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The space $l_\infty(X)$*

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Abstract

In this paper we have shown that the space $c_0^{n\times n}$ cannot be complemented in $l_\infty^{n\times n}$ and $c_0(H)$ cannot be complemented in $l_\infty(H)$ where H is a Hilbert space. Further, extending these results we show that if X is a Banach space then $c_0(X)$ cannot be complemented in $l_\infty(X)$.

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1 Introduction

A closed subspace M of a Banach space X is said to be complementd in X if and only if there exists a bounded linear projection from X onto M. It is not difficult to see that if M is complemented by the closed subspace N, then there exists a $c > 0$ such that $\|m + n\| \geq c\|m\|$ for all $m \in M$ and $n \in N$. Murray [12] showed that $l_p, p > 1, p \neq 2$ has subspaces that cannot be complemented. Philips [14] and Lindenstrauss and Tzafriri [11] proved that c_0 cannot be complemented in l^∞. In fact, Lindenstrauss and Tzafriri [11] established that every infinite dimensional Banach space which is not isomorphic to a Hilbert space contains a closed subspace that cannot be complemented. Similar results were also proved in [2],[4] [6],[15],[17] and [20]. Pelczynski [13] showed that complemented subspaces of l_1 are isomorphic to

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l_1. Lindenstrauss [9] proved that every infinite dimensional complemented subspace of l_∞ is isomorphic to l_∞. This also holds if l_∞ is replaced by $l_p, 1 \leq p < \infty$, c_0 or c. It is also shown by Lindenstrauss [10] that if the Banach space X and its closed subspace Y are generated by weakly compact sets (in particular, if X is reflexive) then Y is complemented in X. Thorp [18] has shown that for X and Y, certain Banach spaces of sequences, the subspace $\mathcal{LC}(X,Y)$ of compact linear operators from X to Y cannot be complemented in $\mathcal{L}(X,Y)$, the space of bounded linear operators from X to Y. Arterburn and Whitley [1] proved similar results for either X an abstract $L-$space or Y a space of type $C(S)$ and considered projections on the subspace $W(X,Y)$ of weakly compact linear operators mapping X to Y. Tong [19] studied the existence of bounded projections from the space $\mathcal{L}(X,Y)$ onto the subspace $\mathcal{LC}(X,Y)$, where X and Y are normed spaces. Johnson [5] showed that if X and Y are Banach spaces, the space X is infinite dimensional and if Y contains a complemented copy of c_0, then $\mathcal{LC}(X,Y)$ cannot be complemented in $\mathcal{L}(X,Y)$. Kuo [7] showed that if X and Z are Banach spaces and (a) if X contains an isomorph of c_0, then $\mathcal{LC}(X,l^\infty)$ cannot be complemented in $\mathcal{L}(X,l^\infty)$ (b) if S is a compact Hausdorff space which is not scattered, then $\mathcal{LC}(C(S),Z)$ cannot be complemented in $W(C(S),Z)$ for $Z=c_0$ or l^∞. In particular, $\mathcal{LC}(l^\infty,c_0)$ cannot be complemented in $\mathcal{L}(l^\infty,c_0)$.

Let $\mathcal{L}(X)$ be the set of all bounded linear operators from the Banach space X into itself and $\mathcal{LC}(X)$ be the set of all compact operators in $\mathcal{L}(X)$. Conway [3] has established that $\mathcal{LC}(H)$ cannot be complemented in $\mathcal{L}(H)$ where H is an infinite dimensional separable Hilbert space.

Let

$$l_\infty^{n\times n} = \{f:\mathbb{N}\to\mathbb{C}^{n\times n} : f(m)=A_m^{n\times n} \text{ and } \sup_m \|f(m)\|_{\mathbb{C}^{n\times n}} < \infty\}$$

and

$$c_0^{n\times n} = \{f:\mathbb{N}\to\mathbb{C}^{n\times n} : f(m)=A_m^{n\times n} \text{ and } \|f(m)\|_{\mathbb{C}^{n\times n}} \to 0 \text{ as } m\to\infty\}.$$

It is not difficult to see that $c_0^{n\times n}\subset l_\infty^{n\times n}$. If $f\in l_\infty^{n\times n}$, we define $\|f\| = \sup_m \|f(m)\|_{\mathbb{C}^{n\times n}}$. The space $l_\infty^{n\times n}$ with the sup norm is a Banach space and $c_0^{n\times n}$ is a closed subspace of $l_\infty^{n\times n}$.

Let H be a Hilbert space and

$$l_\infty(H) = \{f:\mathbb{N}\to H : f(n)=x_n\in H$$

and

$$\sup_n \|f(n)\|_H < \infty\}$$

and

$$c_0(H) = \{f : \mathbb{N} \to H : f(n) = x_n \in H, f \in l_\infty(H)$$

and

$$\|f(n)\|_H \to 0 \text{ as } n \to \infty\}.$$

It is also not difficult to verify that $c_0(H) \subset l_\infty(H)$. The space $l_\infty(H)$ is a Banach space [8] with the norm $\|f\| = \sup_n \|f(n)\|_H$ and $c_0(H)$ is a closed subspace of $l_\infty(H)$.

Now let $(X, \|\cdot\|_X)$ be a Banach space. Define

$$l_\infty(X) = \{f : \mathbb{N} \to X : f(n) = x_n \in X \text{ and } \sup_n \|f(n)\|_X < \infty\}.$$

The space $l_\infty(X)$ is a Banach space with the norm $\|f\| = \sup_n \|f(n)\|_X$ and define

$$c_0(X) = \{f : \mathbb{N} \to X : f(n) = x_n \in X, f \in l_\infty(X)$$

and

$$\|f(n)\|_X \to 0 \text{ as } n \to \infty\}.$$

The space $c_0(X)$ is a closed subspace of $l_\infty(X)$. In this paper we have shown that the space $c_0^{n\times n}$ cannot be complemented in $l_\infty^{n\times n}$ and $c_0(H)$ cannot be complemented in $l_\infty(H)$ where H is a separable Hilbert space. Further, extending these results we show that if X is a Banach space then $c_0(X)$ cannot be complemented in $l_\infty(X)$.

For this we need to introduce the Hardy space. Let $\mathbb{T}$ denote the unit circle in the complex plane $\mathbb{C}$. Let $d\theta$ be the arc-length measure on $\mathbb{T}$. For $1 \le p \le +\infty$, $L^p(\mathbb{T})$ will denote the Lebesgue space of $\mathbb{T}$ induced by $\dfrac{d\theta}{2\pi}$. Given $f \in L^1(\mathbb{T})$, the Fourier coefficients of f are

$$a_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta, \quad n \in \mathbb{Z} \tag{1}$$

where $\mathbb{Z}$ is the set of all integers. Let $\mathbb{Z}_+$ denote the set of nonnegative integers. For $1 \le p \le +\infty$, the Hardy space of $\mathbb{T}$ denoted by $H^p(\mathbb{T})$, is the subspace of $L^p(\mathbb{T})$ consisting of functions f with $a_n(f) = 0$ for all negative integers n. We shall let $H^p(\mathbb{D})$ denote the space of analytic functions on $\mathbb{D}$ which are harmonic extensions of functions in $H^p(\mathbb{T})$. The Hardy space $H^2(\mathbb{D})$ is a reproducing kernel Hilbert space and the reproducing kernel (called the Cauchy or Szego kernel) $K_w(z) = \dfrac{1}{1-\bar{w}z}$, for $z, w \in \mathbb{D}$. It is not so important to distinguish $H^p(\mathbb{D})$ from $H^p(\mathbb{T})$. The sequence of functions $\{e^{in\theta}\}_{n\in\mathbb{Z}_+}$ forms an orthonormal basis for $H^2(\mathbb{T})$.

Let $\mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T})$ denote the Hilbert space of $\mathbb{C}^n$-valued, norm square integrable, measurable functions on $\mathbb{T}$. When endowed with the inner product defined by the equality

$$\langle f, g\rangle = \int_{\mathbb{T}} \langle f(z), g(z)\rangle_{\mathbb{C}^n} dm, f, g \in \mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T}),$$

the space $\mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T})$ becomes a separable Hilbert space. Here the measure m denotes the normalized Lebesgue measure on $\mathbb{T}$. For a function $F \in \mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T})$, we define the nth Fourier coefficient of F as

$$c_n(F) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}F(e^{it})dt, n \in \mathbb{Z}.$$

The integral is understood in the strong sense. Let $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T})$ be the Hardy space of functions in $\mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T})$ with vanishing negative Fourier coefficients. Notice that $\mathcal{L}^2_{\mathbb{C}^n}(\mathbb{T}) = L^2(\mathbb{T}) \otimes \mathbb{C}^n$ and $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}) = H^2(\mathbb{T}) \otimes \mathbb{C}^n$ where the Hilbert space tensor product is used.

2 The space $l_\infty^{n\times n}$

In this section we show that $c_0^{n\times n}$ cannot be complemented in $l_\infty^{n\times n}$. For this purpose, we introduce the operators U_a and V_a acting on a direct sum $\sum\limits_{b\in\mathbb{B}} \oplus H_b$, with each H_b the same Hilbert space H. Define the bounded linear operators

$$U_a : H \longrightarrow \sum \oplus H_b,\ V_a : \sum \oplus H_b \longrightarrow H,$$

for each a in $\mathbb{B}$, as follows. When $x \in H$ and $u = \{x_b\} \in \sum \oplus H_b, V_a u = x_a$ and $U_a x$ is the family $\{z_b\}$ in which $z_a = x$ and all other z_b are 0; H'_a is the range of U_a, and so consists of all elements $\{z_b\}$ of $\sum \oplus H_b$ in which $z_b = 0$ when $b \neq a$. The space H'_a is a closed subspace of $\sum \oplus H_b$, and observe that $V_a U_a$ is the identity operator on H and $U_a V_a$ is the projection E_a from $\sum \oplus H_b$ onto H'_a. Since the subspaces $H'_a (a \in \mathbb{B})$ are pairwise orthogonal, and $\vee H'_a = \sum \oplus H_b$, it follows that the sum $\sum\limits_{a\in\mathbb{B}} E_a$ is strong-operator convergent to I. Note that $U_a = V_a^*$, since

$$\langle U_a x, \{x_b\}\rangle = \langle x, x_a\rangle = \langle x, V_a\{x_b\}\rangle$$

whenever $x \in H$ and $\{x_b\} \in \sum \oplus H_b$. With each bounded linear operator T acting on $\sum \oplus H_b$, we associate a matrix $[T_{ab}]_{a,b\in\mathbb{B}}$, with entries T_{ab} in $\mathcal{L}(\mathcal{H})$ defined by

$$T_{ab} = V_a T U_b. \tag{2}$$

If $u = \{x_b\} \in \sum \oplus H_b$, then Tu is an element $\{y_b\}$ of $\sum \oplus H_b$, and

$$y_a = V_a T u = V_a T \left(\sum_b E_b u \right) = \sum_b V_a T U_b V_b u = \sum_b T_{ab} x_b.$$

Thus

$$T\left(\sum \oplus x_b\right) = \sum \oplus y_b \text{ where } y_a = \sum_{b\in\mathbb{B}} T_{ab} x_b \ (a \in \mathbb{B}). \tag{3}$$

The usual rules of matrix algebra have natural analogues in this situation. From (2.1), the matrix elements T_{ab} depend linearly on T. Since

$$V_a T^* U_b = U_a^* T^* V_b^* = (V_b T U_a)^* = (T_{ba})^*,$$

the matrix of T^* has $(T_{ba})^*$ in the (a, b) position. If S and T are bounded linear operators acting on $\sum \oplus H_b$, and $R = ST$, then

$$R_{ab} = V_a R U_b = V_a S T U_b = \sum_{c\in\mathbb{B}} V_a S E_c T U_b$$

$$= \sum_{c\in\mathbb{B}} V_a S U_c V_c T U_b = \sum_{c\in\mathbb{B}} S_{ac} T_{cb},$$

the sum converging in the strong-operator topology if the index set $\mathbb{B}$ is infinite. In this way, we establish a one-to-one correspondence between elements of $\mathcal{L}(\sum_{b\in\mathbb{B}} \oplus H_b)$ and certain matrices $[T_{ab}]_{a,b\in\mathbb{B}}$ with entries T_{ab} in $\mathcal{L}(\mathcal{H})$. When the index set $\mathbb{B}$ is finite, each such matrix corresponds to some bounded operator T acting on $\sum \oplus H_b$; indeed, T is defined by (2.2), and its boundedness follows at once from the relations

$$||\{y_b\}||^2 = \sum_a ||y_a||^2 = \sum_a ||\sum_b T_{ab} x_b||^2 \leq \sum_a \left(\sum_b ||T_{ab}||\ ||x_b|| \right)^2$$

$$\leq \sum_a \left(\sum_b ||T_{ab}||^2 \right) \left(\sum_b ||x_b||^2 \right) = \left(\sum_a \sum_b ||T_{ab}||^2 \right) ||\{x_b\}||^2.$$

Theorem 1. *The space $c_0^{n\times n}$ cannot be complemented in $l_\infty^{n\times n}$.*

Proof. The set of vectors $\{e_n\}_{n=0}^{\infty}$ form an orthonormal basis for $H^2(\mathbb{T})$ where $e_n(z) = z^n, z \in \mathbb{T}$. It is easy to verify that $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}) = H^2(\mathbb{T}) \oplus H^2(\mathbb{T}) \oplus \cdots \oplus H^2(\mathbb{T})$ (direct sum of n-copies of $H^2(\mathbb{T})$) and if $T \in \mathcal{L}(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}))$ then

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix},$$

$T_{ij} \in \mathcal{L}(H^2(\mathbb{T})), 1 \leq i, j \leq n$. Define the maps V_i and $U_i, 1 \leq i \leq n$, as we have discussed earlier in this section on the space $\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}) = H^2(\mathbb{T}) \oplus H^2(\mathbb{T}) \oplus \cdots \oplus H^2(\mathbb{T})$. It is not difficult to see that $T_{ij} = V_i T U_j, 1 \leq i, j \leq n$. Further, define the map $\sigma : \mathcal{L}(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}))$ into $l_\infty^{n\times n}$ as

$$\sigma(T)(m) = \frac{1}{n} \begin{pmatrix} \langle T_{11}e_m, e_m\rangle & \langle T_{12}e_m, e_m\rangle & \cdots & \langle T_{1n}e_m, e_m\rangle \\ \langle T_{21}e_m, e_m\rangle & \langle T_{22}e_m, e_m\rangle & \cdots & \langle T_{2n}e_m, e_m\rangle \\ \vdots & \vdots & \cdots & \vdots \\ \langle T_{n1}e_m, e_m\rangle & \langle T_{n2}e_m, e_m\rangle & \cdots & \langle T_{nn}e_m, e_m\rangle \end{pmatrix},$$

if $T = (T_{ij})_{1\leq i,j\leq n}$. Notice that $\|T_{ij}\| = \|V_i T U_j\| \leq \|T\|$. Then

$$\begin{aligned} \|\sigma(T)\| &= \sup_m \|\sigma(T)(m)\|_{\mathbb{C}^{n\times n}} \\ &= \sup_m \max_{1\leq j\leq n} \tfrac{1}{n} \textstyle\sum_{i=1}^n |\langle T_{ij}e_m, e_m\rangle| \\ &\leq \sup_m \max_{1\leq j\leq n} \tfrac{1}{n} \textstyle\sum_{i=1}^n \|T_{ij}\| \\ &\leq \sup_m \max_{1\leq j\leq n} \tfrac{1}{n} n\|T\| = \|T\|. \end{aligned}$$

Thus σ is a contraction. Now if $T \in \mathcal{LC}(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}))$, then $T_{ij} \in \mathcal{LC}(H^2(\mathbb{T}))$ for all $i, j \in \{1, 2 \cdots, n\}$ and $|\langle T_{ij}e_m, e_m\rangle| \to 0$ as $m \to \infty$. Hence $\|\sigma(T)(m)\| \to 0$ as $m \to \infty$. Thus $\sigma(\mathcal{LC}(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}))) \subset c_0^{n\times n}$.

Define $\rho : l_\infty^{n\times n} \to \mathcal{L}(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}))$ as $\rho(F) = T$ where

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix},$$

and $T_{ij} = \frac{1}{n} \sum_{m=1}^{\infty} F_{ij}(m) P_{\mathrm{sp}\{e_m\}}$ if

$$F = \begin{pmatrix} F_{11} & F_{12} & \cdots & F_{1n} \\ F_{21} & F_{22} & \cdots & F_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ F_{n1} & F_{n2} & \cdots & F_{nn} \end{pmatrix} \in l_\infty^{n\times n}.$$

Now $\|F\|_{l_\infty^{n\times n}} = \sup_m \|F(m)\|_{\mathbb{C}^{n\times n}} = \sup_m \max_{1\le j\le n} \sum_{i=1}^n |F_{ij}(m)|$. Hence

$$\begin{aligned}
\|\rho(F)\| &= \|T\| \\
&\le (\textstyle\sum_{i,j=1}^n \|T_{ij}\|^2)^{\frac{1}{2}} \\
&\le (\textstyle\sum_{i,j=1}^n (\frac{1}{n}\sup_m \|F(m)\|)^2)^{\frac{1}{2}} \\
&= \left[\textstyle\sum_{i,j=1}^n \left(\frac{1}{n^2}(\sup_m \|F(m)\|)^2\right)\right]^{\frac{1}{2}} \\
&= \sup_m \|F(m)\|_{\mathbb{C}^{n\times n}} \\
&= \|F\|_{l_\infty^{n\times n}}.
\end{aligned}$$

This proves that ρ is a contraction and $\rho(c_0^{n\times n}) \subset \mathcal{LC}(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}))$. Now if there is a projection P from $l_\infty^{n\times n}$ onto $c_0^{n\times n}$, then $Q = \rho\circ P\circ\sigma$ is a projection from $\mathcal{L}(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}))$ onto $\mathcal{LC}(\mathcal{H}^2_{\mathbb{C}^n}(\mathbb{T}))$ since $Q^2 = Q$. This is a contradiction since $\mathcal{LC}(H)$ cannot be complemented [3] in $\mathcal{L}(H)$ if H is an infinite dimensional separable Hilbert space. □

3 Projection onto $c_0(H)$

In this section we shall establish that $c_0(H)$ cannot be complemented in $l_\infty(H)$ if H is an infinite dimensional separable Hilbert space.

Theorem 2. *If H is an infinite dimensional separable Hilbert space, then $c_0(H)$ cannot be complemented in $l_\infty(H)$.*

Proof. Let $\{e_n\}_{n=0}^\infty$ be an orthonormal basis for H. Then it is well known [8] that $e_n \to 0$ weakly. Define $\sigma : \mathcal{L}(H) \to l_\infty(H)$ as $\sigma(T)(n) = Te_n$ for all $n \in \mathbb{N}$. The map σ is well defined and is a contraction since

$$\sup_n \|\sigma(T)(n)\| = \sup_n \|Te_n\| \le \|T\| < \infty.$$

If $T \in \mathcal{LC}(H)$, then $Te_n \to 0$ strongly and $\|\sigma(T)(n)\| = \|Te_n\| \to 0$ as $n \to \infty$. Thus $\sigma(T) \in c_0(H)$ and $\sigma(\mathcal{LC}(H)) \subset c_0(H)$.

Now define $\rho : l_\infty(H) \to \mathcal{L}(H)$ as $\rho(F) = T$ where

$$T = \sum_{n=0}^\infty \langle F(n), e_n\rangle P_{\mathrm{sp}\{e_n\}}$$

if $F = (F(n))_{n=0}^\infty \in l_\infty(H)$. Notice that if $F \in l_\infty(H)$, then

$$\|F\|_{l_\infty(H)} = \sup_n \|F(n)\|_H < \infty$$

and $\|T\| = \sup_n |\langle F(n), e_n\rangle| \leq \sup_n \|F(n)\|_H < \infty$. Hence the map ρ is well-defined, $\rho(c_0(H)) \subseteq \mathcal{LC}(H)$ and

$$\begin{aligned}\|\rho(F)\| &= \|T\| = \sup_n |\langle F(n), e_n\rangle| \leq \sup_n \|F(n)\|_H \\ &= \|F\|_{l_\infty(H)}.\end{aligned}$$

Thus the map ρ is a contraction. Now if there exists a projection P from $l_\infty(H)$ onto $c_0(H)$ then $Q = \rho \circ P \circ \sigma$ is a projection from $\mathcal{L}(H)$ onto $\mathcal{LC}(H)$. Since there exists no projection [3] from $\mathcal{L}(H)$ onto $\mathcal{LC}(H)$, hence there exists no projection from $l_\infty(H)$ onto $c_0(H)$. $\square$

4 Projection onto $c_0(X)$

In this section we shall establish that $c_0(X)$ cannot be complemented in $l_\infty(X)$ where X is an infinite dimensional separable Banach space.

Definition 1. *Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{C}$. A countable subset $\{x_1, x_2, \cdots\}$ of X is called a Schauder basis for X if $\|x_n\| = 1$ for each n and if for every $x \in X$, there are unique $\alpha_1, \alpha_2, \cdots$ in $\mathbb{C}$ such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$ with the series converging in the norm of X.*

If $\{x_1, x_2, \cdots\}$ is a Schauder basis for X, then for $n = 1, 2, \cdots$, define $f_n : X \to \mathbb{C}$ by $f_n(x) = \alpha_n$, for $x = \sum_{n=1}^{\infty} \alpha_n x_n \in X$. The uniqueness condition in the definition of a Schauder basis shows that each f_n is well-defined and linear on X. It is called the nth coefficient functional on X corresponding to the Schauder basis $\{x_1, x_2, \cdots\}$ for X. If X is a Banach space then it is not difficult to verify [8] that each f_n is a continuous linear functional and $\|f_n\| \leq \alpha$ for all $n = 1, 2, \cdots$ and some $\alpha > 0$. If $x = \sum_{n=1}^{\infty} \alpha_n x_n$, define $\|x\|_\infty = \sup_n \|\sum_{k=1}^{n} \alpha_k x_k\|$. Then $\|\cdot\|_\infty$ is a norm and $(X, \|\cdot\|_\infty)$ is complete. The norm $\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent. Let X be a Banach space, $U^* = \{f : f \in X^* \text{ and } \|f\| \leq 1\}$ be the unit ball of X^* and let E be the set of extreme points of U^*.

Theorem 3. *Let X be a separable Banach space and $\{x_n\}_{n=1}^{\infty}$ be a Schauder basis for X. Suppose $\lim_{n\to\infty} f(x_n) = f(0)$ for each f in E. Then $c_0(X)$ cannot be complemented in $l_\infty(X)$.*

Proof. Let X be a separable Banach space with a Schauder basis $\{x_n\}_{n=1}^{\infty}$ such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$. From [16], it follows that $\lim_{n\to\infty} f(x_n) = f(0)$ for each f in E if and only if $x_n \to 0$ weakly. Define $\sigma : \mathcal{L}(X) \to l_\infty(X)$ such that $\sigma(T)(n) = Tx_n$. Since $\|\sigma(T)\| = \sup_n \|\sigma(T)(n)\|_X = \sup_n \|Tx_n\| \leq \|T\|\|x_n\| = \|T\|$. The map σ is well-defined and σ is a contraction. Further, if $T \in \mathcal{LC}(X)$, then $Tx_n \to 0$ strongly. Hence $\|\sigma(T)(n)\| = \|Tx_n\| \to 0$ strongly and in this case $\sigma(T) \in c_0(X)$. Thus $\sigma(\mathcal{LC}(X)) \subset c_0(X)$. Now define a map $\rho : l_\infty(X) \to \mathcal{L}(X)$ as $\rho(F) = T$ where $T = \sum_{n=1}^{\infty} f_n(F(n)) P_{\mathrm{sp}\{x_n\}}$ where f_n's are the coordinate functions. If $F \in l_\infty(X)$ then F is a function from $\mathbb{N}$ to X such that $\sup_n \|F(n)\|_X < \infty$ and $F(n) \in X$. If $F(n) = \sum_{j=1}^{\infty} \alpha_j x_j, \alpha_j \in \mathbb{C}$ then $f_j(F(n)) = \alpha_j$ for all $j \in \mathbb{N}$. The map ρ is well-defined since

$$\begin{aligned} \|\rho(F)\| &= \|T\| \\ &= \sup_n |f_n(F(n))| \\ &\leq \alpha \sup_n \|F(n)\|_X \\ &= \alpha \|F\|_\infty \end{aligned}$$

for some constant $\alpha > 0$ and $\rho(c_0(X)) \subset \mathcal{LC}(X)$. Now if there exists a projection P from $l_\infty(X)$ onto $c_0(X)$ then $Q = \rho \circ P \circ \sigma$ is a projection from $\mathcal{L}(X)$ onto $\mathcal{LC}(X)$. But from [19] it follows that there exists no bounded projection from $\mathcal{L}(X)$ onto $\mathcal{LC}(X)$. Hence there exists no bounded projection P from $l_\infty(X)$ onto $c_0(X)$. This proves the claim. □

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