Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 7, No. 1/2015

SEVERAL ITERATIVE PROCEDURES TO COMPUTE THE STABILIZING SOLUTION OF A DISCRETE-TIME RICCATI EQUATION WITH PERIODIC COEFFICIENTS ARISING IN CONNECTION WITH A STOCHASTIC LINEAR QUADRATIC CONTROL PROBLEM*

Vasile Drăgan† Ivan G. Ivanov‡

Dedicated to the memory of Prof. Dr. Viorel Arnăutu

Abstract

We consider a discrete-time periodic generalized Riccati equation. We investigate a few iterative methods for computing the stabilizing solution. The first method is the Kleinman algorithm which is a special case of the classical Newton-Kantorovich procedure, the second one is a method of consistent iterations and two new Stein iterations. The proposed methods are tested and illustrated via some numerical examples.

MSC: 15A24, 15A45, 49N10, 49N20, 65F35

*Accepted for publication on January 5-th, 2015

†`vasile.dragan@imar.ro` Institute of Mathematics "Simion Stoilow" of the Romanian Academy, Research Unit 2, POBox 1-764, RO-014700, Bucharest, Romania

‡`i_ivanov@feb.uni-sofia.bg` Faculty of Economics and Business Administration, Sofia University, Sofia 1113, Bulgaria, and Pedagogical College Dobrich, Shoumen University, Shoumen, Bulgaria

Keywords: Discrete-time Riccati equations, periodic coefficients, stabilizing solution, stochastic control, numerical computations.

1 Introduction

Since the pioneer Kalman's work [15], the matrix Riccati differential (difference) equations played a central role in the derivation of the solution of various robust linear quadratic control problems as well as H_2 filtering and H_∞-filtering problems, see e.g. [2, 4, 17] for the continuous-time case, or [3, 9] for the discrete-time case. In [20] where introduced the Riccati differential equations of stochastic control in the case of continuous-time stochastic systems. In the case of discrete-time systems affected by sequences of independent random variables, the discrete-time Riccati equations (DTREs) were introduced in [7, 8, 22].

To solve the linear quadratic optimal control problems on infinite time horizon, a crucial role is played by the so called stabilizing solution of a DTRE. An unified approach of the problem of the existence and uniqueness of a wide class of discrete-time Riccati equations both from deterministic and stochastic framework may be found in the Chapter 5 of [5] for the finite dimensional case and in [19] for infinite dimensional case.

Lately, there is an increasing interest in investigation of several control problems for systems with periodic coefficients. For the readers convenience we refer to [1, 3, 6, 18] and the references therein. Based on the uniqueness of the bounded and stabilizing solution one deduces that in the case of a DTRE with periodic coefficients the bounded and stabilizing solution is also a periodic sequence. This fact is important in the applications because it is necessary finite memory for the offline computation of the gain matrix of the optimal control.

It is worth mentioning that we do not know apriori neither an initial value nor a boundary value of the stabilizing solution of a DTRE. Hence, the existing methods for the computation of a solution with given initial values or boundary values problem for a differential (difference) equation cannot be applied to compute the bounded and stabilizing solution of a DTRE. In the deterministic context there exist two important classes of numerical methods to compute the stabilizing solution of a DTRE namely, the method based on invariant subspaces of associated canonical system [1, 3, 18] and iterative methods [16]. In the case of DTREs from stochastic control the methods based on invariant subspaces are not applicable. Therefore, in this case only iterative methods are mainly used to compute the stabilizing solution of a

Riccati differential (difference) equation. The most popular iterative method is an improved version of Kleinman algorithm. Even if the Kleinman type algorithm is a fast convergent method it has the disadvantage that in the stochastic case require that at each step to compute the unique bounded solution (periodic solution in the periodic case) of a perturbed Lyapunov equation. The numerical computation of such a solution becomes difficult in the case of systems of high dimension of their state space and /or large values of periods in the case of systems with periodic coefficients. That is why in practice were proposed other iterative methods which can be easier implementable (see e.g. Chapter 5 [5] or [14]).

In this paper we consider four iterative methods for computing the stabilizing solution of the discrete-time generalized Riccati equations. There are two Stein iterations which we apply for solving the problem. Similar algorithms for solving the discrete-type algebraic Riccati equations have been developed in our previous investigations [11, 12, 13, 14].

In the last part of the paper, we propose a method to compute the periodic solution occurring at each step of a Kleinman type algorithm. Our method is based on the so called H-representation technique recently developed in [21]. This method allows us to reduce the computation of the periodic solution of a Lyapunov type equation to the computation of the periodic solution of a backward affine equation on an euclidian space of dimension $n(n+1)/2$, n being the dimension of the state space of controlled system under consideration. In the last section of the paper, a comparison between several types of numerical methods discussed in the paper is done.

2 A class of discrete-time Riccati equations of stochastic control (DTRE)

2.1 On the stabilizing solution of DTRE

Consider the discrete-time Riccati equation (DTRE):

$$\begin{aligned} X(t) = \mathcal{G}(X(t+1)) &:= \textstyle\sum_{j=0}^{r} A_j^T(t)X(t+1)A_j(t) \\ &-(\textstyle\sum_{j=0}^{r} A_j^T(t)X(t+1)B_j(t) + L(t)) \\ &\times \left(R(t) + \textstyle\sum_{j=0}^{r} B_j^T(t)X(t+1)B_j(t)\right)^{-1} \\ &\times(\textstyle\sum_{j=0}^{r} B_j^T(t)X(t+1)A_j(t) + L^T(t)) + M(t), \quad t \in \mathbb{Z}. \end{aligned} \tag{1}$$

This equation arising in connection with the linear quadratic optimization problem described by the discrete-time linear stochastic system:

$$x(t+1) = [A_0(t) + \sum_{j=1}^{r} w_j(t)\, A_j(t)]x(t) + [B_0(t) + \sum_{j=1}^{r} w_j(t)\, B_j(t)]u(t) \quad (2)$$

and the cost functional

$$J(u, x_0) = \sum_{t=0}^{\infty} E\,[\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^T \begin{pmatrix} M(t) & L(t) \\ L^T(t) & R(t) \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}] \quad (3)$$

with $M(t) = M^T(t), R(t) = R^T(t)$. In (2), $w(t) = (w_1(t), \ldots, w_r(t))^T, t \geq 0$ are independent random vector with zero mean and satisfying $E[w(t)w^T(t)] = I_r$ for all $t \geq 0$. In (2) and (3), $x(t) \in \mathbb{R}^n$ is the state of the system and $u(t) \in \mathbb{R}^m$ are the control parameters.

We make the assumption:

H_1) There exists an integer $\theta \geq 1$ such that $A_j(t+\theta) = A_j(t);\ B_j(t+\theta) = B_j(t);\ 0 \leq j \leq r; M(t+\theta) = M(t);\ L(t+\theta) = L(t); R(t+\theta) = R(t),\ t \in \mathbb{Z}$.

Definition 1 *A solution $\{X_s(t)\}_{t\in\mathbb{Z}}$ of DTRE (1) is named stabilizing solution if the zero state equilibrium of the closed-loop system*

$$x(t+1) = [A_0(t) + B_0(t)F_s(t) + \sum_{j=1}^{r} w_j(t)\,(A_j(t) + B_j(t)F_s(t))]x(t) \quad (4)$$

is exponentially stable in mean square (ESMS), where

$$\begin{aligned} F_s(t) \quad &= -\Big(R(t) + \textstyle\sum_{j=0}^{r} B_j^T(t)X_s(t+1)B_j(t)\Big)^{-1} \\ &\quad \times(\textstyle\sum_{j=0}^{r} B_j^T(t)\, X_s(t)\, A_j(t) + L^T(t)). \end{aligned} \quad (5)$$

From the developments from Section 5.8 in [5] one deduces a set of necessary and sufficient conditions which guarantee the existence and uniqueness of the bounded and stabilizing solution of DTRE (1).

Proposition 2.1 *Under the assumption* H_1), *the following are equivalent:*

(i) *DTRE (1) has a unique bounded and stabilizing solution $\{X_s(t)\}_{t\in\mathbb{Z}}$ with the properties:*

(a) $X_s(\cdot)$ *is periodic with period θ;*

(b)

$$R(t)+\sum_{j=0}^{r} B_j^T(t)X_s(t+1)B_j(t)>0, \quad forall\ t\in\mathbb{Z}\,; \tag{6}$$

(ii) *the system (2) is stochastic stabilizable and there exist symmetric matrices* $\hat{X}(t), 0\le t\le\theta-1$, *satisfying:*

$$\begin{pmatrix} M(t)-\hat{X}(t) & L(t) \\ L^T(t) & R(t)\end{pmatrix}+\sum_{j=0}^{r}(A_j(t)\ \ B_j(t))^T\ \hat{X}(t+1)(A_j(t)\ \ B_j(t))>0 \tag{7}$$

$0\le t\le\theta-1$, *with* $\hat{X}(\theta)=\hat{X}(0)$.

Remark 2.1 a) Since any assumption regarding the sign of the quadratic form from (3) was not made, it is not expected to obtain information about the sign of the bounded and stabilizing solution $X_s(\cdot)$. The only relevant information about the solution of the linear quadratic optimization problem described by (2) and (3) is the sign condition (6). In this case, the quadratic part of the discrete-time Riccati equation (1) has defined sign.

b) Even if the stabilizing solution $X_s(\cdot)$ is defined for all $t\in\mathbb{Z}$, from Proposition 2.1 one obtains that under the assumption $H_1)$ it is sufficient to compute a finite number of values $X_s(t), 0\le t\le\theta-1$.

The next result may be used to compute a stabilizing control in a state feedback form for the system (2).

Proposition 2.2 *Under the assumption* $H_1)$ *the following are equivalent:*

(i) *the system (2) is stochastically stabilizable;*

(ii) *there exist the matrices* $Y(t)=Y^T(t)>0\in\mathbb{R}^{n\times n}, \Gamma(t)\in\mathbb{R}^{m\times n}, 0\le t\le\theta-1$, *satisfying the following system of LMIs:*

$$\begin{pmatrix} -Y(t) & (\tilde{A}_0(t))^T & \dots & (\tilde{A}_r(t))^T \\ \tilde{A}_0(t) & -Y(t+1) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \tilde{A}_r(t) & 0 & \dots & -Y(t+1)\end{pmatrix}<0 \tag{8}$$

$\tilde{A}_j(t)=A_j(t)Y(t)+B_j(t)\Gamma(t), j=0,\dots,r,\ \ 0\le t\le\theta-1$, *with* $Y(\theta)=Y(0)$.

If $(Y(t),\Gamma(t))$, $0\le t\le\theta-1$ *is a solution of the LMIs (8), then the control* $u(t)=F(t)\,x(t)$ *stabilizes the system (2), where*

$$F(t)=\Gamma(t-[\frac{t}{\theta}]\theta)\,Y^{-1}(t-[\frac{t}{\theta}]\theta), \quad t\ge 0. \tag{9}$$

(iii) *there exist the matrices* $Y(t) = Y^T(t) \in \mathbb{R}^{n\times n}, \Gamma(t) \in \mathbb{R}^{m\times n}, 0 \le t \le \theta - 1$, *satisfying the following system of LMIs:*

$$\begin{pmatrix} -Y(t+1) & \tilde{A}_0(t) & \dots & \tilde{A}_r(t) \\ (\tilde{A}_0(t))^T & -Y(t) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ (\tilde{A}_r(t))^T & 0 & \dots & -Y(t) \end{pmatrix} < 0 \tag{10}$$

$0 \le t \le \theta - 1$, *with* $Y(\theta) = Y(0)$. *If* $(Y(t), \Gamma(t))$, $0 \le t \le \theta - 1$ *is a solution of the LMIs (10), then the stabilizing feedback gain can be obtained as in (9).*

Proof. One obtains immediately applying Theorem 3.11 and Theorem 3.12 [5] in the case of the corresponding closed-loop systems completed with the Schur complement technique.

2.2 Several iterative procedures to compute the stabilizing solution of DTRE

Here we recall several iterative methods which allow us to compute the bounded and stabilizing solution of DTRE (1).

I. A Newton-Kantorovich type method

For each $k = 1, 2, \dots$ one computes $X^{(k)}(\cdot)$ as the unique periodic solution of the discrete-time backward affine equation:

$$\begin{aligned} X^{(k)}(t) \;&= \textstyle\sum_{j=0}^{r} (A_j(t) + B_j(t)F^{(k-1)}(t))^T X^{(k)}(t+1) \\ &\quad (A_j(t) + B_j(t)F^{(k-1)}(t)) + Q_{F^{(k-1)}}(t) \end{aligned} \tag{11}$$

where

$$Q_{F^{(k-1)}}(t) = \begin{pmatrix} I_n \\ F^{(k-1)}(t) \end{pmatrix}^T \begin{pmatrix} M(t) & L(t) \\ L^T(t) & R(t) \end{pmatrix} \begin{pmatrix} I_n \\ F^{(k-1)}(t) \end{pmatrix} \tag{12}$$

and

$$\begin{aligned} F^{(k)}(t) \;&= -\textstyle\sum_{j=0}^{r} \left(R(t) + B_j^T(t) X^{(k)}(t+1) B_j(t) \right)^{-1} \\ &\quad \times (\textstyle\sum_{j=0}^{r} B_j^T(t)\, X^{(k)}(t)\, A_j(t) + L^T(t)) \end{aligned} \tag{13}$$

if $k \ge 1$, while $F^{(0)}(t)$ is a stabilizing feedback gain for the system (2). For example $F^{(0)}(t)$ could be computed via formula (9) either based on a solution of the system of LMIs (8) or a solution of the system of LMIs (10).

One may show in a standard way that if the conditions from Proposition 2.1 (ii) are fulfilled, then for each $k \geq 1$ the control $u(t) = F^{(k)}(t)x(t)$ stabilizes the system (2), thus one obtains that (11) has an unique bounded solution and this solution is periodic with period θ. Furthermore, we have $X^{(k)}(t) \geq X^{(k+1)}(t) \geq \ldots \geq \hat{X}(t),\ k \geq 1, t \in \mathbb{Z},\ \hat{X}(\cdot)$ being any θ-periodic sequence satisfying (7) and $\lim_{k\to\infty} X^{(k)}(t) = X_s(t), t \in \mathbb{Z}$.

Even if the Newton-Kantorovich type method described by (11)-(13) has a quadratic convergence rate it is less used being difficult implementable. The difficulties consist in finding the periodic solution of (11) in the case $r \geq 1$ and $\theta \geq 1$ sufficiently large. That is way ofen alternative methods where derived. Even if those alternative methods have only linear convergence rate, they have the advantage to be easier implementable.

Below, we present some alternative methods to compute the stabilizing solution of DTRE (1). In Section 4 we shall present a method which allows us to compute the θ-periodic solution of (11).

II. A successive approximation method

Step 0. We choose a θ-periodic sequence $\{F^{(0)}(t)\}_{t\in\mathbb{Z}}$ with the property that the control $u(t) = F^{(0)}(t)x(t)$ stabilizes the system (2). For the designing of such a stabilizing feedback gain, may be used, for example, the procedure described by Proposition 2.2. One computes $X^{(1)}(\cdot)$ as a solution of the following system of LMIs:

$$\begin{aligned} X^{(1)}(t) &\geq \sum_{j=0}^{r} (A_j(t) + B_j(t)F^{(0)}(t))^T X^{(1)}(t+1) \\ &\quad (A_j(t) + B_j(t)F^{(0)}(t)) + Q_{F^{(0)}}(t) + \varepsilon^2 I_n \end{aligned} \tag{14}$$

$0 \leq t \leq \theta - 1$, with $X^{(1)}(\theta) = X^{(1)}(0)$, ε is a fixed parameter, $Q_{F^{(0)}}(t)$ being computed as in (12) with $F^{(k-1)}(t)$ replaced by $F^{(0)}(t)$.

Step k, $k \geq 1$. Compute $X^{(k+1)}(\cdot)$ by

$$\begin{aligned} X^{(k+1)}(t) &= \sum_{j=0}^{r} (A_j(t) + B_j(t)F^{(k)}(t))^T X^{(k)}(t+1) \\ &\quad (A_j(t) + B_j(t)F^{(k)}(t)) + Q_{F^{(k)}}(t) + \tfrac{\varepsilon^2}{k+1} I_n\,, \end{aligned} \tag{15}$$

$Q_{F^{(k)}}(t)$ being computed as in (12) while $F^{(k)}(t)$ is computed as in (13). Since the algorithm described by (14)-(15) is a special case of that described in Section 5.7 from [5], we may conclude that under the conditions of Proposition 2.1 $X^{(1)}(t) \geq \ldots \geq X^{(k)}(t) \geq X^{(k+1)}(t) \geq \ldots \geq \hat{X}(t)$ and $\lim_{k\to\infty} X^{(k)}(t) = X_s(t),\ 0 \leq t \leq \theta - 1$.

In the next section we shall discuss some procedural aspects regarding the computation of a solution of the system of LMIs (14).

III. Stein iterations

Step 0. Coincides with Step 0 from the previous algorithm. One computes $X^{(1)}(\cdot),\ 0 \leq t \leq \theta - 1$ as a solution of the system of LMIs (14). Also one computes $F^{(1)}(t)$ as in (13)) for $k = 1$.

Step k. $k \geq 1$. Compute $X^{(k+1)}(\cdot)$ as a unique $\theta-$periodic solution of the backward Stein equation:

$$\begin{aligned}
&X^{(k+1)}(t)\\
&= (A_0(t) + B_0(t)F^{(k)}(t))^T X^{(k+1)}(t+1)(A_0(t) + B_0(t)F^{(k)}(t))\\
&\quad + \textstyle\sum_{j=1}^{r} (A_j(t) + B_j(t)F^{(k)}(t))^T X^{(k)}(t+1)\\
&\times(A_j(t) + B_j(t)F^{(k)}(t)) + Q_{F^{(k)}}(t) + \tfrac{\varepsilon^2}{k+1} I_n
\end{aligned} \tag{16}$$

$t \in \mathbb{Z}, F^{(k)}(t)$ being computed as in (13).

Some procedural issues regarding the computation of the $\theta-$periodic solution of (16) will be discussed in the next section.

IV. Modified Stein iterations

Step 0. Coincides with Step 0 from the algorithm described in II. One computes $X^{(1)}(\cdot)$ as a solution of the system of LMIs (14) and $F^{(1)}(t)$ as in (13) for $k = 1$.

Step k. $k \geq 1$. One computes $X^{(k+1)}(\cdot)$ as a unique $\theta-$periodic solution of the backward Stein equation:

$$\begin{aligned}
&X^{(k+1)}(t)\\
&= (A_0(t) + B_0(t)\Gamma^{(k)}(t))^T X^{(k+1)}(t+1)(A_0(t) + B_0(t)\Gamma^{(k)}(t))\\
&\quad + \textstyle\sum_{j=1}^{r} (A_j(t) + B_j(t)\Gamma^{(k)}(t))^T X^{(k)}(t+1)\\
&\times(A_j(t) + B_j(t)\Gamma^{(k)}(t)) + Q_{\Gamma^{(k)}}(t) + \tfrac{\varepsilon^2}{k+1} I_n
\end{aligned} \tag{17}$$

where $\Gamma^{(k)}(t) = F_1(t)$ if $k = 1$ and

$$\begin{aligned}\Gamma^{(k)}(t) \quad &= -\Big(R(t) + B_0^T(t)\, X^{(k)}(t+1)B_0(t) \\ &+\textstyle\sum_{j=1}^{r} B_j^T(t)\, X^{(k-1)}(t+1)B_j(t)\Big)^{-1} \\ &\times(B_0^T(t)\, X^{(k)}(t+1)A_0(t) \\ &+\textstyle\sum_{j=1}^{r} B_j^T(t)\, X^{(k-1)}(t+1)A_j(t) + L^T(t))\end{aligned} \tag{18}$$

if $k \geq 2$ and $Q_{\Gamma^{(k)}}(t)$ one computes as in (12) taking $\Gamma^{(k)}(t)$ instead of $F^{(k-1)}(t)$.

3 Procedural issues

In this section we shall analyze some aspects regarding the computation of the sequences of approximations of the stabilizing solution of DTRE (1) described in the previous section.

3.1 The computation of the θ-periodic solution of a backward Stein equation with periodic coefficients

The discrete-time backward affine equations (16)-(17) can be regarded as special cases of the discrete-time backward affine equation:

$$X(t) = \hat{A}^T(t)X(t+1)\hat{A}(t) + H(t) \tag{19}$$

$t \in \mathbb{Z}$, where $\{\hat{A}(t)\}_{t\in\mathbb{Z}} \subset \mathbb{R}^{n\times n}$, $\{H(t)\}_{t\in\mathbb{Z}} \subset \mathcal{S}_n$ are periodic sequences of period θ. Assume that the discrete-time linear equation

$$X(t+1) = \hat{A}(t)X(t) \tag{20}$$

is exponentially stable.
Let $T(t,s) = \hat{A}(t-1)\hat{A}(t-2)...\hat{A}(s)$ if $t > s$ and $T(t,s) = I_n$ if $t = s$, $t, s \in \mathbb{Z}$.

The solutions of equation (19) have the representation:

$$X(t) = T^T(\theta,t)X(\theta)T(\theta,t) + \sum_{s=t}^{\theta-1} T^T(s,t)H(s)T(s,t), t \leq \theta - 1.$$

The periodicity condition $X(0) = X(\theta)$ yields

$$X(\theta) = T^T(\theta,0)X(\theta)T(\theta,0) + \tilde{H} \tag{21}$$

where

$$\tilde{H} = \sum_{s=0}^{\theta-1} T^T(s,0)H(s)T(s,0). \tag{22}$$

Since the zero solution of (20) is exponentially stable it follows that the spectral radius of the monodromy matrix $T(\theta,0)$ satisfies $\rho(T(\theta,0)) < 1$ (see e.g. [3] or [10]).

Hence (21) has a unique solution which may be computed using any existing solver for time invariant Stein equations. Instead of (22), the last term $\tilde{H}$ from (21) may be computed also as: $\tilde{H} = X(0;\theta,0)$ where $t \to X(t;\theta,0)$ is the solution of (19) satisfying the final condition $X(\theta;\theta,0) = 0$. Then, the other values $X(t)$, $1 \leq t \leq \theta - 1$ of the θperiodic solution of the equation (19) are obtained recursively from this equation.

Remark 3.1. The unique θ-periodic solution of (16) and (17), respectively can be computed according to the procedure described before taking successively $\hat{A}(t) = A_0(t) + B_0(t)F^{(k)}(t)$ in the case of equation (16) or $\hat{A}(t) = A_0(t) + B_0(t)\Gamma^{(k)}(t)$ in the case of equation (17).

3.2 An iterative method for computation of a solution of a system of LMIs (14)

Let $\{F^{(0)}(t)\}_{t\in\mathbb{Z}}$ be a θ-periodic sequence such that the zero solution of the closed-loop system

$$x(t+1) = [A_0(t) + B_0(t)F^{(0)}(t) + \sum_{j=1}^{r} w_j(t)(A_j(t) + B_j(t)F^{(0)}(t))]x(t) \tag{23}$$

is ESMS. Therefore, the discrete-time backward affine equation

$$\begin{aligned} Y(t) \quad = \quad & \sum_{j=0}^{r}(A_j(t) + B_j(t)F^{(0)}(t))^T Y(t+1)(A_j(t) + B_j(t)F^{(0)}(t)) \\ & + Q_{F^{(0)}}(t) + 2\varepsilon^2 I_n \end{aligned} \tag{24}$$

has a unique θ-periodic solution $\{\tilde{Y}(t)\}_{t\in\mathbb{Z}}$.

Let $Y^{(k)}(t)$ be the θ-periodic solution of the discrete-time backward affine equation:

$$\begin{aligned} Y^{(k)}(t) \quad & = [A_0(t) + B_0(t)F^{(0)}(t)]^T Y^{(k)}(t+1) \\ & \times [A_0(t) + B_0(t)F^{(0)}(t)] + H^{(k)}(t) \end{aligned} \tag{25}$$

where

$$H^{(k)}(t) = \sum_{j=1}^{r}(A_j(t)+B_j(t)F^{(0)}(t))^T Y^{(k-1)}(t+1)(A_j(t)+B_j(t)F^{(0)}(t)) + Q_{F^{(0)}}(t) + 2\epsilon^2 I_n, k \geq 1, \tag{26}$$

with

$$Y^{(0)}(t) = 0, t \in \mathbb{Z}. \tag{27}$$

Proposition 3.1. *If the zero solution of (23) is ESMS then the θ-periodic sequences $\{Y^{(k)}(t)\}_{t\in\mathbb{Z}}$, $k = 0, 1, ...$ are well defined via (25)-(27) and have the properties:*

a) $0 = Y^{(0)}(t) \leq Y^{(1)}(t) \leq ... \leq Y^{(k)}(t) \leq ... \leq \tilde{Y}(t)$*;*

b) $\lim_{k\to\infty} Y^{(k)}(t) = \tilde{Y}(t)$*,* $t \in \mathbb{Z}$*,* $\tilde{Y}(\cdot)$ *being the θperiodic solution of 24.*

If k_0 is such that $0 \leq \sum_{j=1}^{r}(A_j(t)+B_j(t)F^{(0)}(t))^T(Y^{(k_0)}(t+1) - Y^{(k_0-1)}(t+1))(A_j(t)+B_j(t)F^{(0)}(t)) \leq \varepsilon^2 I_n$*,* $0 \leq t \leq \theta-1$*, then* $X^{(1)}(t) \triangleq Y^{(k_0)}(t)$*,* $0 \leq t \leq \theta - 1$*, satisfy the system of LMIs (14).*

The proof is a special case of Corollary 5.3 from [5].

Remark 3.2. For the computation of the θ-periodic solution of the equation (25)-(27) one may use the procedure described in Subsection 3.1.

4 The computation of the θ-periodic solution of a discrete-time backward Stein equation of stochastic control

In this section we shall present an alternative method for the computation of the θ-periodic solution of backward affine equations of type (11)-(13). These equations are special cases of a discrete-time backward affine equation of the form:

$$X(t) = \sum_{j=0}^{r} \hat{A}_j^T(t)X(t+1)\hat{A}_j(t) + G(t) \tag{28}$$

where $\{\hat{A}_j(t)\}_{t\in\mathbb{Z}} \subset \mathbb{R}^{n\times n}$, $0 \leq j \leq r$, $\{G(t)\}_{t\in\mathbb{Z}} \subset \mathcal{S}_n$ are periodic sequences of period θ. Assume that the zero solution of the discrete-time stochastic linear equation:

$$x(t+1) = (\hat{A}_0(t) + \sum_{j=1}^{r} w_j(t)\hat{A}_j(t))x(t) \tag{29}$$

is ESMS. Under these condition (28) has a unique bounded on $\mathbb{Z}$ solution $\hat{X}(\cdot)$ and additionally that solution is periodic with period θ.

Reasoning as in the case of the equation (24) one obtains that $\hat{X}(t) = \lim_{k\to\infty} Z^{(k)}(t)$, where $Z^{(k)}(\cdot), k \geq 1$ is the unique θ-periodic solution of the backward Stein equation:

$$\begin{aligned} Z^{(k)}(t) &= \hat{A}_0^T(t)Z^{(k)}(t+1)\hat{A}_0(t)+\sum_{j=1}^{r}\hat{A}_j^T(t)Z^{(k-1)}(t+1)\hat{A}_j(t)+G(t) \\ Z^{(0)}(t) &= 0, \;\; t \in \mathbb{Z}. \end{aligned} \tag{30}$$

In the following, we shall provide an alternative method which allows us to avoid the iterative process described in (30) to obtain the θ-periodic solution of (28).

4.1 The periodic solution of a discrete-time backward affine equation on an Euclidian space

Let us consider the discrete-time equation

$$x(t) = \hat{M}(t)x(t+1) + g(t) \tag{31}$$

where $\{\hat{M}(t)\}_{t\in\mathbb{Z}} \subset \mathbb{R}^{\hat{n}\times\hat{n}}$, $\{g(t)\}_{t\in\mathbb{Z}} \subset \mathbb{R}^{\hat{n}}$ are periodic sequences of period θ. Assume that the linear equation associated to (31):

$$x(t) = \hat{M}(t)x(t+1) \tag{32}$$

has not nonzero solutions which are periodic of period θ. We set $\hat{T}(t,s) = \hat{M}(t)\hat{M}(t+1)...\hat{M}(s-1)$ if $t < s$ and $\hat{T}(t,s) = I_{\hat{n}}$ if $t = s$. $\hat{T}(t,s)$ is the anti-causal evolution operator defined on $\mathbb{R}^{\hat{n}}$ by the discrete-time backward equation (32).

The solutions of (31) have the representation:

$$x(t) = \hat{T}(t,\tau)x(\tau) + \sum_{s=t}^{\tau-1}\hat{T}(t,s)g(s), \;\forall\;\; t \leq \tau - 1 \in \mathbb{Z}.$$

The periodicity condition $x(0) = x(\theta)$ leads to

$$x(0) = \hat{T}(0,\theta)x(0) + \sum_{s=0}^{\theta-1}\hat{T}(0,s)g(s).$$

Hence, the initial condition $x(0)$ of the unique θ-periodic solution of (31) one obtains solving the system of linear equations

$$(I_{\hat{n}} - \hat{T}(0,\theta))\zeta = \tilde{g} \tag{33}$$

where

$$\tilde{g} = \sum_{s=0}^{\theta-1} \hat{T}(0,s)g(s). \tag{34}$$

Since the linear equation (32) has no nonzero solutions which are periodic sequences of period θ we deduce that $det(I_{\hat{n}} - \hat{T}(0,\theta)) \neq 0$. This allows us to conclude that the equation (33)-(34) has a unique solution $\zeta = \tilde{x}(0) = \tilde{x}(\theta)$. The other values $\tilde{x}(t)$, $1 \leq t \leq \theta-1$ of the periodic solution $\tilde{x}(\cdot)$ are obtained directly from (31).

Remark 4.1 The term $\tilde{g}$ from (34) may be obtain also from $\tilde{g} = x(0;\theta,0)$ where $ttox(t;\theta,0)$ is the solution of (31) satisfying the final condition $x(\theta;\theta,0) = 0$.

4.2 The H-representation technique revisited

In this paragraph we briefly recall the method of H-representation of a Lyapunov operator in terms of a matrix on the space of dimension $\hat{n} = \frac{n(n+1)}{2}$. This allows us to rewrite the equation (28) in the form of an equation of type (31).

For details we refer to [21], where this method was introduced. We recall that if $X \in \mathbb{R}^{n\times n}$, then $\Psi(X) = Vec(X) = (x(1), x(2), ..., x(n))^T \in \mathbb{R}^{n^2}$ where $x(i)$ is the i^th line of the matrix X, $1 \leq i \leq n$.
Let $E_{11}, E_{12}, ..., E_{1n}, E_{22}, ..., E_{2n}, ..., E_{n-1n-1}, E_{n-1n}, E_{nn}$ be the standard base of the space of symmetric matrices $\mathcal{S}_n$.
This means that $E_{pq} = (e_{pq}(i,j))_{\overline{i,j=1,n}}$ with $e_{pq}(ij) = 1$ if $(ij) \in \{(pq),(qp)\}$ and $e_{pq}(ij) = 0$ otherwise. If $X \in \mathcal{S}_n$ is an arbitrary symmetric matrix, then

$$X = E_{11}x_1 + ... + E_{1n}x_n + E_{22}x_{n+1} + ... + E_{nn}x_{\hat{n}}. \tag{35}$$

We introduce the linear operator $\varphi : \mathcal{S}_n \to \mathbb{R}^{\hat{n}}$ defined by

$$\varphi(X) = x \tag{36}$$

where $x = (x_1, x_2, ..., x_{\hat{n}})^T$ is the vector whose components occur in the right hand side of (35). We introduce also the matrix

$$H = \begin{pmatrix} \Psi(E_{11}) & \Psi(E_{12}) & ... & \Psi(E_{1n}) & \Psi(E_{22}) & ... & \Psi(E_{n-1n}) & \Psi(E_{nn}) \end{pmatrix}.$$

The matrix H has n^2 lines and $\frac{n(n+1)}{2}$ columns. Also, $rankH = \frac{n(n+1)}{2}$. For details see for example [21].
From the definition of the operators φ, Ψ and of the matrix H, we obtain the following fundamental relation:

$$\Psi(X) = H\varphi(X) \tag{37}$$

for all $X \in \mathcal{S}_n$. Let $\mathcal{L}(t) : \mathcal{S}_n \to \mathcal{S}_n$,

$$\mathcal{L}(t)X = \sum_{j=0}^{r} \hat{A}_j^T(t)XA_j(t). \tag{38}$$

Applying Lemma 2.2. in [21] we may write

$$\Psi(\mathcal{L}(t)X) = (\sum_{j=0}^{r} \hat{A}_j^T(t) \otimes \hat{A}_j^T(t))\Psi(X)$$

for all $X \in \mathcal{S}_n$, $\otimes$ being the Kronecker product. Using (37) we obtain

$$\Psi(\mathcal{L}(t)X) = (\sum_{j=0}^{r} \hat{A}_j^T(t) \otimes \hat{A}_j^T(t))H\varphi(X), \ \ \forall X \in \mathcal{S}_n. \tag{39}$$

4.3 The computation of the θ-periodic solution of the equation (28)

Now we show how the computation of the θ-periodic solution of (28) can be reduced to the computation of the θ-periodic solution of an equation of type (31.

First, let us remark that (38) allows us to write (28) in a compact form:

$$X(t) = \mathcal{L}(t)X(t+1) + G(t) \tag{40}$$

Since $\Psi : \mathbb{R}^{n\times n} \to \mathbb{R}^{n^2}$ is an isomorphism we may deduce that the equation (40) is equivalent to the equation:

$$\Psi(X(t)) = \Psi(\mathcal{L}(t)X(t+1)) + \Psi(G(t)). \tag{41}$$

Based on (37) and (39) we rewrite (41) in the form

$$H\varphi(X(t)) = (\sum_{j=0}^{r} \hat{A}_j^T(t) \otimes \hat{A}_j^T(t))H\varphi(X(t+1)) + H\varphi(G(t)). \tag{42}$$

Multiplying to the left (42) by H^T and taking into account that $H^T H$ is invertible, we obtain that $x(t) \triangleq \varphi(X(t))$ is a solution of the discrete-time backward equation on $\mathbb{R}^{\hat{n}}$:

$$x(t) = M(t)x(t+1) + g(t) \tag{43}$$

where

$$M(t) = \sum_{j=0}^{r} (H^T H)^{-1} H^T (\hat{A}_j^T(t) \otimes \hat{A}_j^T(t))H \tag{44}$$

and

$$g(t) = \varphi(G(t)). \tag{45}$$

We have:

Proposition 4.1 *(i) If $\{X(t)\}_{t\in\mathbb{Z}}$ is a global solution of equation (28) then $\{x(t)\}_{t\in\mathbb{Z}}$ defined by $x(t) = \varphi(X(t))$, $t \in \mathbb{Z}$ is a global solution of equation (43)-(45).*

(ii) If $\{\tilde{x}(t)\}_{t\in\mathbb{Z}}$ is a global solution of the backward affine equation (43)-(45), then $\{\tilde{X}(t)\}_{t\in\mathbb{Z}}$ defined by $\tilde{X}(t) = \varphi^{-1}(\tilde{x}(t))$ is a global solution of equation (28).

Proof. (i) follows immediately from the previous developments.

(ii) Let $\tilde{x}(\cdot)$ be a global solution of (43)-(45). If $\tilde{X}(t) = \varphi^{-1}(\tilde{x}(t))$, $t \in \mathbb{Z}$, we define $\Delta(t) = \tilde{X}(t) - \mathcal{L}(t)\tilde{X}(t+1) - G(t)$. We have to show that $\Delta(t) = 0$, for all $t \in \mathbb{Z}$. The previous equality is rewritten as:

$$\tilde{X}(t) = \mathcal{L}(t)\tilde{X}(t+1) + G(t) + \Delta(t). \tag{46}$$

Using again (37), (39) and taking into account that $\varphi(\tilde{X}(t)) = \tilde{x}(t)$ we deduce from (46) that

$$H\tilde{x}(t) = (\sum_{j=0}^{r} \hat{A}_j^T(t) \otimes \hat{A}_j^T(t))H\tilde{x}(t+1) + H\varphi(G(t)) + H\varphi(\Delta(t)). \tag{47}$$

Multiplying to the left (47) by H^T and taking into account that $H^T H$ is invertible, we obtain via (44) and (45) that:

$$\tilde{X}(t) = M(t)\tilde{x}(t+1) + g(t) + \varphi(\Delta(t)).$$

Since $\tilde{x}(\cdot)$ is a solution of (43)-(45) we infer that $\varphi(\Delta(t)) = 0$, $t \in \mathbb{Z}$. Taking into account that φ is an invertible operator, we may conclude that $\Delta(t) = 0$ for all $t \in \mathbb{Z}$, which ends the proof.

Remark 4.2 It is easy to see that $\tilde{x}(t)$, $t \in \mathbb{Z}$, is a θperiodic solution of (43)-(45) if and only if $\varphi^{-1}(\tilde{x}(t))$, $t \in \mathbb{Z}$ is a θ-periodic solution of (28).

Proposition 4.2 *If the zero solution of equation (29) is ESMS then the zero solution is the only one θ-periodic solution of the backward linear equation*

$$x(t) = M(t)x(t+1) \tag{48}$$

associated to (43)-(45).

Proof. Let $\{\hat{x}(t)\}_{t\in\mathbb{Z}}$ be a θ-periodic solution of (48). Let $\hat{X}(t) = \varphi^{-1}(\hat{x}(t))$, $t \in \mathbb{Z}$. From Proposition 4.1 and Remark 4.2 we deduce that $\hat{X}(\cdot)$ is a θ-periodic solution of the linear backward equation

$$X(t) = \mathcal{L}(t)X(t+1). \tag{49}$$

Applying Theorem 2.5 and Theorem 3.11 in [5] we deduce that if the zero-solution of (29) is ESMS, then the discrete-time backward equation (49) has a unique, periodic solution of period θ. Hence, $\hat{X}(t) = 0$, $t \in \mathbb{Z}$. This allows us to deduce that $\hat{x}(t) = \varphi(0) = 0$, $t \in \mathbb{Z}$. Thus the proof is complete.

So, the computation of the value $\tilde{x}(\theta)$ of the θ-periodic solution of the equation (43)-(45) can be performed solving the linear system of $\frac{n(n+1)}{2}$ scalar equations with $\frac{n(n+1)}{2}$ scalar unknowns:

$$(I - T(0,\theta))\zeta = \tilde{g} \tag{50}$$

where

$$\tilde{g} = \sum_{s=0}^{\theta-1} T(0,s)g(s) \tag{51}$$

$T(t,s)$ being the anticausal linear evolution operator on $\mathbb{R}^{\hat{n}}$ defined by the backward linear equation (48) and $g(s)$ are the ones defined in (45).

If $\tilde{x}(\theta) = \zeta$ is the unique solution of the linear equation (50)-(51) then the value $\tilde{X}(\theta)$ of the θ-periodic solution of (28) is obtained by

$$\tilde{X}(\theta) = \varphi^{-1}(\tilde{x}(\theta)). \tag{52}$$

To this end, the components of the vector $\tilde{x}(\theta)$ are plugged in the right hand side of (35). The other related values $\tilde{X}(t)$, $1 \le t \le \theta - 1$ of the θ-periodic solution $\tilde{X}(\cdot)$ are obtained directly from (28).

5 Numerical experiments

In this section we present how the considered iterations work for finding a stabilizing solution to (1). We will carry out experiments for numerically solving discrete-time generalized Riccati equation (1).

Our experiments are executed in MATLAB on a 2,16GHz Intel(R) Duo CPU computer. We denote *tol*- a small positive real number denoting the accuracy of computation; $E = max_t\ ||X^{(k)}(t) - \mathcal{G}(X^{(k)}(t+1))||_2$. We use the following stop criterion for all algorithms:

$$E \le tol\,.$$

5.1 Example 1

Consider a discrete-time 3-periodic linear system with r=1, t=0,1,2, given by (n=3) the coefficient matrices:

$$A_0(0)=\begin{pmatrix} -0.466 & 0.0100 & 0.002 \\ -0.09 & -0.45 & 0.1 \\ -0.035 & -0.01 & -0.485 \end{pmatrix}, \quad A_0(1)=\begin{pmatrix} -0.33 & -0.03 & -0.004 \\ -0.075 & -0.49 & 0.09 \\ -0.025 & -0.015 & -0.495 \end{pmatrix},$$

$$A_0(2)=\begin{pmatrix} -0.45 & 0 & -0.001 \\ -0.095 & -0.505 & 0.1 \\ 0.033 & -0.02 & -0.473 \end{pmatrix}, \quad A_1(0)=\begin{pmatrix} -0.055 & -0.05 & -0.008 \\ 0.13 & -0.12 & 0 \\ -0.3 & 0.25 & 0 \end{pmatrix},$$

$$A_1(1)=\begin{pmatrix} -0.04 & 0.02 & -0.02 \\ 0.2 & -0.035 & -0.01 \\ -0.1 & -0.25 & -0.06 \end{pmatrix}, \quad A_1(2)=\begin{pmatrix} 0 & -0.01 & 0.04 \\ 0.1 & -0.055 & 0 \\ 0.02 & 0.025 & -0.045 \end{pmatrix},$$

$$B_0(0)=\begin{pmatrix} 1 & 12 & -5 \\ 0.1 & -1 & 1.5 \\ 0.2 & -0.5 & 0 \end{pmatrix}, \quad B_0(1)=\begin{pmatrix} 1 & 8 & 4.5 \\ -0.5 & -3 & -2.5 \\ -1 & -0.8 & -0.6 \end{pmatrix},$$

$$B_0(2)=\begin{pmatrix} 1 & -6.5 & -8 \\ 1 & -2.5 & 6 \\ -0.8 & -0.8 & -0.4 \end{pmatrix}, \quad B_1(0)=\begin{pmatrix} -1 & 10 & -5 \\ 0.2 & -1 & -1.5 \\ -0.2 & -2 & -0.5 \end{pmatrix},$$

$$B_1(1)=\begin{pmatrix} 1 & -6.5 & -8 \\ 1 & -2.5 & 6 \\ -0.8 & -0.8 & -0.4 \end{pmatrix}, \quad B_1(2)=\begin{pmatrix} -1 & 10 & -5 \\ 0.2 & -1 & -1.5 \\ -0.2 & -2 & -0.5 \end{pmatrix}.$$

$$L(0)=\frac{1}{90}\begin{pmatrix} -0.5 & -0.3 & -0.4 \\ -0.25 & -0.4 & -0.6 \\ -0.5 & -0.5 & -0.8 \end{pmatrix}, \quad L(1)=\frac{1}{90}\begin{pmatrix} -0.5 & -0.14 & -0.8 \\ -0.5 & -0.5 & -0.8 \\ -0.6 & -0.8 & -0.3 \end{pmatrix}.$$

$$L(2)=\frac{1}{90}\begin{pmatrix} -0.3 & -0.15 & -0.7 \\ -0.6 & -0.6 & -0.5 \\ -0.4 & -0.7 & -0.4 \end{pmatrix}, \quad \begin{cases} R(0) = diag(1.5; 1.5; 1.5), \\ R(1) = diag(1; 1; 1), \\ R(2) = diag(1.25; 1.25; 1.25), \\ M(0) = M(1) = M(2) = 0\,. \end{cases}.$$

We have found the solutions $Y(0), Y(1), Y(2)$ using inequality (8). Then we compute $F(0), F(1), F(2)$ using (9). Thus, we can apply iteration (11). After one iteration steps we obtain the stabilizing solution to (1). The solution is negative definite. Next, we compute the stabilizing solution using iteration (15). We solve inequality (14) for finding $X^{(1)}(0), X^{(1)}(1), X^{(1)}(2)$. We need three LMI iteration steps for solving (14). We find the solution after 7 iteration steps with (15).

Next iteration (16). The solution is obtained after 5 iteration steps.

Next iteration (17). The solution is obtained after 6 iteration steps.

5.2 Two additional examples

Let us consider the new discrete-time 3-periodic linear system with r=1, t=0,1,2. The matrix coefficients are constructed using the following MATLAB code:

$$\begin{aligned}
&A_j(t) = randn(n,n);\ m1 = max(A_j(t));\ m2 = max(m1);\\
&A_j(t) = A_j(t)/(10 * m2); \qquad j = 0,1\\
&B_j(t) = randn(n,n);\ \ m1 = max(B_j(t));\ m2 = max(m1);\\
&B_j(t) = B_j(t)/(m2); \qquad j = 0,1\\
&L(t) = abs(randn(n,n));\ \ m1 = max(L(t));\ m2 = max(m1);\\
&L(t) = -L(t)/(80 * m2);\\
&M(t) = zeros(n,n);
\end{aligned}$$

5.2.1 Example 2.1

$R(0) = eye(n,n) * 1.05$; $R(1) = eye(n,n) * 0.175$; $R(2) = eye(n,n) * 0.125$.

In this table the full execution time for each iteration is given. This includes the time for computing the initial point $X^{(1)}(0), X^{(1)}(1), X^{(1)}(2)$ and the time for approximating the stabilizing solution using the corresponding iteration formula.

Results for $\mathbf{n = 8}$ and $tol = 1e-5$ for 50 runs are: the CPU time for iteration (11) is 18.0620 seconds; the average number of iteration steps is 2.02 and the maximal error from all runs is $E = 3.7852e-06$.

Results for $\mathbf{n = 8}$ and $tol = 1e-5$ for 50 runs are: the CPU time for iteration (15) is 3.7970 seconds; the average number of iteration steps is 4.1800 and the maximal error from all runs is $E = 4.8106e-06$.

Results for $\mathbf{n = 8}$ and $tol = 1e-5$ for 50 runs are: the CPU time for iteration (16) is 4.6560 seconds; the average number of iteration steps is 4.0 and the maximal error from all runs is $E = 9.0651e-06$.

Results for $\mathbf{n = 8}$ and $tol = 1e-5$ for 50 runs are: the CPU time for iteration (17) is 4.77 seconds; the average number of iteration steps is 5.06 and the maximal error from all runs is $E = 6.9399e-06$.

Results for $\mathbf{n = 12}$ and $tol = 1e-5$ for 50 runs are: the CPU time for iteration (11) is 125.9060 seconds; the average number of iteration steps is 2.08 and the maximal error from all runs is $E = 9.5401e-06$.

Results for $\mathbf{n = 12}$ and $tol = 1e-5$ for 50 runs are: the CPU time for iteration (15) is 11.7350 seconds; the average number of iteration steps is 4.26 and the maximal error from all runs is $E = 9.4270e-06$.

Results for $\mathbf{n = 12}$ and $tol = 1e-5$ for 50 runs are: the CPU time for iteration (16) is 12.8440 seconds; the average number of iteration steps is 4.160 and the maximal error from all runs is $E = 8.4240e-06$.

Results for $\mathbf{n = 12}$ and $tol = 1e-5$ for 50 runs are: the CPU time for iteration (17) is 15.0380 seconds; the average number of iteration steps is 5.36 and the maximal error from all runs is $E = 9.0533e-06$.

5.2.2 Example 2.2

We choose: $R(0) = eye(n,n) * 1.45$; $R(1) = eye(n,n) * 0.175$; $R(2) = eye(n,n) * 0.125$.

We present the full information about each iteration. This includes the time for approximating the stabilizing solution using the corresponding iteration formula. The initial point $X^{(1)}(0), X^{(1)}(1), X^{(1)}(2)$ is the same for all iterations and it is computing via (14).

Results for $\mathbf{n} = \mathbf{18}$ and $tol = 1e-4$ for 50 runs are: the CPU time for iteration (11) is 811.5010 seconds; the average number of iteration steps is 2.36 and the maximal error from all runs is $E = 9.8986e-05$.

Results for $\mathbf{n} = \mathbf{18}$ and $tol = 1e-4$ for 50 runs are: the CPU time for iteration (15) is 0.622 seconds; the average number of iteration steps is 4.5800 and the maximal error from all runs is $E = 8.2157e-05$.

Results for $\mathbf{n} = \mathbf{18}$ and $tol = 1e-4$ for 50 runs are: the CPU time for iteration (16) is 3.312 seconds; the average number of iteration steps is 4.2200 and the maximal error from all runs is $E = 9.3210e-05$.

Results for $\mathbf{n} = \mathbf{18}$ and $tol = 1e-4$ for 50 runs are: the CPU time for iteration (17) is 3.61 seconds; the average number of iteration steps is 5.46 and the maximal error from all runs is $E = 8.6559e-05$.

6 Conclusion

We have considered four iterations for computing the stabilizing solution to (1). In order to execute iterations (11), (16) and (17) we have to solve a linear system with a big dimension, i.e. it has the size $(\theta\, n)^2 \times (\theta\, n)^2$. In the same time iteration (15) gives us a possibility to find $X^{(k+1)}(t)$ in the following way:

$$\begin{aligned}
&X^{(k+1)}(\theta-2) = \\
&= \textstyle\sum_{j=0}^{r} (A_j(\theta-2) + B_j(\theta-2)F^{(k)}(\theta-2))^T X^{(k)}(\theta-1) \\
&\times (A_j(\theta-2) + B_j(\theta-2)F^{(k)}(\theta-2)) + Q_{F^{(k)}}(\theta-2) + \tfrac{\varepsilon^2}{k+1} I_n
\end{aligned}$$

$$\begin{aligned}
&X^{(k+1)}(\theta-3) = \\
&= \textstyle\sum_{j=0}^{r} (A_j(\theta-3) + B_j(\theta-3)F^{(k)}(\theta-3))^T X^{(k+1)}(\theta-2) \\
&\times (A_j(\theta-3) + B_j(\theta-3)F^{(k)}(\theta-3)) + Q_{F^{(k)}}(\theta-3) + \tfrac{\varepsilon^2}{k+1} I_n
\end{aligned}$$

....

$$\begin{aligned}
&X^{(k+1)}(0) = \\
&= \textstyle\sum_{j=0}^{r} (A_j(0) + B_j(0)F^{(k)}(0))^T X^{(k+1)}(1)(A_j(0) + B_j(0)F^{(k)}(0)) \\
&\quad + Q_{F^{(k)}}(0) + \tfrac{\varepsilon^2}{k+1} I_n
\end{aligned}$$

$$X^{(k+1)}(\theta-1) =$$
$$= \sum_{j=0}^{r} (A_j(\theta-1) + B_j(\theta-1)F^{(k)}(\theta-1))^T X^{(k+1)}(0)$$
$$\times (A_j(\theta-1) + B_j(\theta-1)F^{(k)}(\theta-1)) + Q_{F^{(k)}}(\theta-1) + \frac{\varepsilon^2}{k+1} I_n$$

We call the last iteration the improved approximation method. This method is applied to Example 2.2. Results for $\mathbf{n} = \mathbf{18}$ and $tol = 1e-4$ for 50 runs are: the CPU time is 0.02 seconds; the average number of iteration steps is 2.06 and the maximal error from all runs is $E = 3.7096e-05$.

Acknowledgement. The work of the first author was supported by Grant 145/2011 of CNCS Romania.

References

[1] F.A. Aliev, V.B. Larin, Optimization Problems for Periodic Systems, *Int. Appl. Mech.*,45,11, 1162?188, 2009.

[2] B.D.O. Anderson, J.B. Moore, *Optimal Control: Linear Quadratic Methods*, Prentice-Hall, Englewood Cliffs, NJ, 1989.

[3] S. Bittanti, P.Colaneri. *Periodic Systems, Filtering and Control*, Springer- Verlag, London, 2009.

[4] V. Drăgan, A. Halanay, *Stabilization of linear systems* - Birkhauser, Boston, 1999.

[5] V. Drăgan, T. Morozan, A.M.Stoica. *Mathematical Methods in Robust Control of Discrete-time Linear Stochastic Systems.* Springer, New-York, 2010.

[6] V. Drăgan, S. Aberkane, I. G. Ivanov, On computing the stabilizing solution of a class of discrete-time periodic Riccati equations, *Int. J. Robust Nonlinear Control*, (2013), published online DOI: 10.1002/rnc.3131.

[7] A. Halanay, T. Morozan, Stabilization by linear feedback of linear discrete stochastic systems, *Rev. Roumaine Math. Pures Appl.,*23,4, 561-571, 1978.

[8] A. Halanay, T. Morozan, Optimal stabilizing compensators for linear discrete-time systems under independent perturbations,*Revue Roum. Math. Pure et Appl.*, 37, 3, 213-224, 1992.

[9] A. Halanay, V. Ionescu, *Time varying discrete linear systems,* Berlin, Birkhauser, 1994.

[10] A. Halanay, Vl. Rasvan. *Discrete Time Systems. Stability and Stable Oscillations*, Gordon and Breach, 2000.

[11] I. Ivanov, Accelerated LMI solvers for the maximal solution to a set of discrete-time algebraic Riccati equations, *Appl. Math. E-Notes.* 12: 228–238, 2012. http://www.math.nthu.edu.tw/ amen/, open access

[12] I. Ivanov, Iterations for a General Class of Discrete-Time Riccati-Type Equations: A Survey and Comparison, open access, DOI: 10.5772/45718, 2012.

[13] I.Ivanov, An Improved Method for Solving a System of Discrete-Time Generalized Riccati Equations, *Journal of Numerical Mathematics and Stochastics.* 3(1): 57-70, 2011. http://www.jnmas.org/jnmas3-7.pdf

[14] I.Ivanov, V. Dragan, Decoupled Stein Iterations to the Discrete-time Generalized Riccati Equations, *IET Control Theory Appl.* 6(10): 1400–1409, 2012.

[15] R.E. Kalman, Contributions to the theory of optimal control, *Bull. Soc. Math. Mex.*, 5, 102?19, 1960.

[16] D.L. Kleinman, On an iterative technique for Riccati equation computation, *IEEE Transactions on Automatic Control*, 13, 114?15, 1968.

[17] H. Kwakernaak, R. Sivan, *Linear optimal control systems*, New York, Willey Interscience, 1972.

[18] V.B. Larin, High-Accurasy Algorithms for Solving of Discrete Periodic Riccati Equation, *Appl. Comput. Math.*, 6, 1, 10-17, 2007.

[19] V. M. Ungureanu, V. Drăgan, T. Morozan, Global solutions of a class of discrete-time backward nonlinear equations on ordered Banach spaces with applications to Riccati equations of stochastic control, *Optimal Control, Applications and Methods*, 34, 2, 164–190, 2013.

[20] W.M. Wonham, Random differential equations in control theory, *in Probabilistic Methods in Applied Mathematics, vol. 2 (Academic, New York)*, pp. 131?12, 1970.

[21] W. Zhang, B. Chen. H-Representation and Applications to Generalized Lyapunov Equations and Linear Stochastic Systems. *IEEE Trans. on Aut. Contr.*57 (12): 3009–3022, 2012.

[22] J. Zabczyk, Stochastic control of discrete-time systems, *Control Theory and Topics in Funct. Analysis*, 3, IAEA, Vienna, 1976.