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Computing the invariants of discrete dynamical systems of order two*

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Abstract

We review te method of computing invariants for discrete dynamical systems in a birational form (mappings). It is shown that after elimination of singularities by blow ups the mapping is lifted to an automorphism of a rational elliptic surface. The linear action of the bundle mapping on the Picard group of the surface makes possible the computation of the invariant as the strict transform of the eigenvalue one divizor.

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1 Introduction

Discrete integrability is a very hot topic today in the theory of completely integrable systems. Although it started with the study of lattice (partial difference) soliton equations, gradually it has been focusing on discrete ordinary discrete equations called mappings. Here the methods of soliton theory which enabled the study in the case of lattice equations are not working anymore.

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In the case of discrete setting one of the starting point of study was so called the QRT mapping which is an order two mapping defining an elliptic function. This is a 16-parameter autonomous mapping with a solution given by the values of an elliptic function at equidistant points on a line in the complex torus. The utility in the derivation of discrete Painlevé equations is well known [8]. Starting from a mapping with constant coefficients and allowing the coefficients to depend on the independent variable, then using an integrability detector one can select the proper dependence of coefficients. The first integrability detector was *singularity confinement criterion* [11] which imposes a *finite* number of iterations of singular/indeterminate behavior until reaches regular dynamics with recovery of initial condition memory. This criterion was very productive and a big number of discrete Painleve equations have been obtained [9]. Their integrable character has been settled definitively by finding Lax pairs, bilinear forms Backlund/Schlesinger transformations, special solutions etc.[10].

In this paper we give a short review of the problem of integrability and invariants of second order mappings, (based on our papers [4, 5, 6]) which are not in the QRT family. The mappings which are not in the QRT family were discovered by Hirota, Kimura and Yahagy [14] with a peculiar fact that the invariant is given by a biquartic expression. However these are not the only type of non-QRT integrable mappings. Other types have been discovered which are not particular case of Painlevé equations. The idea is to analyse the singularities of the mappings and eliminate them by blowing up. After this the mapping is lifted to an automorphism of a rational elliptic surface whose Picard group is a lattice changing linearly under the action of the bundle induced automorphism. Analysing this linear action one can compute the divisors of eigenvalue one. Then the strict transform of such divisors gives the associated linear system whose fibering gives the invariant. In the case of existence of invariant exceptional curves then the situation is more complicated. These curves need to be blown down and the corresponding transformation will transform the initial dynamical system.

The paper is structured as follows. In the first chapter we give the main preliminaries about rational elliptic surfaces, blowing up and blowing down procedures. Then in the next chapters we discuss examples of various discrete equations.

2 Preliminaries

Notations: cf. [2, 1]

$$
\begin{aligned}
S &: \text{a smooth rational surface}\\
\mathcal{D} &: \text{the linear equivalent class of a divisor } D\\
D\cdot D' &: \text{the intersection number of divisors } D \text{ and } D'\\
\mathcal{O}(D) &: \text{the invertible sheaf corresponding to } D\\
\mathrm{Pic}(S) &= \text{the group of isomorphism classes of invertible sheaves on } S\\
&\simeq \text{the group of linear equivalent classes of divisors on } S\\
\mathcal{E} &: \text{the total transform of divisor class of a line on } \mathbb{P}^2\\
\mathcal{H}_x, \mathcal{H}_y &: \text{the total transform of divisor class of a line } x = \text{constant}\\
&\quad (\text{or } y = \text{constant}) \text{ on } \mathbb{P}^1\times\mathbb{P}^1\\
\mathcal{E}_i &: \text{the total transform of the exceptional divisor class of the } i\text{-th}\\
&\quad blow-up\\
|D| \simeq &(H^0(S,\mathcal{O}(D))-\{0\})/\,\mathbb{C}^\times : \text{ the linear system of } D\\
K_S &: \text{the canonical divisor of a surface } S\\
g(C) &: \text{the genus of an irreducible curve } C\text{, given by the genus formula}\\
&\quad g(C) = 1+\tfrac{1}{2}(C^2+C\cdot K_S) \text{ if } C \text{ is smooth.}
\end{aligned}
$$

Blowing up: Let X be a smooth projective surface and let p be a point on X. There exist a smooth projective surface X' and a morphism $\pi : X' \to X$ such that $\pi^{-1}(p) \cong \mathbb{P}^1$ and π represents a biholomorphic mapping from $X' - \pi^{-1}(p) \to X - (p)$. The morphism is called blowing down and the correspondence π^{-1} is called blowing up of X at p as a rational mapping. For example if X is the space $\mathbb{C}^2$ and p is a point of coordinate (x_0, y_0) then we denote blowing up of X in p

$$X' = \{(x-x_0, y-y_0; \zeta_0 : \zeta_1) \in \mathbb{C}^2\times\mathbb{P}^1\,|(x-x_0)\zeta_0 = (y-y0)\zeta_1\}$$

by

$$\pi : (x, y) \longleftarrow (x-x_0, (y-y_0)/(x-x_0)) \cup ((x-x_0)/(y-y_0), y-y_0).$$

Total transform and proper transform: Let $\pi : Y \to X$ be the blow-down to a point p on X and D be a divisor on X. The divisor $\pi^*(D)$ on Y is called total transform of D and for any analytic subvariety V on X the closure of $\pi^{-1}(V-p)$ in Y is called the proper transform of V.

Let X be a surface obtained by N times blowing up of $\mathbb{P}^2$. Then the Picard group Pic(X) is isomorphic to the $\mathbb{Z}$-module (the Neron-Severi lattice):

$$\mathrm{Pic}(X) = \mathbb{Z}\,\mathcal{E} \oplus \bigoplus_{i=1}^{N} \mathbb{Z}\,\mathcal{E}_i$$

and the intersection of two divisors on X are given by the following basic formulas (valid for any $i, j = 1, \cdots, N$):

$$\mathcal{E}^2 = 1,\ \mathcal{E}_i^2 = -1,\ \mathcal{E}\cdot\mathcal{E}_i = \mathcal{E}_i\cdot\mathcal{E}_j = 0\ (i \neq j).$$

The anti-canonical divisor class is

$$-K_S = 3\mathcal{E} - \sum_{i=1}^{N} \mathcal{E}_i.$$

In the case where X is a surface obtained by N times blowing up of $\mathbb{P}^1 \times \mathbb{P}^1$, the Picard group Pic$(X)$ is

$$\mathrm{Pic}(X) = \mathbb{Z}\,\mathcal{H}_x \oplus \mathbb{Z}\,\mathcal{H}_y \oplus \bigoplus_{i=1}^{N} \mathbb{Z}\,\mathcal{E}_i$$

and the intersection of divisors and the anti-canonical divisor are given by

$$\mathcal{H}_x\cdot\mathcal{H}_y = 1,\ \mathcal{E}_i^2 = -1,\ \mathcal{E}_i\cdot\mathcal{E}_j = \mathcal{E}_i\cdot\mathcal{H}_x = \mathcal{E}_i\cdot\mathcal{H}_y = \mathcal{H}_x^2 = \mathcal{H}_y^2 = 0\ (\imath \neq j),$$

$$-K_S = 2\mathcal{H}_x + 2\mathcal{H}_y - \sum_{i=1}^{N} \mathcal{E}_i.$$

Definition 1: A rational elliptic surface is defined by the following:
a) a complex surface X;
b) a fibration given by the morphism: $\pi : X \to \mathbb{P}^1$ such that:

- for all but finitely many points $k \in \mathbf{P}^1$ the fiber $\pi^{-1}(k)$ is an elliptic curve
- π is not birational to the projection : $E \times \mathbb{P}^1 \to \mathbb{P}^1$
- no fibers contains exceptional curves of first kind.

It is known that a rational elliptic surface can be obtained by 9 blow-ups from $\mathbb{P}^2$ and that the generic fiber of X can be put into a Weierstrass form:

$$f(x,y,k) = y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6$$

where all the coefficients a_i depend on k.

$$\Delta \equiv -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6 = 0,$$

where $b_2 = a_1^2 + 4a_2, b_4 = 2a_4 + a_1a_3, b_6 = a_3^2 + 4a_6, b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2$. The discriminant has degree 12 which gives the number of singular fibers together with their multiplicities. The singularities have been classified by Kodaira according to the type of singularity of the fiber and according to the irreducible components of the resolution of singular fibers. In the non-autonomous case, the role of minimal rational elliptic surfaces are replaced by "generalized Halphen surfaces" [15].

Let $S = S_m$ be a surface obtained by successive m times blowing up from $\mathbb{P}^2$ (or any rational surface) at indeterminate or extremal point of φ, i.e. the Jacobian $\partial(\bar{x},\bar{y})/\partial(x,y)$ in some local coordinates is zero, such that $\tilde{\varphi}$ is analytically stable. Let F_m be a curve on S with self-intersection -1 and $\mathcal{F}_m$ be the corresponding divisor class. Our strategy to write the blow-down S_m along F_m by coordinates is as follows.

Take a divisor class $\mathcal{F}$ such that there exists a *blowing down structure*: $S = S_m \to S_{m-1} \to S_{m-2} \to \cdots \to S_1 \to \mathbb{P}^2$, where $S_m \to S_{m-1}$ is a blow-down along F_m and each $S_i \to S_{i-1}$ is a blow-down along an irreducible curve, such that the divisor class of lines in $\mathbb{P}^2$ is $\mathcal{F}$. Let $|\mathcal{F}| = \alpha_0 f_0 + \alpha_1 f_1 + \alpha_2 f_2 = 0$. Then $(f_0 : f_1 : f_2)$ gives $\mathbb{P}^2$ coordinates.

In order to find such $\mathcal{F}$ we note the following facts.

It is necessary for the existence of such a blow-down structure that there exists a set of divisor classes $\mathcal{F}_1, \ldots, \mathcal{F}_m$ such that

$$\mathcal{F}^2 = 1 \quad \mathcal{F}_i^2 = -1, \quad \mathcal{F}_i \cdot \mathcal{F}_j = 0, \quad \mathcal{F} \cdot \mathcal{F}_i = 0$$

for $(1 \le i, j \le m)$, and further that (i) the genus of divisor F is zero; (ii) the linear system of $\mathcal{F}$ does not have a fixed part in the sense of Zariski decomposition and its dimension is two.

If the linear system of $\mathcal{F}$ does not have fixed part, then by Bertini theorem, its generic divisor is smooth and irreducible (this follows from the fact that two divisors defines a pencil by blowing up at the unique intersection and P. 137 of [1]), and its genus is given by the formula

$$g = 1 + \frac{1}{2}(F^2 + F \cdot K_S).$$

From this fact and Condition (ii), $1 + \frac{1}{2}(F^2 + F \cdot K_S)$ should be zero.

If we want to blow down to $\mathbb{P}^1 \times \mathbb{P}^1$ instead of $\mathbb{P}^2$, our strategy becomes as follows.

Let F_{m-1} be a curve on S with self-intersection -1 and $\mathcal{F}_{m-1}$ be the corresponding divisor class. Take a divisor class $\mathcal{H}_u$ and $\mathcal{H}_v$ such that there exists a blow-down structure: $S = S_{m-1} \to S_{m-2} \to\to \cdots \to S_1 \to \mathbb{P}^1 \times \mathbb{P}^1$, where $S_{m-1} \to S_{m-2}$ is a blow-down along F_{m-1} and each $S_i \to S_{i-1}$ is a blow-down along an irreducible curve, such that the divisor class of lines $u = \text{const}$ and $v = \text{const}$ are $\mathcal{H}_u$ and $\mathcal{H}_v$. Let $|\mathcal{H}_u| = \alpha_0 f_0 + \alpha_1 f_1 = 0$ and $|\mathcal{H}_v| = \beta_0 g_0 + \beta_1 g_1 = 0$. Then $(u, v) = (f_0/f_1, g_0/g_1)$ gives $\mathbb{P}^1 \times \mathbb{P}^1$ coordinates.

Let S be a rational surface and let K_S the canonical divisor class of S. It is known that if S admits an automorphism φ of infinite order, then φ is "linearizable", φ preserves an elliptic fibration of S, or the algebraic (or topological) entropy of φ is positive. In this section we classify the second case.

Let X be a rational elliptic surface obtained by 9 blow-ups from $\mathbb{P}^2$. The main result from [4] is the following classification.

Classification Let $\varphi : X$ be an automorphism of X which preserve the elliptic fibration $\alpha f_0(x, y, z) + \beta g_0(x, y, z) = 0$. Such cases are classified as follows.
i-1) φ preserves $\alpha : \beta$ and the degree of fibers is 3;
i-2) φ does not preserve $\alpha : \beta$ and the degree of fibers is 3;
ii-1) φ preserves $\alpha : \beta$ and the degree of fibers is $3m$, $(m \geq 2)$;
ii-2) φ does not preserve $\alpha : \beta$ and degree of fibers is $3m$, $(m \geq 2)$

The QRT mappings belong to Case 1-i) [16]. In case i-2) and ii-2), elliptic fibrations admit exchange of fibers.

Case ii) occurs only in the case when the linear system $| - kK_X| = 0$ for $k = 1, \ldots, m-1$ and $| - mK_X| = 1$. Such a pencil is called a Halphen pencil of index m or a rational elliptic surface of index m, i.e. those fibers are degree $3m$ curve on $\mathbb{P}^2$ passing through each point of blow-ups with multiplicity m. It is known that every Halphen pencil of index m contains a unique cubic curve with multiplicity m (see Chap. 5 §6 of [3] for more details).

Let X be a generalized Halphen surface, i.e. $-K_X$ is decomposed into effective divisors as $D = \sum m_i D_i$, $[D] = -K_X$ such that $D_i \cdot K_X = 0$. A generalized Halphen surface can be obtaind from $\mathbb{P}^2$ by successive 9 blow-ups. Generalized Halphen surfaces are classified by the type of D to elliptic, multiplicative and additive.

Let Q be the root lattice defined as the orthogonal complement of D (with respect to the intersection form) and let ω be a meromorphic 2-form on X with $\mathrm{Div}(\omega) = -D$. Then, with suitable normalization, ω determines the period mapping χ from Q to the torus for elliptic case, to $\mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}$ for multiplicative and to $\mathbb{C}$ for additive (see [15] for more details).

We can also characterize the Halphen surface in a different way which is more amenable to the case of non-autonomous dynamical systems. Let X be a generalized Halphen surface and Q the root lattice defined as the orthogonal complement of D with respect to the intersection form and ω a meromorphic 2-form on X with $\mathrm{Div}(\omega) = -D_{red}$, where $D_{red} = \sum_i^s D_i$. Then, the 2-form ω determines the period mapping χ from Q to $\mathbb{C}$ by

$$\chi(\alpha) = \int_{\alpha} \omega$$

in modulo $\sum_{\gamma} \mathbb{Z}\, \chi(\gamma)$, where the summation is taken for all the cycles on D_{red} (see examples in the next section and [15] for more details). A generalized Halphen surface is called elliptic, multiplicative, or additive type if the rank of the first homology group of D_{red} is 2, 1, or 0 respectively.

Theorem 1.
(ell) If X is elliptic type, then X is a Halphen pencil of index m iff $k\ \chi(K_X) \neq 0$ for $k = 1, \dots, m-1$ and $\chi(mK_X) = 0$.
(mult) If X is multiplicative type, then the same assertion holds as the elliptic case.
(add) If X is additive type, then X is a Halphen pencil of index 1 iff $\chi(K_X) = 0$, and never be a Halphen pencil of index $m \geq 2$.

Proof. Case (ell) is a classical result (see Remark 5.6.1 in [3] or references therein). Case (mult) and case (add) of index 1 are Proposition 23 in [15]. Similar to that proof, we can vary D and χ continuously to nonsingular case. Indeed, let $P_1, \dots, P_9$ be the points of blow-ups (possibly infinitely near, we assume P_9 is the point for the last blow-up) and f_0 be the cubic polynomial defining D. There exists a pencil of cubic curves $C_\lambda : f_\lambda = f_0 + \lambda f_1 = 0$ $\lambda \in \mathbb{P}^1$ passing through the 8 points $P_1, \dots, P_8$. For small λ, the cubic curve C_λ is close to D, and the meromorphic 2-form ω_λ for C_λ is also close to ω. Let P_9' be a point close to P_9 on C_λ such that

$$\lim_{\lambda \to 0} \chi_\lambda(-mK_{X'}) = \lim_{\lambda \to 0} \int_{-mK_{X'}} \omega' = \int_{-mK_X} \omega' = \chi(-mK_X)$$

holds(here X' is the surface obtained by blow-ups at $P_1, \ldots, P_8$ and P_9' instead of P_9). Thus, $\chi_\lambda(-mK_{X'}) \neq 0$ holds if $\chi(-mK_X) \neq 0$ for small λ, and therefore X does not have a pencil of degree $3m$. Conversely, if $\chi(-mK_X) = 0$, then $\chi'(-mK_{X'})$ is close to zero, and there exists P_9'' close to P_9' on C' such that $\chi_\lambda(-mK_{X''}) = 0$. Thus, we have

$$\lim_{\lambda \to 0} \chi_\lambda(-mK_{X''}) = \chi(-mK_X).$$

Since X'' has (at least) a pencil of curves of degree $3m$ passing through the 9 points with multiplicity m and this condition is closed in the space of coefficients of polynomials defining curves, X also has the same property. □

Remark 1. *Since every elliptic curve over a function field can be written in Weierstrass normal form, Case ii) of our classification also can. However it needs huge calculation [19].*

3 Examples

In this section we are going to give some examples in order to show our results. A typical example of Case i-1 is th QRT mappings. There are some literature about their relation to rational elliptic surfaces [7], and we do not repeat here.

3.1 Case i-2

First we start with the non-QRT mappings which are not particular cases of q-Painleve equations. We choose an example which is a new integrable mapping and we give a full analysis of space of initial conditions and invariants.

$$x_{n+1} = -x_{n-1} \frac{(x_n - a)(x_n - 1/a)}{(x_n + a)(x_n + 1/a)}$$

First of all in order to compactify the space of dependent variables we write the equations in projective space as a two component system:

$$\phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1, \phi(x, y) = (\overline{x}, \overline{y}).$$

We use $P^1 \times P^1$ instead of P^2 just because the parameters of blowing-up points become easy to write.

$$\overline{x} = y$$

$$\overline{y} = -x\frac{(y-a)(y-1/a)}{(y+a)(y+1/a)}$$

The projective space $\mathbb{P}^1 \times \mathbb{P}^1$ is generated by the following coordinate systems $(X = 1/x, Y = 1/y)$:

$$\mathbb{P}^1 \times \mathbb{P}^1 = (x, y) \cup (X, y) \cup (x, Y) \cup (X, Y)$$

The nondeterminate points for the mappings ϕ and ϕ^{-1}

$$p_1 : (x, y) = (0, -a); \qquad p_2 : (x, y) = (0, -1/a)$$

$$p_3 : (X, y) = (0, a); \qquad p_4 : (X, y) = (0, 1/a)$$

$$p_5 : (x, y) = (a, 0); \qquad p_6 : (x, y) = (1/a, 0)$$

$$p_7 : (x, Y) = (-a, 0); \qquad p_8 : (x, Y) = (-1/a, 0)$$

After blowing up these points projective space is transformed into a surface X and the mapping is lifted to a birational mapping

$$\phi : X \to \mathbb{P}^1 \times \mathbb{P}^1 .$$

Moreover it can be shown that ϕ is lifted to an automorphism of X.

The Picard group of X is the following **Z**-module

$$\mathrm{Pic}(X) = \mathbb{Z}\, H_x \oplus \mathbb{Z}\, H_y \oplus \sum_{i=1}^{8} \mathbb{Z}\, \mathcal{E}_i,$$

where H_x, H_y are the total transforms of the lines $x = \text{const.}$, $y = \text{const.}$ and $\mathcal{E}_i$ are the total transforms of the eight blowing up points. The anticanonical divisor of X is

$$-K_X = 2\,\mathcal{H}_x + 2\,\mathcal{H}_y - \sum_{i=1}^{8} \mathcal{E}_i .$$

Also the intersection form of divisors is given by $\mathcal{H}_i \cdot \mathcal{H}_j = 1 - \delta_{ij}$, $\quad \mathcal{E}_i \cdot \mathcal{E}_j = -\delta_{ij}$, $\quad \mathcal{H}_i \cdot \mathcal{E}_k = 0$.

The proper transforms of the lines $(x = 0, X = 0, y = 0, Y = 0)$ are the following:

$$D_0 := (x, y) = (0, y), \quad D_1 := (X, y) = (0, y)$$

$$D_2 := (x, y) = (x, 0), \quad D_3 := (x, Y) = (x, 0)$$

The induced bundle mapping $\phi_* : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ is acting on these curves in a *linear* way as a permutation:

$$(\overline{D}_0, \overline{D}_1, \overline{D}_2, \overline{D}_3) = (D_2, D_3, D_0, D_1)$$

The relations between strict transforms and total transforms are:

$$D_0 = \mathcal{H}_0 - \mathcal{E}_1 - \mathcal{E}_2, \quad D_1 = \mathcal{H}_0 - \mathcal{E}_3 - \mathcal{E}_4$$

$$D_2 = \mathcal{H}_1 - \mathcal{E}_5 - \mathcal{E}_6, \quad D_3 = \mathcal{H}_1 - \mathcal{E}_7 - \mathcal{E}_8$$

Finally the intersections of strict transforms

$$D_i \cdot D_i = -2, \quad D_0 \cdot D_2 = D_0 \cdot D_3 = D_1 \cdot D_2 = D_2 \cdot D_3 = 1$$

$$D_0 \cdot D_1 = D_2 \cdot D_3 = 0$$

show that X is $A_3^{(1)}$-type surface in the Kodaira classification. One can see immediately that the combination $D_0 + D_1 + D_2 + D_3 + D_4$ is an invariant with respect to the action of $\phi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$. The proper transform of such a combination correspond to the following linear system (these curves pass through all E_i for any k).

$$F \equiv \alpha xy - \beta((x^2+1)(y^2+1) + (a+1/a)(y-x)(xy+1)) = 0 \qquad (1)$$

$$\Leftrightarrow kxy - ((x^2+1)(y^2+1) + (a+1/a)(y-x)(xy+1)) = 0.$$

This family of curves defines a rational elliptic surface. One can see that the linear system is not preserved by the mapping. More precisely the action changes k in $-k$.

So, as a conclusion the dimension of the linear system corresponding to the anticanonical divisor is 1. It can be written as $\alpha f_1(x,y) + \beta f_2(x,y) = 0 \Leftrightarrow k f_1(x,y) + f_2(x,y) = 0$ for $\alpha, \beta \in \mathbb{C}$ and $\deg f = \deg g = (2,2)$. This elliptic fibration is preserved by the action of the dynamical system but not trivially in the sense that it exchanges the singular fibers. The conserved quantity becomes higher degree as $(f/g)^\nu$ for some $\nu > 1$. In our case $\nu = 2$ and the invariant is:

$$I = \left(\frac{((x^2+1)(y^2+1) + (a+1/a)(y-x)(xy+1))}{xy} \right)^2$$

Remark 2. *In order to have a Weierstrass model we perform some homographic transformations according to the algorithm of Schwartz [18]. Then, after long but straitforward calculations we can compute the roots of the elliptic discriminant* $\Delta(k)$

$$\begin{aligned} k_1 &= 0, \quad \text{multiplicity} = 2 \\ k_{2,3} &= \pm 4(1+a^2)/a, \quad \text{multiplicity} = 1 \\ k_{4,5} &= \pm (1-a^2)^2/a^2, \quad \text{multiplicity} = 2 \\ k_6 &= \infty, \quad \text{multiplicity} = 4 \end{aligned}$$

Inserting these values in the linear system we get the main singular fibers of the rational elliptic surface. Namely, for k_1 and $k_{4,5}$ we have $A_1^{(1)}$ fiber, for $k_{2,3}$ we have $A_0^{(1)}$ fiber and for k_6 the $A_3^{(1)}$ fiber. The action of the mapping is nothing but exchanging fibers ($k_1 \leftrightarrow k_1, k_2 \rightarrow k_3 \rightarrow k_2, k_4 \rightarrow k_5 \rightarrow k_4, k_6 \leftrightarrow k_6$)

This approach is useful to compute also the symmetries. The symmetry group is related to the orthogonal complement of the space of initial condition $A_3^{(1)}$. In order to see this we note that

$$\operatorname{rank} \operatorname{Pic}(X) = \operatorname{rank} < \mathcal{H}_0, \mathcal{H}_1, \mathcal{E}_1, ... \mathcal{E}_8 >_{\mathbb{Z}} = 10$$

Now we define:

$$< D > = \sum_{i=0}^{3} \mathbb{Z} D_i$$

$$< D >^{\perp} = \{\alpha \in \operatorname{Pic}(X) | \alpha \cdot D_i = 0, i = 0, 3\}$$

which have 6-generators:

$$< D >^{\perp} = < \alpha_0, \alpha_1, ..., \alpha_5 >_{\mathbb{Z}}$$

$$\alpha_0 = \mathcal{E}_4 - \mathcal{E}_3, \alpha_1 = \mathcal{E}_1 - \mathcal{E}_2, \alpha_2 = \mathcal{H}_1 - \mathcal{E}_1 - \mathcal{E}_5$$

$$\alpha_3 = \mathcal{H}_0 - \mathcal{E}_3 - \mathcal{E}_7, \alpha_4 = \mathcal{E}_5 - \mathcal{E}_6, \alpha_5 = \mathcal{E}_8 - \mathcal{E}_7$$

Related to them we define elementary reflections:

$$w_i : \operatorname{Pic}(x) \rightarrow \operatorname{Pic}(X), w_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i$$

where $c_{ji} = 2(\alpha_j \cdot \alpha_i)/(\alpha_i \cdot \alpha_i)$. One can easily see that c_{ij} is a Cartan matrix of $D_5^{(1)}$-type for the root lattice $Q = \bigoplus_{i=0}^{5} \mathbb{Z}\, \alpha_i$. We introduce also permutation of roots:

$$\sigma_{10}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (\alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$$

$$\sigma_{tot}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (\alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0)$$

Hence the group generated by reflections and permutations becomes an extended Weyl group

$$\widetilde{W}(D_5^{(1)}) =< w_0, w_1, ..., w_5, \sigma_{10}, \sigma_{tot} >$$

This extended Weyl group becomes the group of Cremona isometries for the space of initial conditions X since preserves the intersection form, the canonical divisor K_X (which is nothing but the null vector δ of the Cartan matrix) and semigroup of effective classes of divisors. Accordingly our mapping *lives* in a Weyl group $\widetilde{W}(D_5^{(1)})$ and has the following decomposition in elementary reflections:

$$\phi_* = \sigma_{tot} \circ w_3 \circ w_5 \circ w_4 \circ w_3$$

All elements $\omega \in \widetilde{W}(D_5^{(1)})$ which commutes with ϕ_*, namely $(\omega \circ \phi_* = \phi_* \circ \omega)$ form the symmetries of the mapping.

At the last we characterize also the surface using period map $\chi : Q \to \mathbb{C}$. Let

$$\omega = \frac{1}{2\pi i} \frac{dx \wedge dy}{xy}. \tag{2}$$

For example, $\chi(\alpha_0)$ is computed as follows. The exceptional divisors E_1 and E_2 intersect with D_0 at $(x, y) = (0, -a)$ and $(0, -1/a)$, and $\chi(\alpha_0)$ is computed as

$$\begin{aligned}
\chi(\alpha_0) &= \int_{|x|=\varepsilon,\ y:-1/a\sim -a} \frac{1}{2\pi i} \frac{dx \wedge dy}{xy} \\
&= -\int_{-1/a}^{-a} \frac{dy}{y} \\
&= -\log a^2,
\end{aligned}$$

where $y : -1/a \sim -a$ denotes a path from $-1/a$ to a in D_1. According to the ambiguity of paths, the result should be considered in modulo $2\pi i\,\mathbb{Z}$. Similarly, we obtain

$$\begin{aligned}
&\chi(\alpha_0) = -\log a^2,\ \chi(\alpha_1) = \log a^2,\ \chi(\alpha_2) = \pi i, \\
&\chi(\alpha_3) = -\pi i,\ \chi(\alpha_4) = \log a^2,\ \chi(\alpha_5) = -\log a^2,
\end{aligned}$$

and therefore $\chi(-K_X) = 0$, i.e. $q = \exp(\chi(-K_X)) = 1$.

3.2 Case ii-1

We consider the following symmetric reduction of q-P_V for $q = i$ [14].

$$\bar{x} = \frac{(x-t)(x+t)}{y(x-1)}$$

.

$$\bar{y} = x$$

We define the phase space as a rational surface obtained by blow-ups from $\mathbb{P}^1 \times \mathbb{P}^1$ at 8 points

$$\begin{aligned} P_1 &: (x,y) = (t,0)\\ P_2 &: (x,y) = (-t,0)\\ P_3 &: (x,y) = (0,t)\\ P_4 &: (x,y) = (0,-t)\\ P_5 &: (x,y) = (1,\infty)\\ P_6 &: (x,y) = (\infty,1)\\ P_7 &: (x,y) = (\infty,\infty)\\ P_8 &: (x,x/y) = (\infty,1) \end{aligned}$$

.

Then the system acts the phase as a holomorphic automorphism.

The divisor with eigenvalue one in the linear mapping $\phi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ is now exactly the anticanonical divizor. $-K_X = 2\,\mathcal{H}_x + 2\,\mathcal{H}_y - \mathcal{E}_1 - \cdots - \mathcal{E}_8$. However the situation here is different since the strict transform of $-K_X$ is $xy = 0$ which is not a linear system. Accordingly we cannot compute any invariant here. So $\dim|-K_X|$ is zero. In this case we investigate what happens with

$$-2K_X = 4\,\mathcal{H}_x + 4\,\mathcal{H}_y - \sum_{i=1}^{8} 2\,\mathcal{E}_i$$

One can see that $\dim|-2K_X| = 1$. Actually, we have

$$\begin{aligned} |-2K_X| = &\alpha x^2y^2 + \beta(2x^2y^3 + 2x^3y^2 + x^2y^4 + x^4y^2 - 2x^3y^3 - \\ &2xy^4 - 2x^4y + x^4 + y^4 + 2t^2(xy^2 + x^2y - y^2 - x^2) + t^4) \equiv \alpha f + \beta g \end{aligned}$$

and

$$k = \frac{g}{f} = \frac{(2x^2y^3 + 2x^3y^2 + x^2y^4 + x^4y^2 - 2x^3y^3 - 2xy^4 - 2x^4y}{x^2y^2} +$$

$$+\frac{x^4+y^4+2t^2(xy^2+x^2y-y^2-x^2)+t^4)}{x^2y^2}$$

is the conserved quantity. So it belongs to Case ii-1.

Remark 3. *We say a curve $f(x,y)=0$ passing through a point (x_0,y_0) with multiplicity m if $\frac{\partial^j f(x_0,y_0)}{\partial x^p \partial y^q}=0$ for any $j \leq m$ and $p+q=j$.*

3.3 Case ii-2

We consider the mapping φ

$$\varphi:\begin{cases}\bar{x} &= \dfrac{x(-ix(x+1)+y(bx+1)}{y(x(x-b)+iby(x-1))} \\ \bar{y} &= \dfrac{x(x(x+1)+iby(x-1))}{b(x(x+1)-iy(x-1)}\end{cases}. \tag{3}$$

The inverse of φ is

$$\varphi^{-1}:\begin{cases}\underline{x} &= \dfrac{y(bxy-bx-by+1)}{xy-x+by-1} \\ \underline{y} &= \dfrac{-iy(bxy-bx-by+1))(bxy+x-by+1)}{bx(xy-x-y-1)(xy-x+by-1)}\end{cases}. \tag{4}$$

The phase space is obtained by blow-ups from $\mathbb{P}^1\times\mathbb{P}^1$ at 8 points:

$$\begin{aligned}P_1 &: (x,y)=(-1,0)\\ P_2 &: (x,y)=(0,b)\\ P_3 &: (x,y)=(1,\infty)\\ P_4 &: (x,y)=(\infty,1)\\ P_5 &: (x,y)=(0,0)\\ P_6 &: (x,y/x)=(0,i)\\ P_7 &: (x,y)=(\infty,\infty)\\ P_8 &: (x,y/x)=(\infty,i/b)\\ &.\end{aligned}$$

Then the system acts the phase as a holomorphic automorphism.

For above example, $\dim|-K_X|$ is zero and $\dim|-2K_X|$ is one. Actually, we have

$$|-2K_X|: \quad \begin{aligned} 0 = &\ kf_0(x,y) - f_1(x,y) \\ = &\ kx^2y^2 - \Big(ix(x+1)^2 - i(x+i)(x^2-1)y \\ &+ b(x-1)^2y^2\Big)\Big(-ix(y-1)+y(by-1)\Big) \end{aligned}$$

and

$$k = \frac{f_1(x,y)}{f_0(x,y)}$$

is mapped to $-k$. So

$$k^2 = \left(\frac{f_1(x,y)}{f_0(x,y)}\right)^2$$

is the conserved quantity and it belongs to Case ii-2.

The anticanonical divisor consists of

$$\begin{aligned} &\mathcal{H}_x - \mathcal{E}_2 - \mathcal{E}_5,\ \mathcal{E}_5 - \mathcal{E}_6,\ \mathcal{H}_y - \mathcal{E}_1 - \mathcal{E}_5, \\ &\mathcal{H}_x - \mathcal{E}_4 - \mathcal{E}_7,\ \mathcal{E}_7 - \mathcal{E}_8,\ \mathcal{H}_y - \mathcal{E}_3 - \mathcal{E}_7 \end{aligned}$$

and its orthocomplement is generated by

$$\begin{aligned} \alpha_0 &= \mathcal{H}_x + \mathcal{H}_y - \mathcal{E}_5 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8 \\ \alpha_1 &= \mathcal{H}_x - \mathcal{E}_1 - \mathcal{E}_3 \\ \alpha_2 &= \mathcal{H}_y - \mathcal{E}_2 - \mathcal{E}_4 \\ \beta_0 &= \mathcal{H}_x + \mathcal{H}_y - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_7 - \mathcal{E}_8 \\ (\beta_1 &= \mathcal{H}_x + \mathcal{H}_y - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6). \end{aligned}$$

The mapping is an automorphism of the family of surfaces, whose type is $A_2^{(1)} + A_1^{(1)}$.The expression of mapping in terms of the elementary reflections is done in [4].

3.4 Case involving blowing down structure; discrete Nahm equation with icosahedral symmetry

The last example we discuss is the discrete Nahm equation with icosahedral symmetry [20]. It is given by

$$\begin{cases} \bar{x} - x &= \epsilon(2x\bar{x} - y\bar{y}) \\ \bar{y} - y &= -\epsilon(5x\bar{y} + 5y\bar{x} - y\bar{y}) \end{cases} \tag{5}$$

and is integrable as well. However the invariant here is more complicated.

We can simplify the system by the following change of variable

$$x = \frac{1}{5}(X + \frac{y}{2}), \quad \bar{x} = \frac{1}{5}(\bar{X} + \frac{\bar{y}}{2}),$$

then we divide by $y\bar{y}$ both equations and call again $a = X/y, b = 1/y, u = b - \epsilon a, v = b + \epsilon a$ and finally we get a simpler equation but non-QRT type:

$$6\bar{u}\underline{u} - u(\bar{u} + \underline{u}) - \frac{7\epsilon}{2}(\bar{u} - \underline{u}) - 4u^2 + 49\epsilon^2 = 0.$$

We can apply our procedure to this last non-QRT mapping, however, here we demonstrate that our procedure works well even for the original mapping.

The space of initial condition is given by the $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at the following 12 points:

$$\begin{aligned}
&\mathcal{E}_1 : (x, y) = (\infty, \infty),\ \mathcal{E}_2(-1/7\epsilon, -3/7\epsilon),\ \mathcal{E}_3(-1/7\epsilon, 4/7\epsilon),\\
&\mathcal{E}_4(1/7\epsilon, 3/7\epsilon),\ \mathcal{E}_5(1/7\epsilon, -4/7\epsilon)\ \mathcal{E}_6(1/5\epsilon, 0),\\
&\mathcal{E}_7(1/3\epsilon, 0),\ \mathcal{E}_8(1/\epsilon, 0),\ \mathcal{E}_9(-1/\epsilon, 0),\\
&\mathcal{E}_{10}(-1/3\epsilon, 0),\ \mathcal{E}_{11}(-1/5\epsilon, 0). \mathcal{E}_{12} : (1/x, x/y) = (0, 1/3)
\end{aligned}$$

On this surface the dynamical system is neither an automorphism nor analytically stable due to the following topological singularity patterns:

$$\begin{gathered}
\mathcal{H}_y - \mathcal{E}_1\ (y = \infty) \to \text{point} \to \cdots (4 \text{ points}) \cdots \to \text{point} \to \mathcal{H}_y - \mathcal{E}_1\\
\cdots \to \text{point} \to \text{point} \to \mathcal{H}_x - \mathcal{E}_1\ (x = \infty) \to \text{point} \to \text{point} \to \cdots .
\end{gathered}$$

Moreover, the curve $4x + y = 0 : \mathcal{H}_x + \mathcal{H}_y - \mathcal{E}_1 - \mathcal{E}_3 - \mathcal{E}_5$ is invariant. We blow down along these three curves with the blow-down structure

$$\begin{aligned}
&\mathcal{H}_u = \mathcal{H}_x + \mathcal{H}_y - \mathcal{E}_1 - \mathcal{E}_3,\ \mathcal{H}_v = \mathcal{H}_x + \mathcal{H}_y - \mathcal{E}_1 - \mathcal{E}_5,\\
&\mathcal{H}_x - \mathcal{E}_1,\ \mathcal{H}_y - \mathcal{E}_1,\ \mathcal{H}_x + \mathcal{H}_y - \mathcal{E}_1 - \mathcal{E}_3 - \mathcal{E}_5,\\
&\mathcal{F}_1 = \mathcal{E}_{12},\ \mathcal{F}_2 = \mathcal{E}_2,\ \mathcal{F}_3 = \mathcal{E}_4,\ \mathcal{F}_4 = \mathcal{E}_6,\\
&\mathcal{F}_5 = \mathcal{E}_7,\ \mathcal{F}_6 = \mathcal{E}_8,\ \mathcal{F}_7 = \mathcal{E}_9,\ \mathcal{F}_8 = \mathcal{E}_{10},\ \mathcal{F}_9 = \mathcal{E}_{11},
\end{aligned}$$

where the linear systems of $\mathcal{H}_v$ and $\mathcal{H}_v$ are given by

$$\begin{aligned}
|\,\mathcal{H}_u\,| :&u_0(1 + 7\epsilon x) + u_1(4x + y)\\
|\,\mathcal{H}_v\,| :&v_0(1 - 7\epsilon x) + v_1(4x + y).
\end{aligned}$$

If we take the new variables u and v as

$$v = \frac{2(1+7\epsilon x)}{\epsilon(4x+y)},\ v = \frac{2(1-7\epsilon x)}{\epsilon(4x+y)},$$

then we have

$$\begin{aligned}&\mathcal{F}_1 : (u,v) = (2,-2), \mathcal{F}_2 : (0,-4), \mathcal{F}_3 : (4,0), \mathcal{F}_4 : (6,-1), \mathcal{F}_5 : (5,-2),\\ &\mathcal{F}_6 : (4,-3), \mathcal{F}_7 : (3,-4), \mathcal{F}_8 : (2,-5), \mathcal{F}_9 : (1,-6).\end{aligned}$$

The dynamical system becomes an automorphism having the following topological singularity patterns

$$\begin{aligned}&\mathcal{H}_v - \mathcal{F}_9 \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_3 \to \mathcal{H}_u - \mathcal{F}_4\\ &\mathcal{H}_v - \mathcal{F}_3 \to \mathcal{F}_4 \to \mathcal{F}_5 \to \mathcal{F}_6 \to \mathcal{F}_7 \to \mathcal{F}_8 \to \mathcal{F}_9 \to \mathcal{H}_u - \mathcal{F}_2\end{aligned}$$

and $\mathcal{H}_u \to \mathcal{H}_u + \mathcal{H}_v - \mathcal{F}_2 - \mathcal{F}_4$. Hence we find the invariant (-1) curve $\mathcal{H}_u + \mathcal{H}_v - \mathcal{F}_1 - \mathcal{F}_2 - \mathcal{F}_3$, which should be blown down. Again we take the blow-down structure as

$$\begin{aligned}&\mathcal{H}_s = \mathcal{H}_u + \mathcal{H}_v - \mathcal{F}_1 - \mathcal{F}_2,\ \mathcal{H}_t = \mathcal{H}_u + \mathcal{H}_v - \mathcal{F}_1 - \mathcal{F}_3,\\ &\mathcal{H}_u + \mathcal{H}_v - \mathcal{F}_1 - \mathcal{F}_2 - \mathcal{F}_3,\ \mathcal{F}'_1 = \mathcal{H}_a - \mathcal{F}_1,\ \mathcal{F}'_2 = \mathcal{H}_b - \mathcal{F}_1\\ &\mathcal{F}'_3 = \mathcal{F}_4,\ \mathcal{F}'_4 = \mathcal{F}_5,\ \mathcal{F}'_5 = \mathcal{F}_6,\ \mathcal{F}'_6 = \mathcal{F}_7,\\ &\mathcal{F}'_7 = \mathcal{F}_8,\ \mathcal{F}'_8 = \mathcal{F}_9,\end{aligned}$$

where the linear systems of $\mathcal{H}_s$ and $\mathcal{H}_t$ are given by

$$\begin{aligned}&|\,\mathcal{H}_s\,| : s_0 u(v+2) + s_1(u-v-4)\\ &|\,\mathcal{H}_t\,| : t_0 v(u-2) + t_1(u-v-4)\end{aligned}$$

and hence we take the new variables s and t as

$$s = -\frac{3u(v+2)}{2(u-v-4)},\ t = -\frac{3v(u-2)}{2(u-v-4)}.$$

Then we have

$$\begin{aligned}&\mathcal{F}'_1 : (s,t) = (3,0),\ \mathcal{F}'_2(0,3),\ \mathcal{F}'_3(-3,2),\ \mathcal{F}'_4 : (\frac{s}{t-3}, d-3) = (5,0),\\ &\mathcal{F}'_5(2,3),\ \mathcal{F}'_6(3,2),\ \mathcal{F}'_7 : (u-3, \frac{t}{s-3}) = (0,5),\ \mathcal{F}'_8(2,-3)\end{aligned}$$

and

$$\begin{cases} \bar{s} &= \dfrac{2st - 3s - 3t + 9}{s + t - 3} \\ \bar{t} &= \dfrac{2(s-3)(t+3)}{3s - t - 9} \end{cases}.$$

The invariants can be computed by using the the anticanonical divisor as

$$K = \frac{(s-t)^2 + 4(s+t) - 21}{(s-2)(t-2)(2st - 5s - 5t + 15)} = \frac{-56\epsilon^6 y(-3x+y)^2(4x+y)^3}{d_1 d_2 d_3} \tag{6}$$

and

$$\omega = \frac{2\epsilon ds \wedge dt}{(s-t)^2 + 4(s+t) - 21} = \frac{dx \wedge dy}{y(3x-y)(4x+y)}, \tag{7}$$

where

$$\begin{aligned} d_1 &= -3 - 12\epsilon x + 15\epsilon^2 x^2 - 3\epsilon y - 17\epsilon^2 xy + 4\epsilon^2 y^2 \\ d_2 &= -3 + 12\epsilon x + 15\epsilon^2 x^2 + 3\epsilon y - 17\epsilon^2 xy + 4\epsilon^2 y^2 \\ d_3 &= -3 + 27\epsilon^2 x^2 + 10\epsilon^2 xy + 10\epsilon^2 y^2. \end{aligned}$$

Finally the invariant has the following extremely complicated form:

$$K = \frac{y(3x-y)^2(4x+y)^3}{1 + \epsilon^2 c_2 + \epsilon^4 c_4 + \epsilon^6 c_6}, \tag{8}$$

where

$$\begin{aligned} c_2 &= -7(5x^2 - y^2) \\ c_4 &= 7(37x^4 + 22x^2y^2 - 2xy^3 + 2y^4) \\ c_6 &= -225x^6 + 3840x^5y + 80xy^5 - 514x^3y^3 - 19x^4y^2 - 206x^2y^4. \end{aligned}$$

4 Conclusions

In this paper we gave a review of the algebraic-geometric approach (given in [15], [17]) to integrable second order mappings of non-QRT type. The interesting findings are related to the invariants corresponding to Halphen pencils of higher index. The action of the mapping may or may not exchange their singular fibers. We gave a classification of such mappings which preserve an elliptic fibration and also a theorem which charcaterise their space of initial conditions as Halphen pencils of higher index. Finally three examples showed an explicit realizations of the above mentioned results. Also the case of relatively non-minimal rational elliptic surfaces is quite interesting since the invariants in these cases can have very complicated forms.

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