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UNILATERAL CONDITIONS ON THE BOUNDARY FOR SOME SECOND ORDER DIFFERENTIAL EQUATIONS*

Dan TIBA†

Dedicated to the memory of Prof. Dr. Viorel Arnăutu

Abstract

Sufficient conditions for the existence of solutions in strongly nonlinear boundary value problems of elliptic and parabolic type, including ordinary differential equations with unilateral conditions on the boundary, are derived by means of an abstract scheme for continuous perturbations of accretive operators in Banach spaces.

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1 Introduction

This paper is concerned with strongly nonlinear boundary value problems of elliptic type

$$Au + f(x, u, \mathrm{grad}\ u) = 0, \quad a.e.\ \Omega \tag{1}$$

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†`Institute of Mathematics "Simion Stoilow" ( Romanian Academy) and Academy of Romanian Scientists, Bucharest; dan.tiba@imar.ro`

$$-\frac{\partial u}{\partial \nu} \in \beta(u) \quad a.e.\ \Gamma \tag{2}$$

and parabolic type

$$\frac{\partial u}{\partial t} = Au + f(x, u, \text{grad } u) = 0 \quad a.e.\ ]0, T[\times\Omega \tag{3}$$

$$u(0) = u_0 \quad a.e\ \Omega \tag{4}$$

$$-\frac{\partial u}{\partial \nu} \in \beta(u) \quad a.e.\ [0, T] \times \Gamma.$$

We also prove some existence results for the two points and the periodic problem associated with ordinary differential equations

$$-u''(s) + f(s, u(s), u'(s)) = 0 \quad a.e.\ [0, 1], \tag{5}$$

which can be compared with the classical result of Bernstein [7].

Above A denotes a second order elliptic operator, f is function satisfying the Caratheodory assumptions, β is the subdifferential of a convex, lower-semicontinuous, proper function $j : R \to]-\infty, +\infty]$ and Ω is a bounded domain in R^N with sufficiently smooth boundary Γ.

The following notation will be used throughout this paper. If E is a Banach space, we shall denote by $L^p(0, T; E)$, $1 \leq p \leq \infty$, the space of all p - integrable, E - valued functions on $[0, T]$ and by $C(0, T; E)$ the Banach space of all continuous functions from $[0, T]$ to E. We shall denote by $W^{1,p}(0, T; E)$ the space of all p - integrable, E - valued distributions y with derivative y' taken in the sense of vectorial distributions on $]0, T[$, p - integrable. Equivalently, $y' \in W^{1,p}(0, T; E)$ means that $y : [0, T] \to E$ is absolutely continuous, almost everywhere differentiable on $]0, T[$ and $y' \in L^p(0, T; E)$. By $W^{k,p}(\Omega)$ we mean the usual Sobolev space of real distributions in Ω. We shall use the symbols $||\cdot||_p$, $||\cdot||_{k,p}$ for the norms in $L^p(\Omega)$, $W^{k,p}(\Omega)$ respectively. In the case $p = 2$, we put $H^k(\Omega)$ instead of $W^{k,p}(\Omega)$.

We assume familiarity with concepts and methods of nonlinear monotone equations and we refer to Barbu [2], Brezis [3], [4] for significant results in this field. However, for easy references we recall some facts about subdifferentials.

Let $\varphi : E \to]-\infty, +\infty]$ be a convex, lower semicontinuous, proper function. We denote by $\partial\varphi(x)$ the set of all $z \in E'$, the dual of E, such that

$$\varphi(x) \leq \varphi(y) + (x - y, z) \quad \forall\, y \in E,$$

and call it the subdifferential of φ at x, where $(\cdot,\cdot)$ is the pairing between E and E'.

Conditions of type (2), (4) are called unilateral conditions on the boundary and they arise in elasticity. See for instance Duvaut - Lions [10], Goeleven [11], Goeleven et. al. [12].

Problems (1) - (4) are very much discussed in the literature. We mention the papers for Brezis - Haraux [5], Brezis - Nirenberg [6], Vy Khoi Le [14] that deal with the case when the nonlinear term f does not depend on grad u or the elliptic operator is degenerate and with Landesman - Lazer conditions.

Equations of form (1), (3) appear in the paper of Puel [16], but the problem is the Dirichlet one with unilateral constraints in the interior of Ω and certainly the methods are different. Our method of proof is similar to that used in [19], [20]. Regularity results and various extensions are discussed in [8], [9], [13], [18].

Our approach applies to a large class of problems and, in certain cases, quadratic growth with respect to the gradient is allowed.

In the subsequent sections we introduce an abstract scheme based on m - accretive operators and we apply it to elliptic, parabolic and ordinary differential boundary value problems. An Appendix briefly analyzes some properties of the Nemitsky operator.

2 An Abstract Perturbation Scheme

Let W be a Banach space, topologically and algebraically included in X, another Banach space with dual X' uniformly convex.

Proposition 1. *Let $T : X \to X$ be a m - accretive operator with $0 \in T0$, $D(T) \subset W$ and $(\lambda I + T)^{-1} : X \to W$ compact for some $\lambda \geq 0$. Let $S : W \to X$ be a bounded, demicontinuous mapping.*

Then, for every $m \in N$, there is $x_m \in W$, such that

$$\lambda x_m + Tx_m + S_m x_m \ni 0. \tag{6}$$

Here we have defined the truncate S_m of $S - \lambda I$ by

$$S_m x = \begin{cases} Sx - \lambda x, & ||x||_W \leq m \\ S(\frac{mx}{||x||_W}) - \lambda \frac{mx}{||x||_W}, & ||x||_W > m \end{cases}.$$

Proof.

The equation (6) can be written as

$$x_m = (\lambda I + T)^{-1}(-S_m x_m).$$

The operator defined by the right hand side is compact in W because $(\lambda I + T)^{-1}$ is and S_m is bounded. It maps a certain sphere with a sufficiently large radius in itself because S_m is uniformly bounded on W and $(\lambda I + T)^{-1}$ is compact from X in W.

It is continuous. Here is the argument

Let $x_n \to x$ in W, then $S_m x_n \to S_m x$ weakly in X because S_m is also demicontinuous. It yields $\{S_m x_n\}_n$ to be bounded in X that is extracting a convenient subsequence, denoted again by x_n, we have $(\lambda I + T)^{-1} \cdot (-S_m x_n) \to y$ strongly in W. Hence $(\lambda I + T)^{-1}(-S_m x_n) \to y$ strongly in X. Operator $(\lambda I + T)^{-1}$ is single-valued, demiclosed in X, so $(\lambda I + T)^{-1} \cdot (-S_m x) = y$.

Therefore, one can use the Schauder fixed point theorem to obtain the desired solution.

3 Elliptic Problems

Let A be the second order elliptic operator

$$Au = -\sum_{i,j} \frac{\partial}{\partial x_j}(a_{ij}\frac{\partial u}{\partial x_i}) + u \tag{7}$$

with

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2 \quad \text{a.e. } \Omega,\ \alpha > 0,\ \xi \in R^N. \tag{8}$$

Here $a_{ij} \in C^1(\Omega)$, $a_{ij} = a_{ji}$ and Ω is a bounded domain with a sufficiently smooth boundary Γ.

We denote by $\dfrac{\partial u}{\partial \nu}$ the conormal derivative associated to A

$$\frac{\partial u}{\partial \nu} = \sum_{i,j} a_{ij}\frac{\partial u}{\partial x_i}\cos(\bar{n}, x_j) \tag{9}$$

where $\bar{n}$ is the exterior normal to Ω.

Consider two real numbers $2 \geq q \geq 1,\ p > 1$ such that $W^{1,q}(\Omega) \subset L^p(\Omega)$ topologically and $W^{2,p}(\Omega) \subset W^{1,q}(\Omega)$ with compact inclusion. The existence of these numbers is ensured by the wellknown Sobolev embedding theorem.

Assume that $f : \Omega \times R \times R^N \to R$ satisfies the Caratheodory conditions
$f(\cdot, u, v_1, \dots, v_N)$ is measurable for every u, v.
$f(x, \cdot, \cdot)$ is continuous a.e. $x \in \Omega$.
The Nemitsky operator $S : W^{1,q}(\Omega) \to L^p(\Omega)$ defined by

$$(Su)(x) = f(x, u(x), \operatorname{grad} u(x)) \quad \text{a.e. } \Omega \tag{10}$$

satisfies

$$S \textit{ is bounded} \tag{11}$$

$$S \textit{ is demicontinuous.} \tag{12}$$

See the Appendix for a discussion of such hypotheses. Moreover, the following growth restriction is needed

$$f(x, u, v)u \geq K|u|^s - d|v|^2 - \gamma(x) \cdot u \tag{13}$$

where $K > 0$, $s > 1$ is choosen such that $W^{1,q}(\Omega) \subset L^s(\Omega)$, $\gamma \in L^\infty(\Omega)$ and d is a small constant.

Remark 1. Condition $W^{2,p}(\Omega) \subset W^{1,q}(\Omega)$, with compact inclusion, shows that growth order of $f(x, u, \cdot)$, which is $\dfrac{q}{p}$ (see Appendix), cannot exceed 2 when $N = 2$, cannot exceed $\dfrac{3}{2}$ when $N = 3$ and so on, according to the Sobolev inequalities.

Remark 2. We give a simple example of function $f(x, u, v)$, where $v = (v_1, \dots, v_N) \in R^N$

$$f(x, u, v) = |u|^r \cdot u + |v|^{\frac{q}{p}} \eta(u) + \gamma(x)$$

Conditions (11), (12), (13) are fulfilled evidently for an appropriate r (see the Appendix) under assumption that $\eta : R \to R$ is a monotone continuous and bounded function.

Remark 3. The operator A can be more generally

$$A'u = -\sum_{i,j} \frac{\partial}{\partial x_j}(a_{ij}\frac{\partial u}{\partial x_i}) + \sum_i b_i \frac{\partial u}{\partial x_i} + cu.$$

The last two terms can be taken in $f(x, u, v)$ and one can apply the present results under appropriate conditions on b_i, $c > 0$, [4], p. 6.

Theorem 1. *Under the above hypotheses, problem (1), (2) has at least one solution u in $W^{2,p}(\Omega)$.*

Proof.

We apply *Proposition 1* with $W^{1,q}(\Omega)$ and $X = L^p(\Omega)$.

Operator $T : L^p(\Omega) \to L^p(\Omega)$ defined by

$Tu = Au$

$D(T) = \{u \in W^{2,p}(\Omega); Au \in L^p(\Omega), -\dfrac{\partial u}{\partial \nu} \in \beta(u)\}$

is m - accretive and $(T + \lambda I)^{-1}$ is bounded from $L^p(\Omega)$ in $W^{2,p}(\Omega)$ for $\lambda > 0$ large enough, according to Brezis [4], Proposition I.13 and Remark I.22. It yields that $(T + \lambda I)^{-1}$ is compact operator from $L^p(\Omega)$ in $W^{1,q}(\Omega)$.

Then for every natural number m, there is u_m in $W^{2,p}(\Omega)$ such that

$$\lambda u_m + Tu_m + S_m u_m \ni 0. \tag{14}$$

We assume that $||u_m||_{1,q} > m$, otherwise u_m satisfies (1), (2) and the problem is solved.

Equation (14) becomes

$$\lambda u_m + Au_m + f(x, \frac{mu_m}{||u_m||_{1,q}}, \frac{m \text{ grad } u_m}{||u_m||_{1,q}}) - \lambda \frac{mu_m}{||u_m||_{1,q}} \ni 0. \tag{15}$$

Multiply by u_m and integrate over Ω

$$\int_\Omega Au_m \cdot u_m dx + \int_\Omega f(x, \frac{mu_m}{||u_m||_{1,q}}, \frac{m \text{ grad } u_m}{||u_m||_{1,q}}) u_m dx \leq 0.$$

Integrating by parts, using (8) and (2) we get

$$\alpha ||u_m||_{1,2} + \int_\Omega f(x, \frac{mu_m}{||u_m||_{1,q}}, \frac{m \text{ grad } u_m}{||u_m||_{1,q}}) u_m dx \leq 0.$$

From (13) one obtains $\{u_m\}$ to be bounded in $H^1(\Omega)$, that is for m large enough u_m verifies (1), (2) and the proof is finished.

Remark 4. Not only classical problems, but many boundary problems can be expressed in form (2).

Example 1. Let $j : R \to] - \infty, +\infty]$ be a convex, lower semicontinuous, proper function given by

$$j(s) = \begin{cases} 0 & \text{if } s = 0 \\ +\infty & \text{otherwise} \end{cases}.$$

Then $\beta = \partial j$ is

$$\beta(s) = \begin{cases} \mathbb{R} & \text{if } s = 0 \\ \emptyset & \text{otherwise} \end{cases}.$$

and condition (2) is the Dirichlet one.

Example 2. Let $j(s) = 0$ for every s. Then $\beta(s) = 0$ for every s and condition (2) corresponds to the Neumann problem.

Example 3. Consider

$$j(s) = \begin{cases} 0 & s \geq 0 \\ +\infty & s < 0 \end{cases}.$$

Then

$$\beta(s) = \begin{cases} 0 & s > 0 \\ ]-\infty, 0] & s = 0 \\ \emptyset & s < 0 \end{cases}.$$

We obtain for (2) the Signorini boundary conditions.

Example 4. Consider $j(s) = |s|$. In this case

$$\beta(s) = \operatorname{sgn}(s) = \begin{cases} 1 & s > 0 \\ [-1, 1] & s = 0 \\ -1 & s < 0 \end{cases}.$$

The corresponding condition (2) appears in elasticity.

4 Parabolic Problems

For the sake of simplicity we take the problem

$$\frac{\partial u}{\partial t} - \Delta u + f(x, u, \operatorname{grad} u) = 0 \quad \text{a.e. }]0, T[\times\Omega \tag{16}$$

$$u(0, x) = u_0(x) \quad \text{a.e. } \Omega \tag{17}$$

$$-\frac{\partial u(t, x)}{\partial n} \in \beta(u(t, x)) \quad \text{a.e. } [0, T] \times \Gamma. \tag{18}$$

We start with the following lemma

Lemma 1 *The operator $B : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$ defined by*

$$Bu = \frac{\partial u}{\partial t} - \Delta u$$

$$D(B) = \{u \in H^1(0,T;H^2(\Omega)); u(0,x) = u_0(x), -\frac{\partial u(t,x)}{\partial n} \in \beta(u(t,x))\}$$

is maximal monotone and for $u_0 \in D(\varphi)$, B^{-1} is compact from $L^2(0,T;L^2(\Omega))$ in $L^2(0,T;H^1(\Omega))$.

Here $\varphi : L^2(\Omega) \to]-\infty,+\infty]$ is a proper, lower-semicontinuous, convex function given by

$$\varphi(u) = \begin{cases} \frac{1}{2}\int\limits_\Omega |\text{grad } u|^2 dx + \int\limits_\Gamma j(u) d\tau & \text{if } u \in H^1(\Omega),\ j(u) \in L^1(\Gamma) \\ +\infty & \text{otherwise} \end{cases}$$

and $\partial\varphi = -\Delta$ with

$$D(\partial\varphi) = \{u \in H^2(\Omega); -\frac{\partial u}{\partial n} \in \beta(u) \text{ a.e. } \Gamma\}.$$

Proof

One easily can check, using the Green formula, that operator B is monotone. To obtain the maximality it suffices that problem

$$\frac{\partial u}{\partial t} - \Delta u + u(t,x) = \tilde{f}(t,x) \quad \text{a.e. } \Omega\times]0,T[\tag{19}$$

$$u(0,x) = u_0(x) \quad \text{a.e. } \Omega \tag{20}$$

$$-\frac{\partial u(t,x)}{\partial n} \in \beta(u(t,x)) \quad \text{a.e. }]0,T[\times\Gamma \tag{21}$$

has at least one solution for every $\tilde{f} \in L^2(0,T;L^2(\Omega))$.

Operator $Cu = -\Delta u + u$ with

$$D(C) = \{u \in H^2(\Omega); -\frac{\partial u}{\partial n} \in \beta(u)\}$$

is a subdifferential (see Barbu [2], p. 63).

Therefore, we can apply the smoothing effect on initial data and problem (10) - (12) has at least one solution u for every $f \in L^2(0,T;L^2(\Omega))$ and $u_0 \in L^2(\Omega)$ (see Barbu [2], p. 189).

If $u_0 \in D(\varphi)$ then $\dfrac{\partial u}{\partial t} \in L^2(0,T;L^2(\Omega))$, $\Delta u \in L^2(0,T;L^2(\Omega))$ and the mapping $\tilde{f} \to u$ is compact from $L^2(0,T;L^2(\Omega))$ in $L^2(0,T;W^{1,2}(\Omega))$ in the case $u_0 \in D(\varphi)$ and the proof is finished.

Assume now that $f : \Omega \times R \times R^N \to R$ satisfies the Caratheodory conditions and operator $S : L^2(0,T;H^1(\Omega)) \to L^2(0,T;L^2(\Omega))$ defined by

$$(Su)(t,x) = f(x, u(t,x),\ \mathrm{grad}_x u(t,x))$$

satisfies hypotheses (11) - (13) with $p = q = 2$.

One can state

Theorem 2 *Under the above hypotheses, problem (16) - (18) has at least one solution u in $H^1(0,T;H^2(\Omega))$.*

Proof

According to *Lemma 1*, we can apply *Proposition 1* with $\lambda = 0$, $X = L^2(0,T;L^2(\Omega))$, $W = L^2(0,T;H^1(\Omega)))$ and obtain the approximate equations

$$\frac{\partial u_m}{\partial t} - \Delta u_m + S_m u_m \ni 0.$$

Suppose that the norm of u_m in $L^2(0,T;H^1(\Omega))$ denoted $||u_m||_W$ strictly exceeds m, for every natural number m.

The approximate equations become

$$\frac{\partial u_m}{\partial t} - \Delta u_m + f(x, \frac{m u_m}{|||u_m||_W}, \frac{m\ \mathrm{grad}_x u_m}{||u_m||_W}) \ni 0. \tag{22}$$

Multiply by $u_m(s,x)$ and integrate over $[0,t]$

$$\frac{1}{2}|u_m(t,x)|^2 - \frac{1}{2}|u_0(x)|^2 - \int_0^t \Delta u_m(s,x) \cdot u_m(s,x) ds +$$

$$+ \int_0^t f(x, \frac{m u_m}{|||u_m||_W}, \frac{m\ \mathrm{grad}_x u_m}{||u_m||_W}) \cdot u_m ds = 0.$$

Integrating over Ω, using the Green formula, we get

$$\frac{1}{2}\int_\Omega |u_m(t,x)|^2 dx + \int_0^t \int_\Omega |\mathrm{grad}_x u_m(s,x)|^2 dx ds) + \tag{23}$$

$$+\int_0^t \int_\Omega f(x, \frac{mu_m}{|||u_m||_W}, \frac{m\,\mathrm{grad}_x u_m}{||u_m||_W}) \cdot u_m dxds \leq C.$$

From condition (13), when $t = T$ it yields u_m to be bounded in $L^2(0, T; H^1(\Omega))$ and using again (23) we see that u_m is bounded in $C(0, T; H^1(\Omega))$. Then for large m we have $||u_m||_W \leq m$, that is u_m satisfies problem (16) - (18). The regularity is obtained as in (19) - (21).

5 Ordinary Differential Equations

We take into account the two point boundary value problem:

$$-u''(t) + f(t, u(t), u'(t)) = 0 \quad \text{a.e. } t \in [0, 1] \tag{24}$$

$$u(0) = a,\ u(1) = b \tag{25}$$

where f is Caratheodory:
- $f(t, u, v)$ measurable in t for every u, v
- $f(t, u, v)$ continuous in u, v a.e. $t \in [0, 1]$

and a, b are real numbers.

We assume that

$$|f(t, u, v)| \leq g(t, u) + h(t, u)|v|^2 \tag{26}$$

with

$$\sup_{|u|\leq r} |g(t, u)| \in L^2(0, 1)$$

$$\sup_{|u|\leq r} |h(t, u)| \in L^\infty(0, 1)$$

for every $r > 0$, and

$$f(t, u, v) \cdot u \geq K(u) \cdot v - \alpha|u|^2 + \gamma,\ \alpha < 1 \tag{27}$$

where K is a continuous, d - homogeneous function, that is

$$K(\lambda u) = \lambda^d K(u),\ \lambda > 0,\ d \geq 0.$$

Theorem 3 *Under the above hypotheses, problem (24), (25) has at least one solution u in $W^{2,1}(0, 1)$.*

Proof

Shifting the domain of f in u, v one can suppose instead of (25)

$$u(0) = u(1) = 0 \tag{28}$$

(null Dirichlet boundary conditions).

Operator $T : L^2(0,1) \to L^2(0,1)$ defined by

$Tu = -u''$

$D(T) = \{u \in H^2(0,1); u(0) = u(1) = 0\}$

is maximal monotone and $(I+T)^{-1}$ is compact from $L^2(0,1)$ in $W^{1,4}(0,1)$.

Under condition (26) operator $S : W^{1,4}(0,1) \to L^2(0,1)$ defined by $(Su)(t) = f(t, u(t), u'(t))$ is bounded and continuous (see the Appendix).

We can use *Proposition 1* with $\lambda = 1$, $X = L^2(0,1)$, $W = W^{1,4}(0,1)$ to derive the existence of approximating solutions

$$u_m(t) - u_m''(t) + S_m u_m(t) = 0.$$

Assume that $||u_m||_{1,4} > m$ for every m. Then

$$u_m(t) - u_m''(t) + f(t, \frac{mu_m(t)}{|||u_m||_{1,4}}, \frac{mu_m'(t)}{||u_m||_{1,4}}) - \frac{mu_m(t)}{||u_m||_{1,4}} = 0. \tag{29}$$

Multiply by $u_m(t)$ and integrate over $[0,1]$

$$\int_0^1 |u_m'(t)|^2 dt + \int_0^1 f(t, \frac{mu_m(t)}{|||u_m||_{1,4}}, \frac{mu_m'(t)}{||u_m||_{1,4}}) u_m(t) dt \le 0.$$

From condition (27) one gets

$$\int_0^1 |u_m'(t)|^2 dt + \int_0^1 \frac{||u_m||_{1,4}}{m} \{K(\frac{mu_m(t)}{||u_m||_{1,4}}) \times \frac{mu_m'(t)}{||u_m||_{1,4}} - \tag{30}$$

$$-\alpha \left| \frac{mu_m(t)}{||u_m||_{1,4}} \right|^2 + \gamma\} dt \le 0$$

that is

$$\int_0^1 |u_m'(t)|^2 dt - \alpha \int_0^1 |u_m(t)|^2 dt + \tag{31}$$

$$+ \frac{m^d}{||u_m||_{1,4}^d} \int_0^1 K(u_m(t)) \cdot u_m'(t) dt \le C.$$

Let H be the indefinite integral of K. Then $H(u_m(t))$ is the indefinite integral of $K(u_m(t)) \cdot u'_m(t)$ and from (28), (31) we infer

$$\int_0^1 |u'_m(t)|^2 dt - \alpha \int_0^1 |u_m(t)|^2 dt \leq C.$$

From the inequality

$$\int_0^1 |u_m(t)|^2 dt \leq \int_0^1 |u'_m(t)|^2 dt \tag{32}$$

it yields $\{u'_m\}$ to be bounded in $L^2(0,1)$, which combined with (28) gives $\{u_m\}$ bounded in $H^1(0,1)$ and in $C(0,1)$.

Now from (29) and (26) we get $\{u_m\}$ to be bounded in $W^{2,1}(0,1)$ that is, for instance, $\{u_m\}$ is bounded in $W^{1,4}(0,1)$ too.

So for a sufficiently large m we have $||u_m||_{1,4} \leq m$ and u_m verifies (24), (25) which finishes the proof.

Corollary 1 *Under the same hypotheses as Theorem 4, with $\alpha < 0$ in (27), the periodic problem*

$$-u''_m(t) + f(t, u(t), u'(t)) = 0 \quad \text{a.e. } [0,1]$$

$$u(0) = u(1), \ u'(0) = u'(1)$$

has at least one solution $u \in W^{2,1}(0,1)$.

The proof follows the same lines as in *Theorem 4* because the corresponding operators T and S, defined in this case, have the same properties and the estimations can be derived in a similar way.

Remark 5 The classical result of Bernstein [7] ensures the existence of a solution for the two point problem, provided $f(t,u,v)$, $\dfrac{\partial f}{\partial u}(t,u,v)$, $\dfrac{\partial f}{\partial v}(t,u,v)$ continuous on $(0,1) \times R \times R$ and

$$\frac{\partial f}{\partial u}(t,u,v) \geq K > 0 \tag{33}$$

and (26) with $g(t,u)$, $h(t,u)$ continuous in $(0,1) \times R$.

We use the Lagrange theorem

$$f(t,u,v) - f(t,0,v) = \frac{\partial f}{\partial u}(t,\tilde{u},v) \cdot u$$

where $\tilde{u}$ is some point between u and 0.
Multiply by u

$$f(t,u,v) \cdot u = \frac{\partial f}{\partial u}(t,\tilde{u},v) \cdot u^2 + f(t,0,v) \cdot u \geq Ku^2 + f(t,0,v) \cdot u$$

which may be more restrictive than (27).

Example 5 Let $f(t,u,v) = a(t)u + b(t)v + c(t)$. Then (33) requires $a(t) \geq K > 0$, while (27) with $K(u) = u$ is fulfilled when $a(t) \geq 0$ only.

Example 6 We give now an example when f has quadratic growth in v

$$f(t,u,v) = a(t)u^{2n+1} + b(t)u^p v + c(t)v^2 u + d(t).$$

Then $f(t,u,v)u \geq b(t)u^{p+1} \cdot v + d(t) \cdot u$ in case $a(t) \geq 0$, $c(t) \geq 0$ and (27) is fulfilled.
Condition (33) gives

$$(2n+1)a(t)u^{2n} + pb(t)u^{p-1} \cdot v + c(t)v^2 \geq K > 0$$

which fails for $u = v = 0$ for any $a(t), b(t), c(t)$.

6 Appendix

We give a result concerning the Nemitsky operator in Sobolev spaces. See also Marcus and Mizel [15] or Pascali and Sburlan [17], p. 165.

Let s, p, q be real numbers such that $W^{1,q}(\Omega) \subset L^p(\Omega)$ continuously i.e.

$$\frac{1}{s} \geq \frac{1}{q} - \frac{1}{N}.$$

Proposition 2 *Operator $S : W^{1,q}(\Omega) \to L^p(\Omega)$ defined by*

$$(Su)(x) = f(x, u(x), \operatorname{grad} u(x))$$

where f satisfies the assumptions

$$f(x,\cdot,\cdot) \text{ is continuous a.e. } x \in \Omega \tag{34}$$

$$f(\cdot,u,v) \text{ is measurable for. every } u,v \tag{35}$$

$$|f(x,u,v)| \leq l(x) + h(x)|u|^{\frac{s}{p}} + K(x)|v|^{\frac{q}{p}} \tag{36}$$

with $l \in L^p(\Omega)$, $h, K \in L^\infty(\Omega)$, is bounded and continuous.

Proof

Using an argument with simple functions we see that S maps measurable functions in measurable functions. From condition (36) and $W^{1,q}(\Omega) \subset L^s(\Omega)$ continuously it yields that operator S is well-defined and bounded.

Consider now a sequence $\{u_n\} \subset W^{1,q}(\Omega)$ such that $u_n \to u$ in $W^{1,q}(\Omega)$, that is $u_n \to u$ in $L^s(\Omega)$ and grad $u_n \to$ grad u in $L^q(\Omega)$. To show that S is continuous it suffices to show that there is an infinite subsequence such that $S(u_j) \to S(u)$ strongly in $L^p(\Omega)$.

We choose an infinite subsequence of $\{u_n\}$, which we denote $\{u_j\}$, such that

$$\text{grad } u_j \to \text{grad } u \quad \text{a.e. } \Omega.$$

Then, by (34) $\quad S(u_j) \to S(u)$ a.e. in Ω.
From (36) it follows that functions $|f(x, u_j(x), \text{grad } \ u_j(x))|^p$ are equi-integrable over Ω, so the almost everywhere convergence of $S(u_j)$ to $S(u)$ implies that $S(u_j) \to S(u)$ strongly in $L^p(\Omega)$ and the proof is finished.

Corollary 2 *Operator $S : W^{1,q}(\Omega) \to L^p(\Omega)$ defined by*

$$(Su)(x) = g(x, u(x))$$

where g satisfies the assumptions

$$g(x,\cdot) \text{ is continuous a.e. } x \in \Omega \tag{37}$$

$$g(\cdot,u) \text{ is measurable for every } u \tag{38}$$

$$|g(x,u)| \leq l(x) + h(x)|u|^{\frac{S}{p}} \tag{39}$$

with $l \in L^p(\Omega)$, $h \in L^\infty(\Omega)$, is bounded and continuous.

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