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PERMANENT SOLUTIONS FOR SOME AXIAL MOTIONS OF GENERALIZED BURGERS FLUIDS IN CYLINDRICAL DOMAINS*

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Abstract

Closed form permanent solutions are determined for two types of oscillating motions of generalized Burgers fluids through an infinite annulus. These solutions, presented in simple forms in terms of some modified Bessel functions, are periodic in time and independent of the initial conditions. They satisfy boundary conditions and governing equations and can easy be reduced to the solutions of Burgers, Oldroyd-B, Maxwell, second grade and linearly viscous fluids performing the same motions. Further, the solutions corresponding to motions through an infinite circular cylinder are obtained as limiting cases of previous solutions and some graphical representations are included.

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1 Introduction

Exact solutions for different initial-boundary value problems are important for many reasons. Such solutions, in addition to serve as approximations

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to some motion problems of fluids, can be used as tests to verify numerical schemes that are developed to study more complex unsteady flows. Although the computer techniques can make a complete numerical integration of the governing equations, the accuracy of results can be established by a comparison with an exact solution.

The motion of a fluid can be induced by the application of a pressure gradient or a body force and by a solid wall that is moving or applies a shear stress to the fluid. If the fluid is initially at rest, its motion can become steady or remain unsteady. Starting solutions for unsteady motions which become steady or permanent in time are important for those who want to eliminate the transients from their rheological experiments. They describe the fluid motion some time after its initiation. After that time, when the transients disappear, the fluid moves according to the permanent solutions. However, as it results from the existing literature [1-5], the required time to reach the time-dependent permanent state for unsteady motions induced by oscillating boundaries is small enough.

Consequently, an important problem for such motions as well as for those due to an oscillating pressure gradient or induced by a solid wall that applies an oscillatory shear stress to the fluid is to determine the permanent components of their solutions. The first exact permanent solutions for oscillatory motions of non-Newtonian fluids seem to be those of Rajagopal [6, 7] and Rajagopal and Bhatnagar [8]. Of course, a part of these solutions have been extended to larger classes of fluids (see [9-13], for instance) but permanent solutions for some axial flows in cylindrical domains are lack in the existing literature.

The purpose of this work is to remove this drawback and to provide exact time-dependent permanent solutions corresponding to oscillatory motions of generalized Burgers fluids in cylindrical domains induced by an oscillating pressure gradient or a circular cylinder that applies a longitudinal oscillatory shear-stress to the fluid. These solutions, that are periodic in time and independent of the initial conditions, satisfy the boundary conditions and governing equations and can be immediately reduced to the similar solutions for Burgers, Oldroyd-B, Maxwell, second grade and linearly viscous fluids. Furthermore, they can be used to develop time-dependent permanent solutions for some rotational oscillatory motions of the same fluids.

2 Constitutive and governing equations

The Cauchy stress tensor $\mathbf{T}$ corresponding to an incompressible generalized Burgers fluid (IGBF) is given by [11]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S},\ \mathbf{S} + \lambda_1 \frac{\delta \mathbf{S}}{\delta t} + \lambda_2 \frac{\delta^2 \mathbf{S}}{\delta t^2} = \mu \left(\mathbf{A} + \lambda_3 \frac{\delta \mathbf{A}}{\delta t} + \lambda_4 \frac{\delta^2 \mathbf{A}}{\delta t^2} \right), \tag{1}$$

where $-p\mathbf{I}$ is the indeterminate spherical stress, $\mathbf{S}$ is the extra-stress tensor, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ is the first Rivlin-Ericksen tensor with $\mathbf{L} = \operatorname{grad} \mathbf{v}$ (the velocity gradient), μ is the dynamic viscosity, λ_1 and λ_3 ($< \lambda_1$) (see [14, Sect. 7]) are relaxation and retardation times while λ_2 and λ_4 are material constants whose dimension is the square of time. Further, the upper convected derivative of a frame-indifferent tensor

$$\frac{\delta \mathbf{S}}{\delta t} = \dot{\mathbf{S}} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T \quad \text{or} \quad \frac{\delta \mathbf{A}}{\delta t} = \dot{\mathbf{A}} - \mathbf{L}\mathbf{A} - \mathbf{A}\mathbf{L}^T, \tag{2}$$

is also frame-indifferent. Here, the superposed dot denotes the material time derivative and the superscript 'T' indicates the transpose operation.

This fluid model contains as special cases Burgers, Oldroyd-B, Maxwell and linearly viscous fluids for $\lambda_4 = 0$, $\lambda_2 = \lambda_4 = 0$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$, respectively $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. In some special flows, like those to be here considered, the governing equations corresponding to IGBF resemble those for second grade fluids. Consequently, the solutions corresponding to the above mentioned fluids performing the same motions have to be obtained as limiting cases of present solutions.

As the fluid is incompressible, it can undergo only isochoric motions and therefore $\operatorname{tr} \mathbf{A} = \operatorname{div} \mathbf{v} = 0$. The balance of linear momentum in the absence of body forces becomes

$$-\operatorname{grad} p + \operatorname{div} \mathbf{S} = \rho \dot{\mathbf{v}}, \tag{3}$$

where $\mathbf{v}$ denotes the fluid velocity and ρ is its constant density. In the following we shall consider oscillatory motions of an IGBF in circular cylindrical domains. For such motions we assume a velocity field of the form

$$\mathbf{v} = \mathbf{v}(r,t) = v(r,t)\mathbf{k}, \tag{4}$$

where $\mathbf{k}$ is the unit vector along the z-direction of the cylindrical coordinate system r, θ and z. For such motions, the constraint of incompressibility is automatically satisfied. We also assume that the extra-stress tensor $\mathbf{S}$, as well as the velocity $\mathbf{v}$, is a function of r and t only.

If the fluid has been at rest up to the moment $t = 0$, Eqs. $(1)_2$ and (4) imply $S_{rr} = S_{r\theta} = S_{\theta\theta} = S_{\theta z} = 0$ while the non-trivial shear stress $\tau(r,t) = S_{rz}(r,t)$ satisfies the partial differential equation

$$\left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right) \tau(r,t) = \mu \left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right) \frac{\partial v(r,t)}{\partial r} \cdot \tag{5}$$

Proceeding with the analysis, the momentum equation (3) reduces to [8, Eq. (22.3)]

$$\rho \frac{\partial v(r,t)}{\partial t} = -\frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} [r\tau(r,t)], \tag{6}$$

where $\partial p/\partial z$ is at most a function of time.

By now eliminating $\tau(r,t)$ between Eqs. (5) and (6) we obtain the governing equation for velocity, namely

$$\begin{aligned} \left(1 + \lambda_1 \tfrac{\partial}{\partial t} + \lambda_2 \tfrac{\partial^2}{\partial t^2}\right) \tfrac{\partial v(r,t)}{\partial t} &= -\tfrac{1}{\rho} \left(1 + \lambda_1 \tfrac{\partial}{\partial t} + \lambda_2 \tfrac{\partial^2}{\partial t^2}\right) \tfrac{\partial p}{\partial z} \\ &+ \nu \left(1 + \lambda_3 \tfrac{\partial}{\partial t} + \lambda_4 \tfrac{\partial^2}{\partial t^2}\right) \left(\tfrac{\partial^2}{\partial r^2} + \tfrac{1}{r} \tfrac{\partial}{\partial r}\right) v(r,t), \end{aligned} \tag{7}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid.

3 Motions due to an oscillating pressure gradient

Let us assume that an IGBF is at rest in an annular region between two infinite coaxial circular cylinders of radii R_0 and R $(> R_0)$. After time $t = 0^+$ an oscillating pressure gradient

$$-\frac{\partial p}{\partial z} = P\cos(\omega t) \quad \text{or} \quad -\frac{\partial p}{\partial z} = P\sin(\omega t), \tag{8}$$

acts on the inner fluid along the common axis of cylinders. Here, P is the amplitude and ω is the frequency of oscillations. The fluid is gradually moved and its velocity is of the form (4). In order to solve such a problem, we have to determine the solution of the linear partial differential equation (7) with the initial and boundary conditions

$$v(r,0) = \left.\frac{\partial v(r,t)}{\partial t}\right|_{t=0} = \left.\frac{\partial^2 v(r,t)}{\partial t^2}\right|_{t=0} = 0, \tag{9}$$

$$v(R_0, t) = v(R, t) = 0. \tag{10}$$

The starting solutions corresponding to such problems, as it results from the existing literature, are usually presented as a sum of permanent and transient solutions. They describe the fluid motion some time after its initiation. After this time, when the transients disappear, the fluid flows according to the permanent solutions which are independent of the initial conditions. Denoting by $v_c(r, t)$ and $v_s(r, t)$ the time-dependent permanent solutions corresponding to the cosine or sine oscillations of the pressure gradient and by

$$u(r, t) = v_c(r, t) + iv_s(r, t), \tag{11}$$

the complex velocity, it results that $u(r, t)$ has to satisfy the partial differential equation

$$\begin{aligned}\left(1 + \lambda_1 \tfrac{\partial}{\partial t} + \lambda_2 \tfrac{\partial^2}{\partial t^2}\right) \tfrac{\partial u(r,t)}{\partial t} &= \tfrac{P}{\rho} (1 - \omega^2\lambda_2 + i\omega\lambda_1)e^{i\omega t} \\ &+\nu \left(1 + \lambda_3 \tfrac{\partial}{\partial t} + \lambda_4 \tfrac{\partial^2}{\partial t^2}\right) \left(\tfrac{\partial^2}{\partial r^2} + \tfrac{1}{r} \tfrac{\partial}{\partial r}\right) u(r, t),\end{aligned} \tag{12}$$

with the boundary conditions

$$u(R_0, t) = u(R, t) = 0. \tag{13}$$

Due to the previous assumptions concerning the pressure gradient (see Eqs. (8)), we are looking for a solution of the form

$$u(r, t) = U(r)e^{i\omega t}, \tag{14}$$

and determine $U(r)$ from Eq. (12) and the boundary conditions (13). Direct computations show that $U(r)$ has to satisfy the ordinary differential equation

$$\begin{aligned}\tfrac{d^2U(r)}{dr^2} + \tfrac{1}{r} \tfrac{dU(r)}{dr} - \tfrac{i\omega}{\nu} \tfrac{1-\omega^2\lambda_2+i\omega\lambda_1}{1-\omega^2\lambda_4+i\omega\lambda_3} U(r) \\ + \tfrac{P}{\mu} \tfrac{1-\omega^2\lambda_2+i\omega\lambda_1}{1-\omega^2\lambda_4+i\omega\lambda_3} = 0,\end{aligned} \tag{15}$$

with the boundary conditions

$$U(R_0) = U(R) = 0. \tag{16}$$

The general solution of Eq. (15) is of the form

$$U(r) = C_1 I_0(\gamma r) + C_2 K_0(\gamma r) - i\, \frac{P}{\rho\omega}\,, \tag{17}$$

where $I_0(\cdot)$ and $K_0(\cdot)$ are modified Bessel functions of the first and second kind, C_1 and C_2 are arbitrary constants, $-iP/(\rho\omega)$ is a particular solution of this equation and

$$\gamma = \sqrt{\frac{i\omega}{\nu}\,\frac{1-\omega^2\lambda_2+i\omega\lambda_1}{1-\omega^2\lambda_4+i\omega\lambda_3}}\,.$$

Using the boundary conditions (16) and bearing in mind the notation (11), we find that

$$v_c(r,t) = \frac{P}{\rho\omega}\,\mathrm{Re}\{[1+AI_0(\gamma r)+BK_0(\gamma r)]e^{i(\omega t-\pi/2)}\}, \qquad (18)$$

$$v_s(r,t) = \frac{P}{\rho\omega}\,\mathrm{Im}\{[1+AI_0(\gamma r)+BK_0(\gamma r)]e^{i(\omega t-\pi/2)}\}, \qquad (19)$$

where Re and Im denote the real and imaginary parts of that which follows and

$$A = \frac{K_0(\gamma R)-K_0(\gamma R_0)}{I_0(\gamma R)K_0(\gamma R_0)-I_0(\gamma R_0)K_0(\gamma R)},$$

$$B = \frac{I_0(\gamma R_0)-I_0(\gamma R)}{I_0(\gamma R)K_0(\gamma R_0)-I_0(\gamma R_0)K_0(\gamma R)}\,.$$

A simple analysis clearly shows that $v_c(r,t)$ and $v_s(r,t)$, given by Eqs. (18) and (19), satisfy the boundary conditions (10).

Now, for completion, we also present the similar solutions

$$v_c(r,t) = \frac{P}{\rho\omega}\,\mathrm{Re}\left\{\left[1-\frac{I_0(\gamma r)}{I_0(\gamma R)}\right]e^{i(\omega t-\pi/2)}\right\}, \qquad (20)$$

$$v_s(r,t) = \frac{P}{\rho\omega}\,\mathrm{Im}\left\{\left[1-\frac{I_0(\gamma r)}{I_0(\gamma R)}\right]e^{i(\omega t-\pi/2)}\right\}, \qquad (21)$$

corresponding to the same motions through an infinite circular cylinder of radius R. These solutions can be obtained as a limiting case of Eqs. (18) and (19) (by making $R_0 \to 0$) or following the same way as before. By now letting $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ into Eqs. (18), (19) or (20), (21), the solutions corresponding to linearly viscous fluids performing the same motion are obtained. Eqs. (20) and (21), for instance, become

$$v_c(r,t) = \frac{P}{\rho\omega}\,\mathrm{Re}\left\{\left[1-\frac{I_0\left((i+1)r\sqrt{\omega/(2\nu)}\right)}{I_0\left((i+1)R\sqrt{\omega/(2\nu)}\right)}\right]e^{i(\omega t-\pi/2)}\right\}, \qquad (22)$$

$$v_s(r,t) = \frac{P}{\rho\omega}\,\mathrm{Im}\left\{\left[1-\frac{I_0\left((i+1)r\sqrt{\omega/(2\nu)}\right)}{I_0\left((i+1)R\sqrt{\omega/(2\nu)}\right)}\right]e^{i(\omega t-\pi/2)}\right\}, \qquad (23)$$

In Figs. 1 and 2, for comparison, the profiles of velocities $v_c(r,t)$ and $v_s(r,t)$ corresponding to motions through an infinite circular cylinder are presented for different values of physical parameters. As expected, the velocity diagrams corresponding to generalized Burgers fluids tend to superpose over those of Newtonian fluids when $\lambda_i \to 0$ $(i = 1, 2, 3, 4)$.

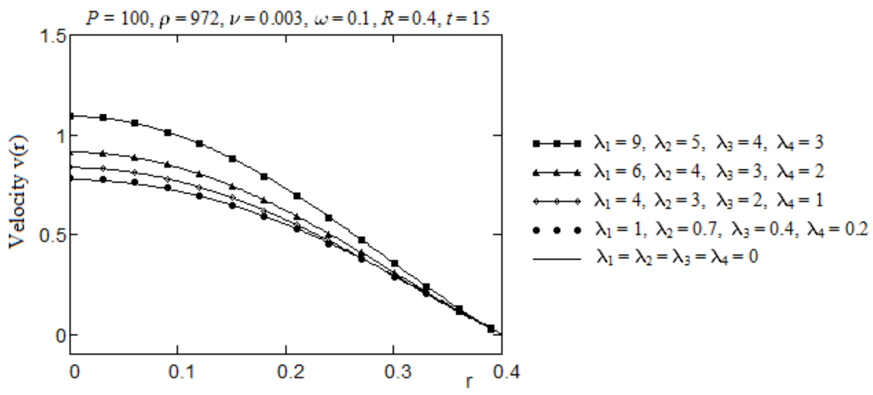

Fig. 1. Profiles of the velocity $v_c(r, t)$ given by Eqs. (20) and (22).

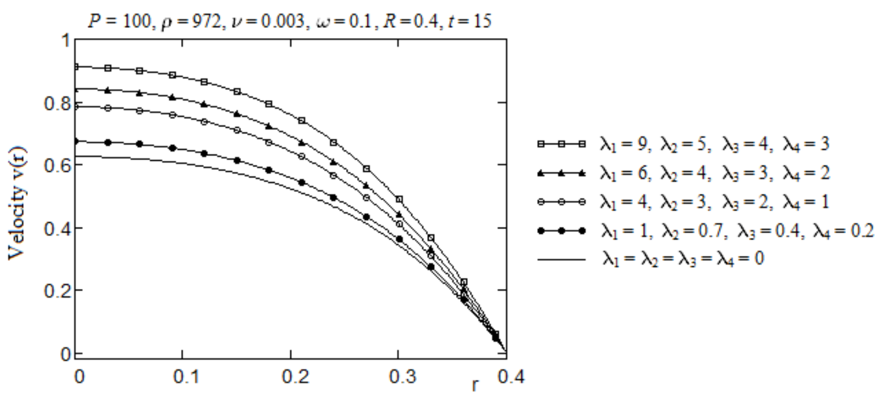

Fig. 2. Profiles of the velocity $v_s(r, t)$ given by Eqs. (21) and (23).

4 Motion induced by an infinite cylinder that applies an oscillatory longitudinal shear stress to the fluid

The flow between circular cylinders or through a cylinder is one of the most important and interesting problems of motion. It has been intensively studied and during recent years many papers of this type have been published. In the following, unlike the previous works, we shall consider the motion of an IGBF produced by an oscillatory shear stress on the boundary.

4.1 Motion between circular cylinders

Consider again an IGBF at rest in the same annular region as before. At time $t = 0^+$ the outer cylinder of radius R applies an oscillatory longitudinal shear stress $f\sin(\omega t)$ or $f\cos(\omega t)$ to the fluid while the inner one of radius R_0 is fixed. Owing to the shear the fluid between cylinders is gradually moved and its velocity is again of the form (4). Assuming that the extra-stress $\mathbf{S}$ is also a function of r and t only, we find the same partial differential equation (5) for the non-trivial shear stress $\tau(r,t)$. In the absence of a pressure gradient in the flow direction, the motion equations reduce to

$$\rho\,\frac{\partial v(r,t)}{\partial t} = \frac{1}{r}\,\frac{\partial}{\partial r}\,[r\tau(r,t)], \tag{24}$$

while the boundary conditions are

$$\tau(R_0,t) = 0;\ \ \tau(R,t) = f\cos(\omega t) \quad \text{or} \quad \tau(R,t) = f\sin(\omega t). \tag{25}$$

In order to solve a problem with shear stress on the boundary, we eliminate the velocity $v(r,t)$ between Eqs. (5) and (24) and find that

$$\begin{aligned} &\left(1+\lambda_1\,\tfrac{\partial}{\partial t}+\lambda_2\,\tfrac{\partial^2}{\partial t^2}\right)\tfrac{\partial\tau(r,t)}{\partial t}\\ &= \nu\left(1+\lambda_3\,\tfrac{\partial}{\partial t}+\lambda_4\,\tfrac{\partial^2}{\partial t^2}\right)\left(\tfrac{\partial^2}{\partial r^2}+\tfrac{1}{r}\,\tfrac{\partial}{\partial r}-\tfrac{1}{r^2}\right)\tau(r,t), \end{aligned} \tag{26}$$

Denoting by $\tau_c(r,t)$ and $\tau_s(r,t)$ the time-dependent permanent shear stresses corresponding to the motion due to the cosine or sine oscillations of the shear stress on the boundary and by

$$T(r,t) = \tau_c(r,t) + i\tau_s(r,t), \tag{27}$$

the complex shear stress, we attain to the next boundary value problem

$$\begin{aligned}&\left(1+\lambda_1\frac{\partial}{\partial t}+\lambda_2\frac{\partial^2}{\partial t^2}\right)\frac{\partial T(r,t)}{\partial t}\\&=\nu\left(1+\lambda_3\frac{\partial}{\partial t}+\lambda_4\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)T(r,t),\end{aligned} \tag{28}$$

$$T(R_0,t)=0,\ T(R,t)=fe^{i\omega t}. \tag{29}$$

We now seek a separable solution of the form

$$T(r,t)=F(r)e^{i\omega t} \tag{30}$$

and follow the same way as before. A simple analysis shows that the function $F(\cdot)$ has to satisfy the Bessel equation

$$s^2\frac{d^2F(s)}{ds^2}+s\frac{dF(s)}{ds}-(1+s^2)F(s)=0, \tag{31}$$

where $s=\gamma r$. The general solution of Eq. (31) is

$$F(s)=C_1I_1(s)+C_2K_1(s), \tag{32}$$

where C_1 and C_2 are again arbitrary constants while $I_1(\cdot)$ and $K_1(\cdot)$ are modified Bessel functions of one order.

Introducing Eq. (32) into (30) and using the boundary conditions (29), we find that

$$\tau_c(r,t)=f\,\mathrm{Re}\left\{\frac{K_1(\gamma R_0)I_1(\gamma r)-I_1(\gamma R_0)K_1(\gamma r)}{K_1(\gamma R_0)I_1(\gamma R)-I_1(\gamma R_0)K_1(\gamma R)}\,e^{i\omega t}\right\}, \tag{33}$$

$$\tau_s(r,t)=f\,\mathrm{Im}\left\{\frac{K_1(\gamma R_0)I_1(\gamma r)-I_1(\gamma R_0)K_1(\gamma r)}{K_1(\gamma R_0)I_1(\gamma R)-I_1(\gamma R_0)K_1(\gamma R)}\,e^{i\omega t}\right\}, \tag{34}$$

Direct computations show that $\tau_c(r,t)$ and $\tau_s(r,t)$ satisfy both the boundary conditions and the governing equation (26) (see for instance [15, Eq. (1.1)]).

4.2 Motion through a circular cylinder

The solutions corresponding to the motion within an infinite circular cylinder that applies an oscillatory longitudinal shear stress $f\cos(\omega t)$ or $f\sin(\omega t)$ to the fluid, namely

$$\tau_c(r,t)=f\,\mathrm{Re}\left\{\frac{I_1(\gamma r)}{I_1(\gamma R)}\,e^{i\omega t}\right\},\ \tau_s(r,t)=f\,\mathrm{Im}\left\{\frac{I_1(\gamma r)}{I_1(\gamma R)}\,e^{i\omega t}\right\}, \tag{35}$$

can be obtained following the same way as before and bearing in mind the fact that the fluid velocity has to remain finite along the axis of cylinder. The solutions (35) can be also obtained as a limiting case of Eqs. (33) and (34) when $R_0 \to 0$. The velocity fields corresponding to these motions, namely

$$\begin{aligned} v_c(r,t) &= \frac{f}{\rho\omega}\,\mathrm{Re}\left\{\frac{I_0(\gamma r)}{I_1(\gamma R)}\ \gamma e^{i(\omega t-\pi/2)}\right\}, \\ v_s(r,t) &= \frac{f}{\rho\omega}\,\mathrm{Im}\left\{\frac{I_0(\gamma r)}{I_1(\gamma R)}\ \gamma e^{i(\omega t-\pi/2)}\right\}, \end{aligned} \tag{36}$$

are immediately obtained introducing Eqs. (35) into (24) and integrating with respect to t.

Finally, for completion, the diagrams of the shear stresses $\tau_c(r,t)$ and $\tau_s(r,t)$ corresponding to motions through an infinite circular cylinder that applies oscillating shears to the fluid are depicted in Figs. 3 and 4 both for Newtonian and generalized Burgers fluids. It is clearly seen from these figures that the shear stress profiles corresponding to generalized Burgers fluids tend to superpose over those of Newtonian fluids when $\lambda_i \to 0$ $(i = 1, 2, 3, 4)$.

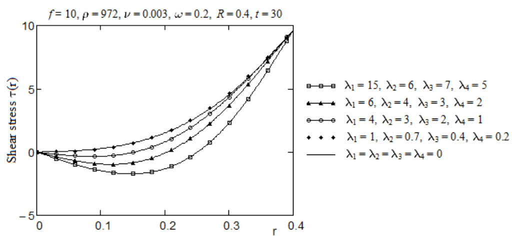

Fig. 3. Profiles of the shear stress $\tau_c(r, t)$ given by Eq. $(35)_1$.

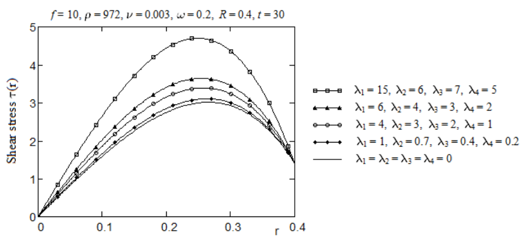

Fig. 4. Profiles of the shear stress $\tau_s(r, t)$ given by Eq. $(35)_2$.

4.3 Applications to oscillatory rotational motions

Motions between rotating cylinders have been intensively studied since Taylor [16] reported the results of his famous investigations. Here, we consider the rotational motion of an IGBF due to the outer cylinder that oscillates around its axis with the angular velocities $W\cos(\omega t)$ or $W\sin(\omega t)$. In this case the velocity of the fluid is of the form

$$\mathbf{v} = \mathbf{v}(r,t) = w(r,t)\mathbf{e}_\theta, \tag{37}$$

where $\mathbf{e}_\theta$ is the unit vector along the θ-direction. The constraint of incompressibility is again satisfied and the governing equation for the fluid velocity, namely [12, Eq. (13)]

$$\begin{aligned}&\left(1+\lambda_1\tfrac{\partial}{\partial t}+\lambda_2\tfrac{\partial^2}{\partial t^2}\right)\tfrac{\partial w(r,t)}{\partial t}\\ &=\nu\left(1+\lambda_3\tfrac{\partial}{\partial t}+\lambda_4\tfrac{\partial^2}{\partial t^2}\right)\left(\tfrac{\partial^2}{\partial r^2}+\tfrac{1}{r}\tfrac{\partial}{\partial r}-\tfrac{1}{r^2}\right)w(r,t),\end{aligned} \tag{38}$$

has the same form as Eq. (26) for the shear stress $\tau(r,t)$.

As the associated boundary conditions

$$w(R_0,t)=0;\ w(R,t)=W\cos(\omega t) \quad \text{or} \quad w(R,t)=W\sin(\omega t), \tag{39}$$

are identical to those from Eqs. (25), it results that the time-dependent permanent solutions corresponding to such motions are given by

$$w_c(r,t) = W\,\mathrm{Re}\left\{\frac{K_1(\gamma R_0)I_1(\gamma r)-I_1(\gamma R_0)K_1(\gamma r)}{K_1(\gamma R_0)I_1(\gamma R)-I_1(\gamma R_0)K_1(\gamma R)}\,e^{i\omega t}\right\}, \tag{40}$$

$$w_s(r,t) = W\,\mathrm{Im}\left\{\frac{K_1(\gamma R_0)I_1(\gamma r)-I_1(\gamma R_0)K_1(\gamma r)}{K_1(\gamma R_0)I_1(\gamma R)-I_1(\gamma R_0)K_1(\gamma R)}\,e^{i\omega t}\right\}. \tag{41}$$

Of course, the similar solutions corresponding to the motion through an infinite circular cylinder, namely

$$w_c(r,t)=W\,\mathrm{Re}\left\{\frac{I_1(\gamma r)}{I_1(\gamma R)}\,e^{i\omega t}\right\},\ w_s(r,t)=W\,\mathrm{Im}\left\{\frac{I_1(\gamma r)}{I_1(\gamma R)}\,e^{i\omega t}\right\}, \tag{42}$$

are immediately obtained using Eqs. (35).

5 Conclusions

In this note two unsteady oscillatory motions of an IGBF in an annulus are considered and closed form time-dependent permanent solutions are established in terms of the modified Bessel functions $I_0(\cdot), I_1(\cdot), K_0(\cdot)$ and $K_1(\cdot)$.

These solutions, which are periodic in time and independent of the initial conditions, satisfy the boundary conditions and governing equations and can be easy reduced to the similar solutions for Burgers, Oldroyd-B, Maxwell, second grade and linearly viscous fluids performing the same motions. By now taking $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ into Eqs. (42), for instance, the solutions

$$w_c(r,t) = W\,\mathrm{Re}\left\{\frac{I_1[(1+i)r\sqrt{\omega/(2\nu)}]}{I_1[(1+i)R\sqrt{\omega/(2\nu)}]}\,e^{i\omega t}\right\},$$
$$w_s(r,t) = W\,\mathrm{Im}\left\{\frac{I_1[(1+i)r\sqrt{\omega/(2\nu)}]}{I_1[(1+i)R\sqrt{\omega/(2\nu)}]}\,e^{i\omega t}\right\}, \tag{43}$$

corresponding to Newtonian fluids (see [10, Eqs. $(35)_2$]) are recovered. Furthermore, in the view of some asymptotic expansions of modified Bessel functions (see for instance Bandelli and Rajagopal [17, Sect. 4], all these solutions can be well enough approximated by simple expressions containing the elementary functions $\cos(\cdot), \sin(\cdot), \cosh(\cdot)$ and $\sinh(\cdot)$. Indeed, following [4] we can show that the shear stress $\tau_c(r,t)$ given by Eq. $(35)_1$ can be approximated by

$$f\sqrt{\frac{R}{r}}\,\cos(\omega t)\,\frac{\cosh[(r+R)a]\cos[(r-R)b]-\cosh[(r-R)a]\sin[(r+R)b]}{\cosh(2Ra)-\sin(2Rb)}$$
$$-f\sqrt{\frac{R}{r}}\,\sin(\omega t)\,\frac{\sinh[(r+R)a]\sin[(r-R)b]+\sinh[(r-R)a]\cos[(r+R)b]}{\cosh(2Ra)-\sin(2Rb)} \tag{44}$$

where

$$a = \sqrt{\delta}\cos\left(\frac{\varphi}{2}\right),\ \ b = \sqrt{\delta}\sin\left(\frac{\varphi}{2}\right),$$
$$\varphi = \mathrm{arctg}\left(\frac{\omega^2\lambda_1\lambda_3+(1-\omega^2\lambda_2)(1-\omega^2\lambda_4)}{\omega\lambda_3(1-\omega^2\lambda_2)-\omega\lambda_1(1-\omega^2\lambda_4)}\right)$$
$$\delta = \frac{\omega}{\nu}\ \frac{\sqrt{[\omega\lambda_3(1-\omega^2\lambda_2)-\omega\lambda_1(1-\omega^2\lambda_4)]^2+[\omega^2\lambda_1\lambda_3+(1-\omega^2\lambda_2)(1-\omega^2\lambda_4)]^2}}{(1-\omega^2\lambda_4)^2+\omega^2\lambda_3^2}\,.$$

Finally, it is worth pointing out that based on a simple remark regarding the governing equations corresponding to the shear stress $\tau(r,t)$ in longitudinal motions and the velocity $\omega(r,t)$ in the case of rotational motions in cylindrical domains, some important applications of our results have been brought to light.

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