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A Survey of the P Function Method for Higher Order Equations and Some Applications*

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Dedicated to the memory of Prof. Dr. Viorel Arnăutu

Abstract

This paper gives a survey of an extension of the classical maximum principle, namely the P function method.

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1 Introduction

The intention of this paper is to survey some extensions (the P function method) and applications of the classical maximum principle for elliptic operators.

It is well-known that every subharmonic function in a bounded domain Ω (i.e. $\Delta u \geq 0$ in Ω) satisfies the classical maximum principle

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

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The subbiharmonic function $u(x) = -x_1^4 - |x|^2$ in the ball $\Omega = \{(x_1, \ldots, x_n) \,|\, |x| < R\}$ (i.e. $\Delta^2 u \leq 0$ in Ω) shows that there are no classical maximum principles for the biharmonic operator $\Delta^2 u$ (and for higher-order elliptic operators at all). Still some results can be proven.

The first proof of a maximum principle for an elliptic equation of higher-order that has a similar form to the classical maximum principle was given by Miranda [33].

Miranda showed that for the biharmonic equation $\Delta^2 u = 0$, where u is a smooth function defined on a plane domain the function $|\nabla u|^2 - u\Delta u$ takes its maximum value on the boundary of the domain. Later, in [37], Payne uses functionals containing the square of the second gradient of the solution to semilinear equations of the form

$$\Delta^2 u = f(u)$$

to deduce integral bounds on $(\Delta^2 u)^2$.

Since then many authors have extended the Miranda's result. For example, maximum principles for fourth order equations containing nonlinearities in u or Δu can be found in works of Payne [37], Schaefer [57], [60], [61]. Similar results are proved by Zhang [72], Mareno [30], [31] (studied some equations from plate theory), Danet [5], [6], [7], [9], Tseng and Lin [68], etc. (see the references cited here).

Most recently the authors in [9] obtain maximum principles results for the more general variable coefficient m- metaharmonic equation

$$\Delta^m u - a_{m-1}(x)\Delta^{m-1} u + a_{m-2}(x)\Delta^{m-2} u - \cdots + a_0(x)u = 0 \quad \text{in } \Omega. \qquad (1)$$

using P functions containing terms of the form $(\Delta^i u)$. Here Ω is a bounded domain in $\mathbb{R}^n$.

The survey paper [8] is devoted to the P function method and gives a presentation of research of the past years on applications of the P function method in second order elliptic problems. Historical notes and an extensive survey of the literature is added. The present paper intends to continue our previous work [8] by presenting contributions to the P function method for higher order elliptic equations.

2 Main Results

2.1 The general case (m arbitrary)

First we present a maximum principle for the general equation of order $2m$.

Theorem 2.1 *([9]) Let u be a classical solution of (1), i.e. $C^{2m}(\Omega) \cap C^{2m-2}(\overline{\Omega})$. We consider the function* P_1

$$\mathrm{P}_1 = (\Delta^{m-1}u)^2 + 2a_{m-2}(\Delta^{m-2}u)^2 + (\Delta^{m-3}u)^2 + \cdots + u^2.$$

Suppose that a_{m-1}, $a_{m-2} > 0$ and $\Delta(1/a_{m-2}) \leq 0$ in Ω. If

$$\sup_{\Omega} \left\{ \frac{a_0^2}{2a_{m-1}+1} \right\} < \frac{4n+4}{(\operatorname{diam}\Omega)^2},$$

$$\frac{a_0^2}{2a_{m-1}} > \max\left\{1 + \sup_{\Omega} a_1, \ldots, 1 + \sup_{\Omega} a_{m-3}\right\},$$

$$\frac{a_0^2}{2a_{m-1}+1} > \sup\left\{|a_1| + \cdots + |a_{m-3}|\right\},$$

and

$$\left(\frac{a_0^2}{2a_{m-1}} + 1\right) a_{m-2} > 1 \quad \text{in } \Omega$$

then, either there exists a constant $k \in \mathbb{R}$ such that $\mathrm{P}_1/w_1 \equiv k$ in Ω or P_1/w_1 does not attain a nonnegative maximum in Ω. Here $w_1(x) = 1 - \alpha(x_1^2 + \cdots + x_n^2) \in C^\infty(\mathbb{R}^n)$, and α is a positive constant.

Remark. The coefficient a_0 can be replaced by a_{m-j}, $j = 4, \ldots, m-1$ if there exists a $j = 4, \ldots, m-1$ such that

$$\frac{a_{m-j}^2}{2a_{m-1}} > \max_{k=3,\ldots,m} \left\{2 + \sup_{\Omega} a_k\right\},$$

$$\frac{a_{m-j}^2}{2a_{m-1}} + 2 > \sup_{\Omega}\left\{|a_0| + \cdots |a_{m-j-1}| + |a_{m-j+1}| + \cdots + |a_{m-3}|\right\},$$

and

$$\left(\frac{a_{m-j}^2}{2a_{m-2}} + 1\right) a_{m-2} > 1 \quad \text{in } \Omega.$$

We now show that the uniqueness result and the maximum principle holds ([9]).

Theorem 2.2 *There is at most one classical solution of the boundary value problem*

$$\begin{cases} \Delta^m u - a_{m-1}(x)\Delta^{m-1}u + a_{m-2}(x)\Delta^{m-2}u + \cdots + (-1)^m a_0(x)u = f & \text{in } \Omega \\ u = g_1,\ \Delta u = g_2, \ldots,\ \Delta^{m-1}u = g_m & \text{on } \partial\Omega, \end{cases} \tag{2}$$

provided the coefficients $a_{m-1}, \ldots, a_0$ satisfy the conditions imposed in Theorem 2.1.

Remark. The boundary value problem

$$\begin{cases} \Delta^m u + 2^m u = 0 & \text{in } \Omega = (0,\pi) \times (0,\pi) \\ u = \Delta u = \cdots = \Delta^{m-1}u = 0 & \text{on } \partial\Omega, \end{cases}$$

has (at least) the solutions $u_1(x,y) \equiv 0$ and $u_2(x,y) = \sin x \sin y$ in Ω. This example shows that if we do not impose some restrictions on the coefficients $a_{m-1}, \ldots, a_0$, then the uniqueness result might be violated.

In [2] the authors pose an interesting open problem: If $f = 0$ in $\Omega, g_2 = \cdots = g_m = 0$ on $\partial\Omega, m \geq 3, n \geq 2, a_{m-1} = \cdots = a_1 = 0$ in Ω do all the solutions of (2) satisfy the maximum principle (3) where $C > 1$ is a constant? This problem, as it turns out, can be solved when Ω is a class C^2 domain ([63]). Here we present a version for arbitrary domains ([9]).

Theorem 2.3 *We consider the boundary value problem (2), where $f = 0$ in Ω and $g_2 = \cdots = g_m = 0$ on $\partial\Omega$.*
Then

$$\max_{\overline{\Omega}} |u| \leq C \max_{\partial\Omega} |u|, \tag{3}$$

holds for all solutions of (2) provided the coefficients $a_{m-1}, \ldots, a_0$ are subject to one of the conditions imposed in theorem 2.1.

Theorem 2.4 *([32])*
Suppose that $u \in C^{2m+1}(\Omega) \cap C^{2m-1}(\overline{\Omega})$ is a solution of (1). Furthermore for $n > 4$ one defines

$$\mathrm{P}_2 = \nabla^2(\Delta^{m-2}u) \cdot \nabla^2(\Delta^{m-2}u) - \nabla(\Delta^{m-2}u) \cdot \nabla(\Delta^{m-1}u) +$$

$$+ \frac{a_{m-1}}{2}\nabla(\Delta^{m-2}u) \cdot \nabla(\Delta^{m-2}u) + \frac{a_{m-3}}{2}\nabla(\Delta^{m-3}u) \cdot \nabla(\Delta^{m-3}u)$$

$$+ a_{m-2}\left[\frac{n-4}{n+2}\right](\Delta^{m-2}u)^2 - \left[\frac{4-n}{2(n+2)}\right](\Delta^{m-1}u)^2 + \sum_{i=0}^{m-2}\phi_i(\Delta^i u)^2,$$

where the functions $\phi_0, \ldots, \phi_{m-2} \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ *satisfy* $\sum_{i=0}^{m-2}\phi_i^2 + 1 \leq \alpha$ *for some positive constant* α*, and* $|\nabla^2 u|^2 = u_{,ij}\, u_{,ij}$*. Additionally, one imposes the conditions*

$$\sum_{i=0}^{m-3} a_i^2 \leq \beta,\ \sum_{i=0}^{m-1} \nabla a_i \cdot \nabla a_i \leq \gamma,$$

for constants $\gamma \geq 0$, $\beta > 0$.

$$\phi_i \geq \frac{\beta}{2}$$

$$a_{m-2} \geq 1,\ a_{m-1} - \frac{1}{2} \geq \frac{\gamma(n+2)}{2(n-4)},$$

$$\frac{\Delta a_i}{2} - \frac{\nabla a_{m-i} \cdot \nabla a_{m-i}}{a_{m-i}} \geq 0,\ i = 1, 3,$$

$$\Delta a_{m-2} - 4\frac{\nabla a_{m-2} \cdot \nabla a_{m-2}}{a_{m-2}} \geq 0,$$

$$\Delta\phi_i \geq 3\max\left\{\frac{\beta(n-4)}{2(n+2)} + \frac{\gamma}{2}, \alpha, 4\frac{\nabla\phi_i \cdot \nabla\phi_i}{\phi_i}\right\},\ i = 0, \ldots, m-2.$$

Then, P_2 *is subharmonic in* Ω.

We briefly indicate how theorem 2.4 can be used to obtain integral bounds on the square of the second gradient of $\Delta^{m-2}u$. Suppose that the hypotheses of theorem 2.4 are satisfied and the m conditions

$$\Delta^i u = 0,\ i = 0, \ldots, m-5,$$

$$\Delta^{m-2}u = \frac{\Delta^{m-2}u}{\partial n} = 0,\ \Delta^{m-3}u = \frac{\Delta^{m-3}u}{\partial n} = 0,$$

hold on $\partial\Omega$. Let A denote the area of Ω. Using theorem 2.4 and integration by parts we get

$$\int_{\Omega} |\nabla^2(\Delta^{m-2}u)|^2 dx \le \frac{A}{2} \max_{\partial\Omega} \Big\{ |\nabla^2(\Delta^{m-2}u)|^2 + \phi_{m-4}(x)(\Delta^{m-4}u)^2 +$$

$$\frac{n-4}{2(n+2)}(\Delta((\Delta^{m-2}u)))^2 \Big\}.$$

Before treating some particular cases, we shift our attention from n dimensional to one dimensional case and mention the following result (Ω denotes an open interval (α, β)) ([5])

Theorem 2.5 *There can be at most one classical solution of the problem*

$$\begin{cases} u^{(2m)} - du^{(6)} + c(x)u^{(4)} - b(x)u'' + a(x)u = f & \text{in } \Omega \\ u = g_1,\ u'' = g_2,\ u''' = g_3, \ldots, u^{(m)} = g_m & \text{on } \partial\Omega, \end{cases}$$

where $m \ge 6$ *is even,* $d \ge 0$ *and* $b \ge 0,\ a, c > 0,\ (1/a)'', (1/c)'' \le 0$ *in* Ω.

The result follows since the function

$$\begin{aligned} \mathrm{P}_3 \;=\; & u''u^{(2m-2)} - 2u'''u^{(2m-3)} + 3u^{(4)}u^{(2m-4)} - \cdots + (m-3)u^{(m-2)}u^{(m+2)} \\ & -(m-3)u^{(m-1)}u^{(m+1)}/2 - ((m-3)/2+1)[\,(u^{(m)})^2 - u^{(m-1)}u^{(m+1)}] \\ & +[(u^{'''})^2 - du^{''}u^{(4)}] + c(x)(u'')^2/2 + a(x)u^2/2 \end{aligned}$$

assumes its maximum value on $\partial\Omega$, where u is a solution of

$$u^{(2m)} - du^{(6)} + c(x)u^{(4)} - b(x)u'' + a(x)u = 0 \quad \text{in } \Omega.$$

Similarly, we can treat the problem

$$\begin{cases} u^{(2m)} + du^{(6)} - c(x)u^{(4)} + b(x)u'' - a(x)u = f & \text{in } \Omega \\ u = g_1,\ u'' = g_2,\ u''' = g_3, \ldots, u^{(m)} = g_m & \text{on } \partial\Omega, \end{cases}$$

where $m \ge 5$ is odd.

2.2 The particular case $m = 4$

In this section we consider classical solutions (i.e., $C^8(\Omega) \cap C^6(\overline{\Omega})$) of

$$\Delta^4 u - a(x)\Delta^3 u + b(x)\Delta^2 u - c(x)\Delta u + du = 0, \tag{4}$$

in the bounded plane domain Ω, and present ([5]) a maximum principle for a certain function defined on the solutions of (4). Then we use the maximum principle to prove a uniqueness result for the corresponding boundary value problem.

Theorem 2.6 *Let u be a classical solution of (4). Assume that*

$$a > 0, \quad \Delta(1/a) \leq 0 \quad in\ \Omega,$$

$$b \geq 0 \quad in\ \Omega,$$

$$c > 0, \quad \Delta(1/c) \leq 0 \quad in\ \Omega,$$

and

$$d > 0$$

are satisfied. Then the functional

$$\mathrm{P}_4 = \frac{c(x)}{2}(\Delta u)^2 + \frac{a(x)}{2}\left(\Delta^2 u\right)^2 + d(|\nabla u|^2 - u\Delta u) + |\nabla(\Delta^2 u)|^2 - \Delta^2 u \Delta^3 u$$

assumes its maximum value on $\partial\Omega$. The result also holds if a and c are nonnegative constants.

An important application of the above presented maximum principle is the following uniqueness result:

Theorem 2.7 *There is at most one classical solution of the boundary value problem*

$$\begin{cases} \Delta^4 u - a\Delta^3 u + b(x)\Delta^2 u - c\Delta u + du = f & \text{in } \Omega, \\ u = g, \quad \Delta u = h, \quad \Delta^2 u = i, \quad \Delta^3 u = j & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where $a, c \geq 0$, b and d satisfy the hypotheses of theorem 2.6, and the curvature k of $\partial\Omega$ (Ω is a smooth plane domain) is strictly positive.

We suppose that u_1 and u_2 are two solutions of (5). Defining $v = u_1 - u_2$, we see that v satisfies (4) and

$$v = \Delta v = \Delta^2 v = \Delta^3 v = 0 \quad \text{on } \partial\Omega. \tag{6}$$

By virtue of theorem 2.6

$$\mathrm{P}_4 \leq \max_{\partial\Omega} \mathrm{P}_4 \quad \text{in } \Omega. \tag{7}$$

Since $v = \Delta^2 v = 0$ on $\partial\Omega$, we have

$$|\nabla v| = \left|\frac{\partial v}{\partial n}\right| \quad \text{on } \partial\Omega \tag{8}$$

and

$$|\nabla(\Delta^2 v)| = \left|\frac{\partial(\Delta^2 v)}{\partial n}\right| \quad \text{on } \partial\Omega, \tag{9}$$

where $\partial/\partial n$ denotes the outward directed normal derivative operator.
It can be shown that (introducing a normal coordinate system)

$$\frac{\partial v}{\partial n} = \frac{\partial(\Delta^2 v)}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{10}$$

By (6), (7), (8), (9) and (10) we get

$$\mathrm{P}_4 \leq 0 \quad \text{in } \Omega,$$

which gives

$$-v\Delta v - \Delta^2 v \Delta^3 v \leq 0 \quad \text{in } \Omega. \tag{11}$$

Integrating (11) over Ω and using Green's identity we obtain

$$\int_\Omega |\nabla v|^2 + \int_\Omega |\nabla(\Delta^2 v)|^2 \leq 0.$$

Hence $v \equiv 0$ in $\overline{\Omega}$ by continuity.

It is known that once we have a maximum principle for an equation, the nonexistence of a nontrivial solution of the zero-boundary problem will be a consequence.
An inverse result, of establishing a maximum principle from some nonexistence results was carried out by Schaefer and Walter (Theorem 2, [63]).
Using their result and our theorem 2.7, we obtain the following maximum principle

Corollary 2.1 *Suppose that u is a classical solution of the boundary value problem*

$$\begin{cases} \Delta^4 u - a\Delta^3 u + b\Delta^2 u - c\Delta u + du = 0 & \text{in } \Omega, \\ \Delta u = 0, \quad \Delta^2 u = 0, \quad \Delta^3 u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a, b, c \geq 0$, d satisfy the hypotheses of theorem 2.6, and the curvature k of $\partial\Omega$ (Ω is a smooth domain) is strictly positive. Then there exists a constant $K > 0$ such that

$$\max_{\overline{\Omega}} |u| \leq K \max_{\partial\Omega} |u|.$$

Remark.

1. Similar uniqueness results can be inferred using theorem 2.6. It can be shown (see [5]) that there is at most one classical solution of the boundary value problem

$$\begin{cases} \Delta^4 u - a\Delta^3 u + b(x)\Delta^2 u - c\Delta u + du = f & \text{in } \Omega \\ u = g, \quad \Delta u = h, \quad \Delta^2 u = i, \quad \frac{\partial(\Delta^2 u)}{\partial n} = j & \text{on } \partial\Omega, \end{cases}$$

2. We note that Dunninger [11] developed a maximum principle from which follows the uniqueness for the classical solution of the boundary value problem

$$\begin{cases} \Delta^2 u + cu = f & \text{in } \Omega \subset \mathbb{R}^n, \\ u = g, \Delta u = h & \text{on } \partial\Omega, \end{cases}$$

where $c > 0$ is a constant.

An uniqueness result for solutions of a more general fourth-order elliptic equation, under the same boundary conditions follows from Corollary 1, [72].

The uniqueness question for solutions of the boundary value problem (here $a, b \geq 0$ and $c > 0$ in Ω)

$$\begin{cases} \Delta^3 u - a(x)\Delta^2 u + b(x)\Delta u - c(x)u = f & \text{in } \Omega \subset \mathbb{R}^n, \\ u = g, \ \Delta u = h, \ \Delta^2 u = i & \text{on } \partial\Omega, \end{cases}$$

has been settled in a satisfactory way by Schaefer [58] (the constant coefficient case with n=2) and Goyal and Goyal [17] (the constant and variable coefficient case).

We see that our uniqueness result (theorem 2.7) is a generalization of results of Dunninger, Goyal and Schaefer.

2.3 The particular case $m = 3$.

This subsection is dedicated to maximum principles for a class of linear equations of sixth order. As a consequence of these maximum principles we will obtain uniqueness results for boundary value problems of sixth order. This section is based on the paper [7].

Schaefer [58] investigated the uniqueness of the solution for the boundary value problems

$$\begin{cases} \Delta^3 u - a(x)\Delta^2 u + b(x)\Delta u - c(x)u = f & \text{in } \Omega \subset \mathbb{R}^n \\ u = g,\ \Delta u = h,\ \Delta^2 u = i & \text{on } \partial\Omega, \end{cases} \tag{12}$$

where $a, b, \geq 0$, $c > 0$ are constants, and the curvature of $\partial\Omega$ is positive.

Our aim here is to remove via the P function method dimension and geometry conditions (convexity and smoothness) with, of course, further conditions on the coefficients a, b and c.

We deal with classical solutions (i.e. $u \in C^6(\Omega) \cap C^4(\overline{\Omega})$) of

$$\Delta^3 u - a(x)\Delta^2 u + b(x)\Delta u - c(x)u = 0 \quad \text{in } \Omega \subset \mathbb{R}^n, n \geq 2. \tag{13}$$

The uniqueness results can be inferred from the following maximum principles.

Theorem 2.8 *Let u be a classical solution of (13) and suppose that*

$$\frac{a(b+c)^2}{b^2(a-1)} < \frac{8n+8}{(\operatorname{diam}\Omega)^2}, \tag{14}$$

holds, where $a > 1, b, c$ are constants. We consider the function P_5 given by

$$\mathrm{P}_5 = (a\Delta^2 u + bu)^2 + ab(a-1)(\Delta u)^2 + b^2(a-1)u^2.$$

Then, either there exists a constant $k \in \mathbb{R}$ such that $\mathrm{P}_5/w_1 \equiv k$ in Ω or P_5/w_1 does not attain a nonnegative maximum in Ω.

By computation and using equation (13) we have in Ω

$$\begin{aligned} \Delta\big((a\Delta^2 u + bu)^2\big) &\geq 2(a\Delta^2 u + bu)(a\Delta^3 u + b\Delta u) \\ &= 2\big(a^3(\Delta^2 u)^2 + abcu^2 + a^2(b+c)u\Delta^2 u + \\ &\quad ab(1-a)\Delta u\Delta^2 u + b^2(1-a)u\Delta u\big), \end{aligned}$$

$$\Delta\big(ab(a-1)(\Delta u)^2\big) \geq 2ab(a-1)\Delta u\Delta^2 u,$$

$$\Delta\big(b^2(a-1)u^2\big) \geq 2b^2(a-1)u\Delta u.$$

That means that

$$\begin{aligned}\Delta \mathrm{P}_5 &\geq 2a\big(a^2(\Delta^2 u)^2 u + a(b+c)u\Delta^2 u + bcu^2\big)\\ &= 2a\left(a\Delta^2 u + \frac{b+c}{2}u\right)^2 + 2a\left(bc - \frac{(b+c)^2}{4}\right)u^2 \\ &\geq -\frac{a(b+c)^2}{2}u^2.\end{aligned} \tag{15}$$

Hence P_5 satisfies the differential inequality

$$\Delta \mathrm{P}_5 + \frac{a(b+c)^2}{2b^2(a-1)}\mathrm{P}_5 \geq 0 \quad \text{in}\, \Omega.$$

Since (14) holds, we can use a version of the generalized maximum principle (lemma 2.1, [7]) to obtain the desired result.

Theorem 2.9 *Let u be a classical solution of (13) and suppose that*

$$\sup_{\Omega} \frac{(a+c)^2}{a(b-1)} < \frac{8n+8}{(\operatorname{diam}\Omega)^2}, \tag{16}$$

$$b > 1 \quad \text{in}\, \overline{\Omega},\ \ \Delta(1/(\mathrm{b}-1)) \leq 0 \quad \text{in}\, \Omega$$

holds.

If

$$\mathrm{P}_6 = (\Delta^2 u + u)^2 + (b-1)(\Delta u)^2 + (b-1)u^2$$

then, either there exists a constant $k \in \mathbb{R}$ such that $\mathrm{P}_6/w_1 \equiv k$ in Ω or P_6/w_1 does not attain a nonnegative maximum in Ω.

If $a = c$ in Ω then, P_6 attains its maximum value on $\partial\Omega$ (the restriction (16) is not needed).

Theorem 2.10 *Let u be a classical solution of (13), where $a > 0$ in $\overline{\Omega}$, and c is of arbitrary sign in Ω. Suppose that*

$$\sup_{\Omega} \frac{c^2}{2a} + 1 < \frac{4n+4}{(\operatorname{diam}\Omega)^2},$$

$$b > 0,\ \Delta(1/b) \leq 0 \quad \text{in}\, \Omega,$$

$$b\left(\frac{c^2}{2a}+1\right) \geq 1 \quad \text{in}\, \Omega$$

holds.

If

$$\mathrm{P}_7 = (\Delta^2 u)^2 + b(\Delta u)^2 + u^2$$

then, either there exists a constant $k \in \mathbb{R}$ *such that* $\mathrm{P}_7/w_1 \equiv k$ *in* Ω *or* P_7/w_1 *does not attain a nonnegative maximum in* Ω.

Theorem 2.11 *Let* u *be a classical solution of (13) and suppose that*

$$\sup_{\Omega} \frac{1}{a}\left(c + \frac{(c+1)^2}{4(a-1)}\right) < \frac{2n+2}{(\operatorname{diam}\Omega)^2},$$

$$b = 0,\ a > 1 \quad \text{in}\, \overline{\Omega},\ \Delta(1/\mathrm{a}) \leq 0 \quad \text{in}\, \Omega,$$

$$c > 0,\ \Delta(1/c) \leq 0 \quad \text{in}\, \Omega$$

holds.

We consider the function P_8 *given by*

$$\mathrm{P}_8 = (\Delta^2 u - \Delta u)^2 + c(\Delta u - u)^2 + a(\Delta u)^2.$$

Then, either there exists a constant $k \in \mathbb{R}$ *such that* $\mathrm{P}_8/w_1 \equiv k$ *in* Ω *or* P_8/w_1 *does not attain a nonnegative maximum in* Ω.

Now an uniqueness result follows from the above mentioned maximum principles.

Theorem 2.12 *There is at most one classical solution of the boundary value problem (12), where* a, b *and* c *satisfy the conditions of Theorem 2.8 or Theorem 2.9 or Theorem 2.10 or Theorem 2.11.*

For various uniqueness results for sixth order boundary value problems the reader is referred to [7].

2.4 The particular case $m = 2$.

In 1971, J. Serrin [65] and H. Weinberger [71] proved that if Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary

$$\begin{cases} \Delta u = -1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial n} = c & \text{on } \partial\Omega, \end{cases}$$

(where c is a constant) then Ω is a ball of radius $|nc|$ and the solution is radially symmetric about the center.
Serrin's proof is based on the classical maximum principle and on the method of moving parallel planes.Weinberger's method is more elementary. It also uses the maximum principle but relies on Green's theorem to establish certain identities. Unfortunately, Weinberger's argument does not extend to more general results.

Using the following maximum principle Benett [1] was able to show that an analogous result holds for a fourth order problem.

Theorem 2.13 *The function*

$$\mathrm{P}_9 = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial u_i \partial u_j} - \nabla u \cdot \nabla(\Delta u) + \frac{n-4}{n+2}\int_0^u f(y)dy + \frac{n-4}{2(n+2)}(\Delta u)^2$$

assumes its maximum value on $\partial\Omega$, where u is a solution of $\Delta^2 u + f(u) = 0$ in $\Omega \subset \mathbb{R}^n, f' \leq 0$ in $\mathbb{R}$.

Corollary 2.2 *([1]) Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{4+\varepsilon}$ boundary, and suppose that the following overdetermined problem has a solution in $u \in C^4(\partial\Omega)$*

$$\begin{cases} \Delta^2 u = -1 & \textit{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \textit{on } \partial\Omega, \\ \Delta u \equiv c & \textit{on } \partial\Omega \ (c \textit{ - constant}). \end{cases}$$

Then Ω is an open ball of radius $[|c|(n^2+2n)]^{\frac{1}{2}}$ and u is radially symetric.

The above mentioned result allows a characterization of open balls in $\mathbb{R}^n$ by means of an integral identity:
Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and suppose that there is a real constant M so that

$$\int_\Omega B dx = M \int_{\partial\Omega} \frac{\partial B}{\partial n} ds$$

holds for any function B in $C^4(\overline{\Omega})$ satisfying

$$\begin{cases} \Delta^2 B = 0 & \text{in } \Omega, \\ B = 0 & \text{on } \partial\Omega. \end{cases}$$

Then Ω is an open ball.

Finally, we state our last maximum principle for an fourth order equation.

Theorem 2.14 *([7])*
Let u be a classical solution of

$$\Delta^2 u - a_1 \Delta u + a_0(x) u = 0 \quad \textit{in } \Omega \subset \mathbb{R}^n, \tag{17}$$

where $a_1 \equiv const. > 0$, $a_0 > 0$ in $\overline{\Omega}$.
Suppose that

$$\sup_{\Omega} \left(a_1 - \frac{1}{a_1} \left(\frac{a_0 - 1}{a_0} \right)^2 \right) < \frac{2n+2}{(\operatorname{diam} \Omega)^2}. \tag{18}$$

Let

$$\mathrm{P}_{10} = \frac{1}{2}(\Delta u - a u)^2 + \frac{1}{2}(\Delta u)^2 + u^2.$$

Then, either there exists a constant $k \in \mathbb{R}$ such that $\mathrm{P}_{10}/w_1 \equiv k$ in Ω or P_{10}/w_1 does not attain a nonnegative maximum in Ω.
If

$$a_1^2 \geq \left(\frac{a_0 - 1}{a_0} \right)^2 \quad \text{in } \Omega, \tag{19}$$

then the function P_{10} attains its maximum value on $\partial\Omega$ (here the assumption (18) is not needed).

Remark. A classical result ([2]) tells us that the boundary value problem

$$\begin{cases} \Delta^2 u - a_1(x) \Delta u + a_0(x) u = f & \text{in } \Omega \subset \mathbb{R}^n \\ u = g, \ \Delta u = h & \text{on } \partial\Omega, \end{cases} \tag{20}$$

has a unique solution if $a_1, a_0 > 0$ and if $\Delta a_0 < 0$ or $\Delta(1/a_0) < 0$ in Ω.

Theorem 2.14 tells us that if $a_1 \geq 1$ and $a_0 > 0$ then the boundary value problem (20) has a unique solution. We see that no smoothness restrictions are needed on the coefficient a_0.

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