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An algorithm for the computation of the generalized solution for implicit systems*

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Abstract

We discuss the solution of implicit systems in the critical case, i.e. when the classical assumptions of the implicit functions theorem are not satisfied. The generalized solution introduced bellow solves such cases and it may not be a manifold. In certain examples, it may have a complex structure and its approximation is nontrivial. We present here an algorithm for the approximation of the generalized solution. Numerical tests are also included.

MSC:26B10, 34A12, 53A05

keywords: implicit function theorem, differential equations, parametrization, generalized solution, critical case, approximation

1 Introduction

In this paper, we discuss the approximation of the solution for implicit functions systems, in the critical case. The method we use was introduced in [6] and was further studied in [4] and [7]. It is based on iterated systems of ordinary differential equations, to obtain the solution in parametric form.

We investigate a new algorithm solving this question.

This paper is organized as follows. In section two we recall some preliminary notions and results from [6] and [4]. Section three describes the

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algorithm. In section four we give some numerical examples in dimension two and three, computed with MatLab.

For general references on the subject of implicit functions and parametrizations, we quote [3], [2]. In [3], Ch. 5 it is specified that a general solution of the critical case is not known.

2 Preliminaries

In dimension two, the problem we study is given by the implicit equation:

$$f(x,y)=0, f(x_0,y_0)=0, \tag{1}$$

where $(x_0,y_0)\in\Omega\subset\mathcal{R}^2$, an open subset and $f\in C^1(\Omega)$. Consider the critical case, i.e.:

$$\nabla f(x_0,y_0)=0, \tag{2}$$

There exists $(x^n,y^n)\in\Omega$, $(x^n,y^n)\to(x_0,y_0)$, such that $\nabla f(x^n,y^n)\neq 0$, $\forall n$. Otherwise f is identically null in a neighborhood of (x_0,y_0).

We solve (1) with the initial condition (x^n,y^n). We use the Hamiltonian system (see [6]):

$$\begin{aligned} x_n'(t) &= -\frac{\partial f}{\partial y}(x_n,y_n) \\ y_n'(t) &= \frac{\partial f}{\partial x}(x_n,y_n) \end{aligned} \tag{3}$$

with the initial condition: $x_n(0)=x^n,\ y_n(0)=y^n$.

By Peano's theorem [1], we know that system (3) has a local solution on some interval I_{max} that may be chosen independent of n.

We consider a closed disc D, with the center in $(x_0,y_0)\in\Omega\subset R^2$ and the set:

$$T_n=\{(x,y)\in D;(x,y)=(x_n(t),y_n(t)),t\in\bar{I}_{max}\}$$

.

We consider a convergent subsequence $T_{n_k}\to T_\alpha$ in the Hausdorff-Pompeiu sense and put $T=\cup_\alpha T_\alpha$, where α is the subsequence.

If $f(x_0,y_0)=0$ and relation (2) is true, then T is called the local generalized solution for (1).

In dimension three, we consider the following problem:

$$f(x,y,z)=0, f(x_0,y_0,z_0)=0, \tag{4}$$

where $(x_0,y_0,z_0)\in\Omega\subset R^3$, f is in $C^1(\Omega)$.

Because we discuss the critical case, we have the following condition:

$$\nabla f(x_0, y_0, z_0) = 0.$$

Let $(\tilde{x}_n, \tilde{y}_n, \tilde{z}_n) \to (x_0, y_0, z_0)$ in Ω, such that $\nabla f(\tilde{x}_n, \tilde{y}_n, \tilde{z}_n) \neq 0$ (in fact, we may without loss of generality, fix here $f_x(\tilde{x}_n, \tilde{y}_n, \tilde{z}_n) \neq 0$). The existence of such a sequence follows as before (otherwise f is identically null in a neighborhood of (x_0, y_0, z_0)).

Consider two iterated Hamiltonian systems:

$$\begin{aligned} x_n' &= -f_y(x_n, y_n, z_n), && t \in I_1^n, \\ y_n' &= f_x(x_n, y_n, z_n), && t \in I_1^n, \\ z_n' &= 0, && t \in I_1^n, \\ x_n(0) &= \tilde{x}_n,\ y_n(0) = \tilde{y}_n,\ z_n(0) = \tilde{z}_n; \end{aligned} \tag{5}$$

and

$$\begin{aligned} \dot{\varphi}_n &= -f_z(\varphi_n, \psi_n, \xi_n), && s \in I_2^n(t), \\ \dot{\psi}_n &= 0, && s \in I_2^n(t), \\ \dot{\xi}_n &= f_x(\varphi_n, \psi_n, \xi_n), && s \in I_2^n(t), \\ \varphi_n(0) &= x_n(t),\ \psi_n(0) = y_n(t),\ \xi_n(0) = z_n(t). \end{aligned} \tag{6}$$

where I_1^n and $I_2^n(t)$ are real closed intervals containing 0. It is proved in [4] that I_1^n, $I_2^n(t)$ may be chosen independent of n and t, that is $I_1^n = I_1$ $I_2^n(t) = I_2$.

For $(\varphi_n, \psi_n, \xi_n) : I_1 \times I_2 \to \mathcal{R}^3$, we denote

$$T_n = \{(\varphi_n(t,s), \psi_n(t,s), \xi_n(t,s)); (t,s) \in I_1 \times I_2\}.$$

Like in dimension two, there is a convergent subsequence such that $T_n \to T_\alpha$ in the Hausdorff-Pompeiu metric, where α denotes the subsequence.

So, we can again define the set called the generalized solution of (4):

$$T = \cup_\alpha T_\alpha.$$

Moreover, if we also have $f_y(\tilde{x}_n, \tilde{y}_n, \tilde{z}_n) \neq 0$, one can also consider the following supplementary iterated Hamiltonian system:

$$\begin{aligned} x_n' &= -f_y(x_n, y_n, z_n), && t \in I_1^n, \\ y_n' &= f_x(x_n, y_n, z_n), && t \in I_1^n, \\ z_n' &= 0, && t \in I_1^n, \\ x_n(0) &= \tilde{x}_n,\ y_n(0) = \tilde{y}_n,\ z_n(0) = \tilde{z}_n; \end{aligned} \tag{7}$$

and

$$\begin{aligned} \dot{\varphi}_n &= 0, & s &\in I_2^n(t) \\ \dot{\psi}_n &= -f_z(\varphi_n, \psi_n, \xi_n), & s &\in I_2^n(t), \\ \dot{\xi}_n &= f_y(\varphi_n, \psi_n, \xi_n), & s &\in I_2^n(t), \\ \varphi_n(0) &= x_n(t),\ \psi_n(0) = y_n(t),\ \xi_n(0) &= z_n(t). \end{aligned} \tag{8}$$

Notice that if $\nabla f(\tilde{x}_n, \tilde{y}_n, \tilde{z}_n) \neq 0$, then by a good choice of the axes, one may obtain both $f_x(\tilde{x}_n, \tilde{y}_n, \tilde{z}_n) \neq 0$ and $f_y(\tilde{x}_n, \tilde{y}_n, \tilde{z}_n) \neq 0$. The solutions provided by the Hamiltonian systems or the iterated Hamiltonian systems are local around the initial condition. In dimension three, the solution of the supplementary system (7)- (8) should be taken together with the one of (5)-(6) in order to obtain more information.

3 The Algorithm

In this section we describe the steps of our algorithm. The question is related to the choice of the approximating initial conditions used in the definition of the generalized solution, according to Section 2. We write a unified algorithm in dimension two and three and we denote by $x_0 \in \Omega \subset R^d$, $d = 2$ or $d = 3$, the critical point in the implicit functions problem (9):

$$f(x) = 0, x \in \Omega, f(x_0) = 0,$$

$$\nabla f(x_0) = 0, \tag{9}$$

where $f \in C^1(\Omega)$.

Similar algorithms may be formulated for general implicit systems, as discussed in [7].

Algorithm 2.1

Step 1: Consider $\varepsilon > 0$ and a division of a neighborhood of the initial condition x_0, of dimension ε, in equal parts. We can choose for the neighborhood a sphere or a cube of "dimension" ε and a division of this neighborhood in k parts.

Step 2: We compute the solution for (3) in dimension two and for (5)-(6) in dimension three. These solutions are computed in each of some k points chosen as initial conditions, fixed respectively in the k parts of the division.

Step 3: We make a refinement of the neighborhood by dividing it in $2k$ parts and/or we take its new dimension $\varepsilon/2$.

Step 4: We again compute the approximate solutions for (3) or (5)-(6), in each of the corresponding new $2k$ points.

Step 5: After each new iteration, we compute the Hausdorff-Pompeiu distance between the corresponding obtained solutions. The trajectories taken into account for this computations are truncated to a certain neighborhood of x_0, prescribed from the beginning.

Step 6: If the Hausdorff-Pompeiu distance is less then a certain fixed tolerance, then the algorithm stops. If it is greater than the tolerance, we return to step 3.

For the stopping criterion one can use different conditions. For example, we can also fix from the beginning a maximum number of iterations. In Step 2 and 4 one can also use the systems (7)-(8).

4 Numerical examples

All the computations were performed with MatLab.

Example 1 *Let* $f(x,y) = (x^2-y^2)\left(x^2-\frac{1}{4}y^2\right)\left(x^2-\frac{1}{16}y^2\right)$, *with the critical point* $(x_0,y_0)=(0,0)$.

We have $f(x_0,y_0)=0$ and $\nabla f(x_0,y_0)=0$.

In this example, the initial neighborhood of $(0,0)$ is fixed as $\left(-\frac{3}{16},\frac{3}{16}\right)\times\left(-\frac{1}{8},\frac{1}{8}\right)$.

For the first iteration we take the approximate initial conditions as the four corners of this rectangle. We compute the corresponding truncated solution trajectories that lie in the square $\left[-\frac{1}{4},\frac{1}{4}\right]\times\left[-\frac{1}{4},\frac{1}{4}\right]$.

For the second iteration we add four more approximate initial conditions, the middles of the edges of the above rectangle.

For the third iteration we consider the rectangle $\left(-\frac{3}{32},\frac{3}{32}\right)\times\left(-\frac{1}{16},\frac{1}{16}\right)$ with initial conditions given by the corners and the middles of the edges. In the fourth iteration we add as initial conditions the middles of all the segments formed in the previous iteration. The last computed iteration again halves the edges of the rectangle and takes as approximate initial conditions sixteen similar points as in iteration four.

In Fig. 1, 2, 3, we show the computed trajectories in iterations 1, 3, respectively 5 (in Fig. 3, we also include the exact solution for comparison).

We have stopped the algorithm after five iterations since the result is already satisfactory.

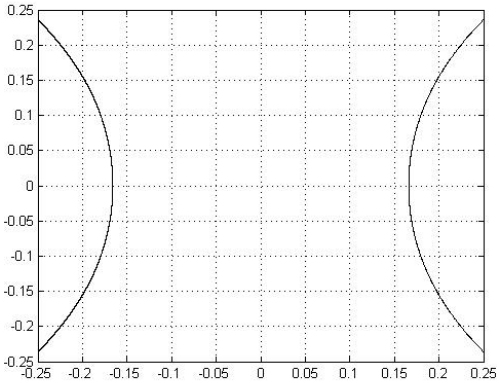

Figure 1: Iteration 1

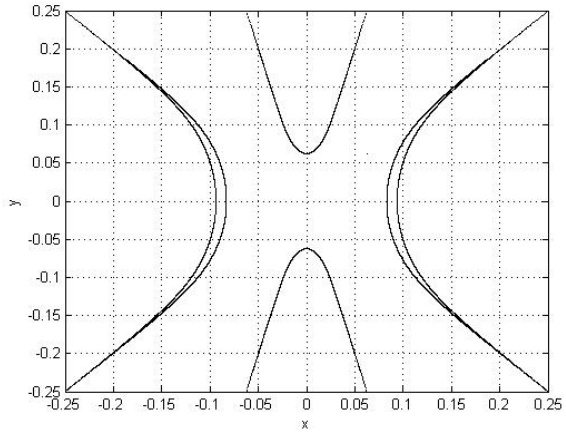

Figure 2: Iteration 3

We have computed the Hausdorff-Pompeiu distance between the obtained trajectories in two consecutive iterations and we have got the values: $h_{12} = 0.191$, $h_{23} = 0.093$, $h_{34} = 0.060$, $h_{45} = 0.046$. We have used the *hausdorff* routine of Hassan Radvar-Esfahlan [5].
In solving the Hamiltonian system (3) we have used the *ode45* routine of MatLab with a fine discretization. Since we work in the neighborhood of a

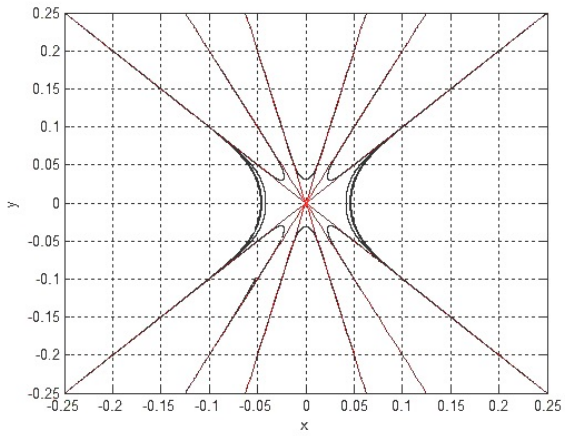

Figure 3: Iteration 5 and the exact solution

critical point, the gradient (the speed) is very small and it is necessary to integrate the equation over long time intervals in order to obtain significant trajectories.

Another variant of our **algorithm** is the following.

We keep fixed in all the iterations the same initial neighborhood of $(0,0)$, for instance as the one in the previous example: $\left(-\frac{3}{16},\frac{3}{16}\right)\times\left(-\frac{1}{8},\frac{1}{8}\right)$.

For the first iteration, we take as approximate initial conditions the corners of the rectangle and compute the solutions that are in the square $\left[-\frac{1}{4},\frac{1}{4}\right]\times\left[-\frac{1}{4},\frac{1}{4}\right]$.

For the second iteration we add the ones that have as approximate initial conditions the middles of the edges of the rectangle.

In the third iteration, we supplement the approximate initial conditions by the middles of all the segments formed in iterations one and two. This process can be, of course, continued.

In Fig. 4 we show the trajectories in iteration three together with the exact solution. In Fig. 5, the fourth iteration together with the exact solution is shown. The computed branches of the solution, in Fig. 5, intersect the true solution, which is impossible theoretically since they represent different level lines. This is due to the very small initial value $f\left(-\frac{3}{64},\frac{1}{8}\right)=2.8012e-8$ and the routine does not distinguish between very close level lines.

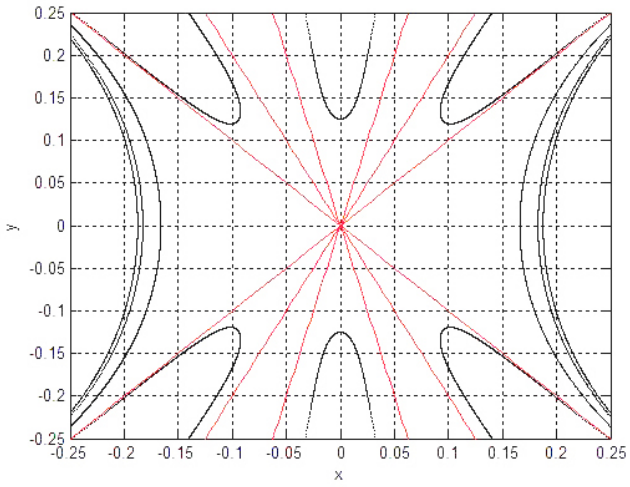

Figure 4: Iteration 3

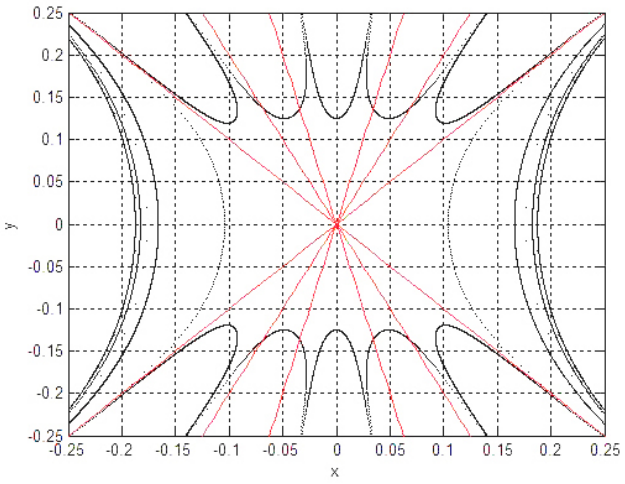

Figure 5: Iteration 4

We compute the Hausdorff-Pompeiu distance between two consecutive iterations, and we obtain the following result: $h_{12} = 0.1913$, $h_{23} = 0.0978$, $h_{34} = 0.0617$. We again used the *hausdorff* routine of Hassan Radvar-Esfahlan [5].

Example 2 *Let* $f(x, y) = (x^2 - y^2)(x^2 + y^2 - 1)$.

We compute the corresponding solutions of (3), with four different initial conditions: $\left(0, \frac{1}{8}\right)$, $\left(\frac{1}{8}, 0\right)$, $\left(0, -\frac{1}{8}\right)$, $\left(-\frac{1}{8}, 0\right)$, around the critical point $(0, 0)$.

This choice of $f(x, y)$ has five critical points and the obtained trajectories, see Fig. 6, look very differently with respect to the previous example.

In case the second variant of the algorithm is used, in this example, in not a very fine neighborhood of $(0, 0)$, with the approximating initial conditions $\left(0, \frac{9}{8}\right)$, $\left(\frac{9}{8}, 0\right)$, $\left(0, -\frac{9}{8}\right)$, $\left(-\frac{9}{8}, 0\right)$, the numerical result would look like in Fig. 7 and would be incorrect. Clearly, if we work with $f_\delta(x, y) = (x^2 - y^2)(x^2 + y^2 - \delta^2)$, $\delta > 0$ small, then such confusions may arise easily. We recommend in each possible example to use various choices of the approximating initial conditions in order to get a good description of the searched solution.

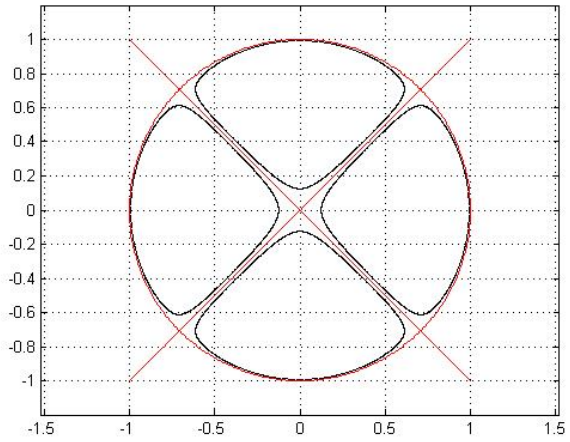

Figure 6: Example 2

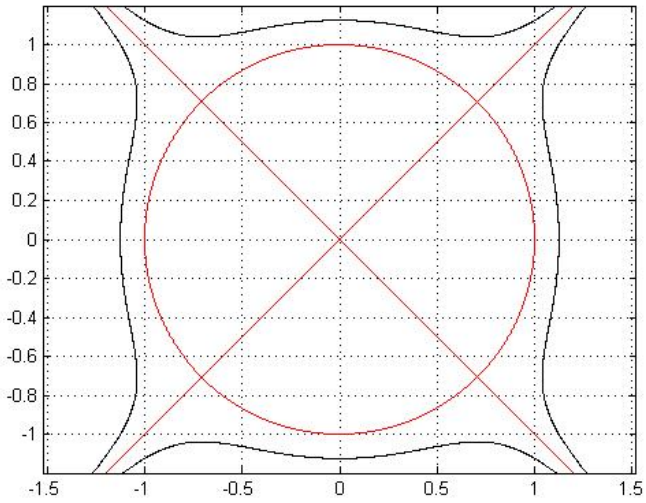

Figure 7: initial conditions $\left(0, \frac{9}{8}\right)$, $\left(\frac{9}{8}, 0\right)$, $\left(0, -\frac{9}{8}\right)$, $\left(-\frac{9}{8}, 0\right)$

Example 3 *Let*

$$f(x, y, z) = (x^2 + y^2 - z^2)(x^2 + y^2 - 4z^2)(x^2 + y^2 - 16z^2), \tag{10}$$

with the critical point $(x_0, y_0, z_0) = (0, 0, 0)$.

We have:

$$f(x_0, y_0, z_0) = (0, 0, 0) \text{ and } \nabla f(x_0, y_0, z_0) = 0.$$

For this example we choose four approximate initial conditions: $\left(\frac{3}{32}, \frac{1}{16}, \frac{1}{10}\right)$, $\left(\frac{3}{32}, \frac{1}{16}, \frac{15}{100}\right)$, $\left(\frac{3}{32}, \frac{1}{16}, \frac{3}{10}\right)$ and $\left(\frac{3}{32}, \frac{1}{16}, \frac{6}{10}\right)$, which are on a vertical line through $\left(\frac{3}{32}, \frac{1}{16}\right)$.

In Fig. 8 we show the upper part (with positive z) of the exact solution for (10) together with the approximating initial conditions and the solutions of the first Hamiltonian system (5). This figure can be rotated to see all

such details. The other half is symmetrical with it. In Fig 9 and Fig. 10 we show the solutions of the second Hamiltonian system (6) corresponding to initial conditions $\left(\frac{3}{32}, \frac{1}{16}, \frac{1}{10}\right)$, respectively $\left(\frac{3}{32}, \frac{1}{16}, \frac{6}{10}\right)$ (the other two initial conditions give similar graphical representations as in Fig. 10). Due to the second equation in (6), the surfaces represented in Fig. 9 and 10 are limited by the planes defined when the second coordinate is constant. One can remove this constraint by using as well the system (7), (8).

Remark 1 *The above numerical examples use various choices of the approximate initial conditions. The chosen points are in a neighborhood of the critical point and should create a "net" around it. The neighborhood should be sufficiently small around the critical point, otherwise the obtained result may be flawed, as explained in Example 2.*

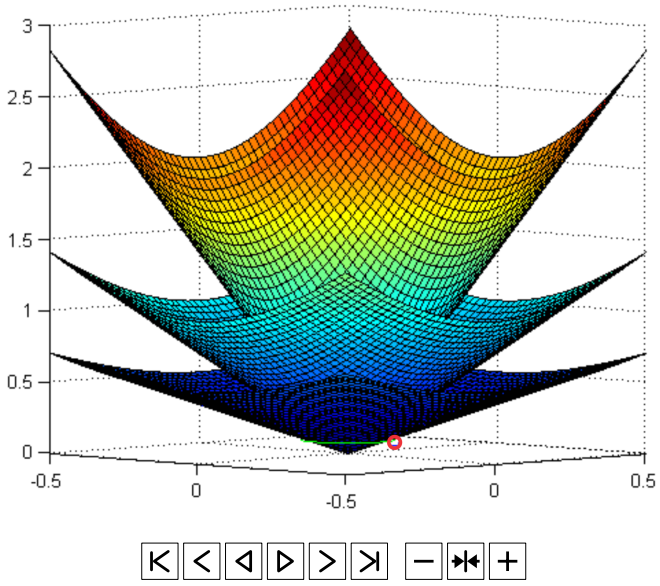

Figure 8: Example 3

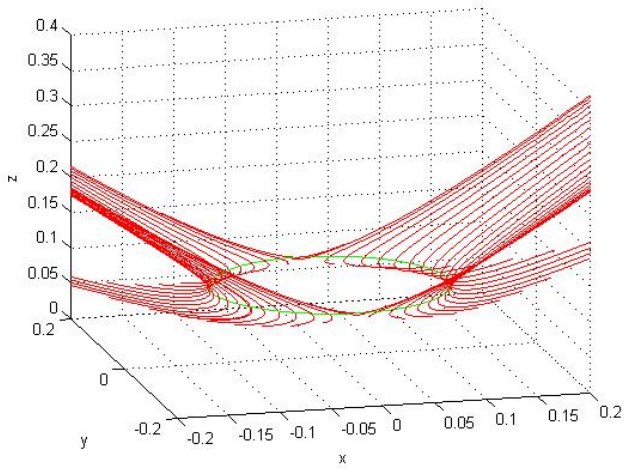

Figure 9: $\left(\frac{3}{32}, \frac{1}{16}, \frac{1}{10}\right)$

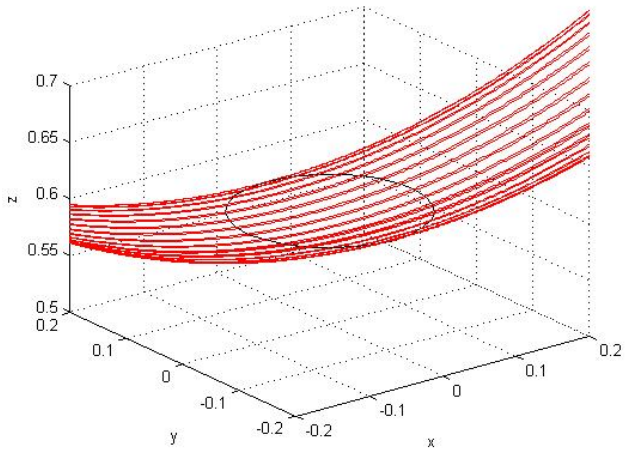

Figure 10: $\left(\frac{3}{32}, \frac{1}{16}, \frac{6}{10}\right)$

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