t_2)y_{11}(t_1,t_2) - 12)
\end{aligned}$$

$$\dot{y}_{11}(t_1, t_2) = 0$$
$$\dot{y}_{12}(t_1, t_2) = -1$$

where $\dot{y}$ denotes the derivative with respect to t_2,

$$y_1(t_1, 0) = x_1(t_1), y_2(t_1, 0) = x_2(t_1), y_3(t_1, 0) = x_3(t_1), y_4(t_1, 0) = x_4(t_1),$$
$$y_5(t_1, 0) = x_5(t_1), y_6(t_1, 0) = x_6(t_1), y_7(t_1, 0) = x_7(t_1), y_8(t_1, 0) = x_8(t_1),$$
$$y_9(t_1, 0) = x_9(t_1), y_{10}(t_1, 0) = x_{10}(t_1), y_{11}(t_1, 0) = x_{11}(t_1),$$
$$y_{12}(t_1, 0) = x_{12}(t_1).$$

These equations were obtained using the Symbolic Math Toolbox from MatLab in (11). We apply *ode*45 to solve (12), (13). We used a step of 0.5 in the first system and 1 in the second one, with the intervals $[-3, 3]$, respectively $[-2000, 2000]$. The discretization here is very rough, but it happens that the numerical results are very good.

The values obtained for the minimum of the cost function is $-1.5988 \cdot 10^{10}$ and the coordinates of the solution point are $[-2.4967 \cdot 10^{13}, 6.0715 \cdot 10^{18}, -2.4967 \cdot 10^{13}, -2.4967 \cdot 10^{13}, 2.4964 \cdot 10^{13}, 3998, -5.6136 \cdot 10^{-6}, 3.9960 \cdot 10^{6}, -7.9879 \cdot 10^{9}, -8.0039 \cdot 10^{9}, 1.2490 \cdot 10^{-16}, -1999]$. The result was obtained in 4 seconds and the restrictions are satisfied: $F_1 = 0, F_2 = -6.0715 \cdot 10^{-18}, F_3 = -6.0715 \cdot 10^{-18}, F_4 = -6.0715 \cdot 10^{-18}, F_5 = 3.3827 \cdot 10^{-17}, F_6 = 0, F_7 = 5.6136 \cdot 10^{-6}, F_8 = 0, F_9 = -1.9073 \cdot 10^{-6}, F_{10} = 1.9073 \cdot 10^{-6}$.

Working the same minimization problem with the routine *MultiStart* from MatLab, we get (for 400 initial guesses):

$$f_{min} = -7.9092 \cdot 10^6,$$
$$time = 10\ minutes,$$
$$\hat{x} = [-0.0030, -0.5916, 0.5886, 0.5886, 0.5947, -38149.2109, 0.2071,$$
$$38149.555, -114252.3108, -7794834.4490, -195.3221, 1.5509 \cdot 10^{-5}]$$
$$(F_i(\hat{x}))_{i=\overline{1,12}} = [0, -2.2204 \cdot 10^{-16}, -1.4528 \cdot 10^{-16}, -1.1102 \cdot 10^{-16},$$
$$-1.1796 \cdot 10^{-16}, 8.4314 \cdot 10^{-12}, -1.1752 \cdot 10^{-16},$$
$$4.3956 \cdot 10^{-13}, 5.0059 \cdot 10^{-12}, -2.4054 \cdot 10^{-10}].$$

If we take the second cost functional and we fix the integration intervals

$[-1,1]$, $[-9,9]$, with the step 0.0005, respectively 0.00001, we obtain:

$f_{min} = -3.2253 \cdot 10^{20}$,
$time = 11\ minutes$,
$\overline{x} = [10, 190, -80, -80, 10^3, 3.7999 \cdot 10^4, -6.8579 \cdot 10^6, -6.3478 \cdot 10^6, 6.4608 \cdot 10^6, 3.2447 \cdot 10^7, 1, 10]$,
$(F_i(\overline{x}))_{i=\overline{1,12}} = [0, 1.1191 \cdot 10^{-13}, 7.2831 \cdot 10^{-14}, 3.0198 \cdot 10^{-14}, 5.6133 \cdot 10^{-13}, 7.2724 \cdot 10^{-12}, -0.0389, 0.0049, -0.0049, 3.7252 \cdot 10^{-9}]$.

Using MultiStart for the same problem, we get:

$f_{min} = -4.859 \cdot 10^{10}$,
$time = 9\ minutes$,
$\tilde{x} = [-15.7970, -1.4006, 235.1496, 235.1496, -3.4258 \cdot 10^3, -8.1356 \cdot 10^3, -3.6898 \cdot 10^3, -1.2999 \cdot 10^7, 1.2986 \cdot 10^7, 5.1921 \cdot 10^7, -31.6827, 0.4986]$,
$(F_i(\tilde{x}))_{i=\overline{1,12}} = [0, 0, 1.7764 \cdot 10^{-15}, 0, -1.9895 \cdot 10^{-13}, 3.1630 \cdot 10^{-13}, 3.7126 \cdot 10^{-13}, 8.6710 \cdot 10^{-10}, 4.2308 \cdot 10^{-10}, 1.5013 \cdot 10^{-9}]$.

Finally, in $\mathbb{R}^{50}$, consider the following example:

Example 3.

$$\min(x_1 + x_2 + \cdots + x_{50}), \tag{14}$$

with the restrictions given by the relations:

$$\begin{aligned} F_{i=2k} &= x_1^2 + x_{i-1} - x_i - r_1,\ k = \overline{1,24}, \\ F_{i=2k-1} &= x_1^2 + 2x_{i-1} - x_i - r_2,\ k = \overline{1,24}, \end{aligned}$$

where $r_1 = x_{49} + x_{50}$, $r_2 = x_{49}^2 + x_{50}^2$.

We again choose the initial condition $x^0 = [1, 1, \cdots, 1] \in \mathbb{R}^{50}$ and verify that

$$F_i(x^0) = 0,\ i = \overline{1,48}$$

and

$$\frac{D(F_1, \cdots, F_{48})}{D(x_1, \cdots, x_{48})}(x^0) \neq 0. \tag{15}$$

Let $A(x)$ be the corresponding 48×48 nonsingular submatrix of the Jacobian (15). We solve the linear system:

$$v_j(x) \cdot \nabla F_i(x) = 0,\ i = \overline{1,48},\ j = 1,2, \tag{16}$$

where the last two components of $v_j(x)$ are the rows of $I_2 \cdot det(A(x))$ and we obtain $v_j(x) \in \mathbb{R}^{50},\ j = 1,2,\ x \in \mathbb{R}^{50}$.

We again use the Symbolic Math Toolbox of MatLab to solve (16). The two corresponding systems of differential equations (that we skip) were solved with the routine *ode*45 and with the integration intervals $[-100, 100]$ for both systems and the discretization step 0.5, respectively 0.1. We obtain a local solution of (16):

$$\begin{aligned}
&f_{min} = -3.4229 \cdot 10^{11}\\
&time = 2\ minutes\\
&x^\star = [0, -3.8857 \cdot 10^{-16}, -10201, -10201, -30603, -30603, -71407,\\
&-71407, -153015, -153015, -316231, -316231, -642663, -642663,\\
&-1295527, -1295527, -2601255, -2601255, -5212711, -5212711,\\
&-10435623, -10435623, -20881447, -20881447, -41773095, -41773095,\\
&-83556391, -83556391, -167122983, -167122983, -334256167,\\
&-334256167, -668522535, -668522535, -1337055271, -1337055271,\\
&-2674120743, -2674120743, -5348251687, -5348251687, -10696513575,\\
&-10696513575, -21393037351, -21393037351, -42786084903,\\
&-42786084903, -85572180007, -85572180007, 0, 101];
\end{aligned}$$

with constraints satisfied as follows:

$$\begin{aligned}
&(F_i(x^\star))_{i=\overline{1,48}} = [0, 3.8857 \cdot 10^{-16}, 0, 0, 3.63797 \cdot 10^{-12}, 0, -7.2759 \cdot 10^{-12},\\
&0, 0, 0, 0, 0, 1.1641 \cdot 10^{-10}, 0, -2.3283 \cdot 10^{-10}, 0, 0, 0, 9.3132 \cdot 10^{-10}, 0,\\
&-1.8626 \cdot 10^{-9}, 0, 3.7252 \cdot 10^{-9}, 0, 0, 0, -1.4901 \cdot 10^{-8}, 0, 0, 0, 0, 0, 1.1920 \cdot\\
&10^{-7}, 0, -2.3841 \cdot 10^{-7}, 0, 0, 0, 0, 0, 0, 0, 3.8146 \cdot 10^{-6}, 0, 0, 0,\\
&-1.5258 \cdot 10^{-5}, 0].
\end{aligned}$$

Solving the same problem with MultiStart, with 400 initial guesses, we get:

$$\begin{aligned}
&f_{min} = -1.0538 \cdot 10^4,\\
&time = 7\ minutes
\end{aligned}$$

The solution point is:

$$x = [9.3801, 87.9865, 9.3797, 87.9861, 9.3789, 87.9853, 9.3774, 87.9838, 9.3743, 87.9808, 9.3683, 87.9747, 9.3561, 87.9625, 9.3317, 87.9381, 9.2829, 87.8894, 9.1854, 87.7918, 8.990, 87.5968, 8.6004, 87.2068, 7.8203, 86.4267, 6.2602, 84.8666, 3.1399, 81.7463, -3.1006, 75.5057, -15.5817, 63.0246, -40.5439, 38.0624, -90.4684, -11.8620, -190.3174, -111.7110, -390.0153, -311.4089, -789.4111, -710.8047, -1588.2028, -1509.5964, -3185.7861, -3107.1797, -0.5882, -15.9447]$$

and the constraints values are:

$$(F_i(x))_{i=\overline{1,48}} = [1.1474 \cdot 10^{-6}, 1.3228 \cdot 10^{-6}, -1.410 \cdot 10^{-5}, 1.2036 \cdot 10^{-6}, -1.4166 \cdot 10^{-5}, 1.1446 \cdot 10^{-6}, -1.4195 \cdot 10^{-5}, 1.1166 \cdot 10^{-6}, -1.4216 \cdot 10^{-5}, 1.1007 \cdot 10^{-6}, -1.4222 \cdot 10^{-5}, 1.0944 \cdot 10^{-6}, -1.4221 \cdot 10^{-5}, 1.0902 \cdot 10^{-6}, -1.4219 \cdot 10^{-5}, 1.0890 \cdot 10^{-6}, -1.4223 \cdot 10^{-5}, 1.0899 \cdot 10^{-6}, -1.4216 \cdot 10^{-5}, 1.0838 \cdot 10^{-6}, -1.4219 \cdot 10^{-5}, 1.0859 \cdot 10^{-6}, -1.4249 \cdot 10^{-5}, 1.0779 \cdot 10^{-6}, -1.4247 \cdot 10^{-5}, 1.0378 \cdot 10^{-6}, -1.4177 \cdot 10^{-5}, 1.0165 \cdot 10^{-6}, -1.4206 \cdot 10^{-5}, 1.1074 \cdot 10^{-6}, -1.6069 \cdot 10^{-5}, 6.7033 \cdot 10^{-7}, -1.3447 \cdot 10^{-5}, 2.3012 \cdot 10^{-6}, -1.4711 \cdot 10^{-5}, 1.6266 \cdot 10^{-6}, -1.4845 \cdot 10^{-5}, 8.4856 \cdot 10^{-7}, -1.4017 \cdot 10^{-5}, 1.3591 \cdot 10^{-6}, -1.7655 \cdot 10^{-5}, 5.3842 \cdot 10^{-7}, -1.4211 \cdot 10^{-5}, 3.4216 \cdot 10^{-6}, -1.8340 \cdot 10^{-5}, 5.4224 \cdot 10^{-6}, -3.8360 \cdot 10^{-6}, 4.7253 \cdot 10^{-6}].$$

Remark 1. *In all the considered examples, the implicit parametrization met- hod produces better numerical results comparing with the other indicated approaches. As explained in [6], the parametrization method can be extended to the critical case as well (i.e. when (3)) is not satisfied.*

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