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NONLINEAR DELAY EVOLUTION INCLUSIONS WITH GENERAL NONLOCAL INITIAL CONDITIONS *

Mihai Necula[†] Ioan I. Vrabie[‡]

Dedicated to the memory of Prof. Dr. Viorel Arnăutu

Abstract

We consider a nonlinear delay differential evolution inclusion subjected to nonlocal implicit initial conditions and we prove an existence result for bounded C^0-solutions.

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1 Introduction

The goal of this paper is to prove an existence result for bounded C^0-solutions to a class of nonlinear delay differential evolution inclusions sub-

*Accepted for publication on December 21-st, 2014
[†]necula@uaic.ro Department of Mathematics, "Al. I. Cuza" University Iaşi 700506, Romania
[‡]ivrabie@uaic.ro Department of Mathematics, "Al. I. Cuza" University Iaşi 700506, Romania and Octav Mayer Mathemtics Institute (Romanian Academy) Iaşi 700505, Romania

jected to nonlocal implicit initial conditions of the form

$$\begin{cases} u'(t) \in Au(t) + f(t), & t \in \mathbf{R}_+, \\ f(t) \in F(t, u_t), & t \in \mathbf{R}_+, \\ u(t) = g(u)(t), & t \in [\,-\tau, 0\,], \end{cases} \tag{1}$$

where X is a Banach space, $\tau \geq 0$, $A : D(A) \subseteq X \hookrightarrow X$ is the infinitesimal generator of a nonlinear semigroup of contractions, the multifunction $F : \mathbf{R}_+ \times C([\,-\tau, 0\,]; \overline{D(A)}) \hookrightarrow X$ is nonempty, convex weakly compact valued and strongly-weakly u.s.c., and $g : C_b([\,-\tau, +\infty); \overline{D(A)}) \to C([\,-\tau, 0\,]; \overline{D(A)})$ is nonexpansive and has *affine growth*, i.e. there exists $m_0 \geq 0$ such that

$$\|g(u)\|_{C([\,-\tau,0\,];X)} \leq \|u\|_{C_b([\,0,+\infty);X)} + m_0 \tag{2}$$

for each $u \in C_b([\,-\tau, +\infty); \overline{D(A)})$.

If I is an interval, $C_b(I; X)$ denotes the space of all bounded and continuous functions from I, equipped with the sup-norm $\|\cdot\|_{C_b(I;X)}$, while $C_b(I; \overline{D(A)})$ denotes the closed subset in $C_b(I; X)$ consisting of all elements $u \in C_b(I; X)$ satisfying $u(t) \in \overline{D(A)}$ for each $t \in I$. Let $a \in \mathbf{R}$. On the linear space $C_b([\,a, +\infty); X)$ let us consider the family of seminorms $\{\|\cdot\|_k;\ k \in \mathbf{N},\ k \geq a\}$, defined by $\|u\|_k = \sup\{\|u(t)\|;\ t \in [\,a, k\,]\}$ for each $k \in \mathbf{N}$, $k \geq a$. Endowed with this family of seminorms, $C_b([\,a, +\infty); X)$ is a separated locally convex space, denoted by $\widetilde{C}_b([\,a, +\infty); X)$. Further, $C([\,a, b\,]; X)$ stands for the space of all continuous functions from $[\,a, b\,]$ to X endowed with the sup-norm $\|\cdot\|_{C([\,a,b\,];X)}$ and $C([\,a, b\,]; \overline{D(A)})$ is the closed subset of $C([\,a, b\,]; X)$ containing all $u \in C([\,a, b\,]; X)$ with $u(t) \in \overline{D(A)}$ for each $t \in [\,a, b\,]$. Finally, if $u \in C_b([\,-\tau, +\infty); X)$ and $t \in \mathbf{R}_+$, $u_t \in C([\,-\tau, 0\,]; X)$ is defined by

$$u_t(s) := u(t + s)$$

for each $s \in [\,-\tau, 0\,]$.

The existence problem on the standard compact interval $[\,0, 2\pi\,]$, in the simplest case when $\tau = 0$, i.e. when the delay is absent, was studied by Paicu, Vrabie [41]. In this case $C([\,-\tau, 0\,]; \overline{D(A)})$ identifies with $\overline{D(A)}$, F identifies with a multifunction from $[\,0, 2\pi\,] \times X$ to X. By using an interplay between compactness arguments and invariance techniques, they have proved an existence result handling periodic, anti-periodic, mean-value evolution inclusions subjected to initial condition expressed by an integral with respect to a Radon measure μ. A very important specific case concerns T-periodic problems, which corresponds to the choice of g as $g(u) = u(T)$, was studied by Paicu [39]. For F single-valued, this case was analyzed by Aizicovici, Papageorgiou, Staicu [3], Caşcaval, Vrabie [18], Hirano, Shioji [34], Paicu [40], Vrabie [44]. For a survey concerning: periodic, anti-periodic, quasi-periodic and almost periodic solutions to differential inclusions, see Andres [6]. As long as differential inclusions subjected to general nonlocal initial conditions without delay are concerned, we mention the papers of Aizicovici, Staicu [5] and Paicu, Vrabie [41]. The case of periodic retarded equations and inclusions subjected to nonlocal initial conditions were studied by Vrabie [46], and Chen, Wang, Zhou [20], while the general delay equations was considered by Burlică, Roşu [14] and Vrabie [48], [49] and [50].

Existence results in the periodic abstract undelayed case were obtained by Aizicovici, Papageorgiou, Staicu [3], Caşcaval, Vrabie [18], Hirano, Shioji [34], Paicu [40], Vrabie [44], while the anti-periodic case was considered by Aizicovici, Pavel, Vrabie [4]. The semilinear case of undelayed differential equations subjected to nonlocal initial data, was initiated by the pioneering work of Byszewski [15]. Further steps in this direction were made by Byszewski [16], Byszewski, Lakshmikantham [17], Aizicovici, Lee [1], Aizicovici, McKibben [2], Zhenbin Fan, Qixiang Dong, Gang Li [27], García-Falset [29] and García-Falset, Reich [30]. All these studies are strongly motivated by the fact that specific problems of this kind describe the evolution of various phenomena in Physics, Meteorology, Thermodynamics, Population Dynamics. A model of the gas flow through a thin transparent tube, expressed as a problem with nonlocal initial conditions, was analyzed in Deng [24]. Some models in Pharmacokinetics were discussed in the monograph of McKibben [35, Section 10.2, pp. 394–398]. Models arising from Physics were analyzed by Olmstead, Roberts [38] and Shelukhin [43]. Linear second order evolution equations subjected to linear nonlocal initial conditions in Hilbert triples were considered in Avalishvili, Avalishvili [8] and motivated by mathematical models for long-term reliable weather forecasting as mentioned in Rabier, Courtier, Ehrendorfer [42]. For Navier-Stokes equations subjected to initial nonlocal conditions see Gordeziani [32]. Classical nonlinear delay evolution initial-value problems, i.e. when $g \equiv \psi$ with $\psi \in C([\tau, 0]; \overline{D(A)})$, were considered by Mitidieri, Vrabie [36] and [37], also by using compactness arguments. It should be emphasized that in Mitidieri, Vrabie [36] and [37] the general assumptions on the forcing term F are very general allowing – in certain specific cases when A is a second order elliptic operator – F to depend on Au as well.

Our paper extends the main result in Vrabie [47] to cover the more general case in which g has affine rather than linear growth. This case is important in applications and does not follow by a simple modification of the arguments used in Vrabie [47].

The paper is divided into 7 sections. In Section 2 we have included some concepts and results widely used subsequently. In Section 3 we prove an existence and uniqueness result for the unperturbed problem (1) which, although auxiliary, is important by its own. Section 4 collects the hypotheses used and provides some comments on several remarkable particular cases handled by the general frame considered. Section 5 is devoted to the statement of the main result, i.e. Theorem 7 and to a short description of the idea of the proof. Section 6 is concerned with the proof of the main result and the last Section 7 contains an example illustrating the possibilities of the abstract developed theory.

2 Preliminaries

Although the paper is almost self-contained, some familiarity with the basic concepts and results on nonlinear evolution equations governed by m-dissipative operators, delay evolution equations and on multifunction theory would be welcome. For details in these three topics, we refer the reader, in order, to Barbu [11], Hale [33] and Vrabie [45]. However, we recall for easy reference the most important notions and results we will use in the sequel.

Definition 1 If X is a Banach space and $\mathcal{C} \subseteq X$, the multifunction $F : \mathcal{C} \hookrightarrow X$ is said (*strongly-weakly*) *upper semicontinuous* (*u.s.c.*) *at* $\xi \in \mathcal{C}$ if for every (weakly) open neighborhood V of $F(\xi)$ there exists an open neighborhood U of ξ such that $F(\eta) \subseteq V$ for each $\eta \in U \cap \mathcal{C}$. We say that F is (*strongly-weakly*) *u.s.c. on* $\mathcal{C}$ if it is (strongly-weakly) u.s.c. at each $\xi \in \mathcal{C}$.

Definition 2 A multifunction $F : I \times \mathcal{C} \hookrightarrow X$ is said to be *almost strongly-weakly u.s.c.* if for each $\gamma > 0$ there exists a Lebesgue measurable subset $E_\gamma \subseteq I$ whose Lebesgue measure $\lambda(E_\gamma) \leq \gamma$ and such that F it is strongly-weakly u.s.c. from $(I \setminus E_\gamma) \times \mathcal{C}$ to X.

Remark 1 If the sequence $(\varepsilon_n)_n$ is strictly decreasing to 0, we can always choose the sequence $(E_{\varepsilon_n})_n$, where E_{ε_n} corresponds to ε_n as specified in Definition 2, such that $E_{\varepsilon_{n+1}} \subseteq E_{\varepsilon_n}$, for $n = 0, 1, \ldots$.

We also need the following general fixed point theorem for multifunctions obtained independently by Ky Fan [28] and Glicksberg [31].

Theorem 1 (Ky Fan-Glicksberg) *Let K be a nonempty, convex and compact set in a separated locally convex space and let $\Gamma : K \hookrightarrow K$ be a nonempty, closed and convex valued multifunction with closed graph. Then Γ has at least one fixed point, i.e. there exists $f \in K$ such that $f \in \Gamma(f)$.*

A very useful variant of Theorem 1, is

Theorem 2 *Let K be a nonempty, convex and closed set in a separated locally convex space and let $\Gamma : K \hookrightarrow K$ be a nonempty, closed and convex valued multifunction with closed graph. If $\Gamma(K) := \cup_{x\in K}\Gamma(x)$ is relatively compact, then Γ has at least one fixed point, i.e. there exists $f \in K$ such that $f \in \Gamma(f)$.*

Proof. Since K is closed, convex and $\Gamma(K) \subseteq K$, we have

$$\overline{\operatorname{conv}\Gamma(K)} \subseteq \overline{\operatorname{conv} K} = K.$$

So,

$$\Gamma(\overline{\operatorname{conv}\Gamma(K)}) \subseteq \Gamma(K) \subseteq \overline{\operatorname{conv}\Gamma(K)},$$

which shows that the set $\mathcal{C} := \overline{\operatorname{conv}\Gamma(K)}$, which by Mazur's Theorem, i.e. Dunford, Schwartz [22, Theorem 6, p. 416] is compact, is nonempty, closed, convex and $\Gamma(\mathcal{C}) \subseteq \mathcal{C}$. So, we are in the hypotheses of Theorem 1, with K substituted by $\mathcal{C} \subseteq K$, wherefrom the conclusion. $\square$

Since, by Edwards [23, Theorem 8.12.1, p. 549], the weak closure of a weakly relatively compact set, in a Banach space, coincides with its weak sequential closure, Theorem 2 implies:

Theorem 3 *Let K be a nonempty, convex and weakly compact set in Banach space and let $\Gamma : K \hookrightarrow K$ be a nonempty, closed and convex valued multifunction with sequentially closed graph. Then Γ has at least one fixed point, i.e. there exists $f \in K$ such that $f \in \Gamma(f)$.*

In the single-valued case, Theorem 3 is due to Arino, Gautier, Penot [7].

If $x, y \in X$, we denote by $[x, y]_\pm$ the *right* (*left*) *directional derivative of the norm* calculated at x in the direction y, i.e.

$$[x,y]_+ = \lim_{h\downarrow 0}\frac{\|x+hy\|-\|x\|}{h} \qquad \left([x,y]_- = \lim_{h\uparrow 0}\frac{\|x+hy\|-\|x\|}{h}\right).$$

We recall that:

$$[x, y + ax]_\pm = [x, y]_\pm + a\|x\| \tag{3}$$

for $a \in \mathbf{R}$. See Barbu [11, Proposition 3.7, p. 101].

We say that the operator $A : D(A) \subseteq X \hookrightarrow X$ is *dissipative* if

$$[\,x_1 - x_2, y_1 - y_2\,]_- \leq 0$$

for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, and m-*dissipative* if it is dissipative and, for each $\lambda > 0$, or equivalently for some $\lambda > 0$, $R(I - \lambda A) = X$.

Let $A : D(A) \subseteq X \hookrightarrow X$ be an m-dissipative operator, let $\xi \in \overline{D(A)}$, $f \in L^1(a, b; X)$ and let us consider the differential equation

$$u'(t) \in Au(t) + f(t). \tag{4}$$

Theorem 4 (Benilan) *Let $\omega \in \mathbf{R}$ and let $A : D(A) \subseteq X \hookrightarrow X$ be an m-dissipative operator such that $A + \omega I$ is dissipative. Then, for each $\xi \in \overline{D(A)}$ and $f \in L^1(a, b\,; X)$, there exists a unique C^0-solution of* (4) *on* $[\,a, b\,]$ *which satisfies $u(a) = \xi$. Furthermore, if $f, g \in L^1(a, b\,; X)$ and u, v are the two C^0-solutions of* (4) *corresponding to f and g respectively, then*:

$$\|u(t) - v(t)\| \leq e^{-\omega(t-s)}\|u(s) - v(s)\| + \int_s^t e^{-\omega(t-\theta)}\|f(\theta) - g(\theta)\| d\theta \tag{5}$$

for each $a \leq s \leq t \leq b$.

See Benilan [12], or Barbu [11, Theorem 4.1, p. 128].

We denote by $u(\cdot, a, \xi, f)$ the unique C^0-solution of the problem (4) satisfying

$$u(a, a, \xi, f) = \xi$$

and we notice that $u(t, 0, \xi, 0) = S(t)\xi$, where $\{S(t);\ S(t) : \overline{D(A)} \to \overline{D(A)}\}$ is the semigroup of nonexpansive mappings generated by A via the Crandall-Liggett Exponential Formula. See Crandall, Liggett [21].

We recall that the semigroup $\{S(t);\ S(t) : \overline{D(A)} \to \overline{D(A)}\}$ is called *compact* if, for each $t > 0$, $S(t)$ is a compact operator.

We conclude this section with some compactness results concerning the set of C^0-solutions of the problem (4) whose initial data $u(a)$ and forcing terms f belong to some subsets B, in $\overline{D(A)}$, and respectively $\mathcal{F}$, in $L^1(a, b; X)$. First, we introduce:

Definition 3 Let (Ω, Σ, μ) be a complete measure space, $\mu(\Omega) < +\infty$. A subset $\mathcal{F} \subseteq L^1(\Omega, \mu; X)$ is called *uniformly integrable* if for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\int_E \|f(t)\|\, d\mu(t) \leq \varepsilon$$

for each $f \in \mathcal{F}$ and each $E \in \Sigma$ satisfying $\mu(E) \leq \delta(\varepsilon)$.

The next result is an extension of a compactness theorem due to Baras [10].

Theorem 5 *Let X be a Banach space, let $A : D(A) \subseteq X \hookrightarrow X$ be an m-dissipative operator and let us assume that A generates a compact semigroup. Let $B \subseteq \overline{D(A)}$ be bounded and let $\mathcal{F}$ be uniformly integrable in $L^1(a, b; X)$. Then, for each $\sigma \in (a, b)$, the set $\{u(\cdot, a, \xi, f)\,;\ (\xi, f) \in B \times \mathcal{F}\}$ is relatively compact in $C([\,\sigma, b\,]; X)$. If, in addition, B is relatively compact, then $\{u(\cdot, a, \xi, f)\,;\ (\xi, f) \in B \times \mathcal{F}\}$ is relatively compact even in $C([\,a, b\,]; X)$.*

See Vrabie [45, Theorems 2.3.2 and 2.3.3, pp. 46–47].

Definition 4 An m-dissipative operator A is called of *complete continuous type* if for each $a < b$ and each sequences $(f_n)_n$ in $L^1(a, b; X)$ and $(u_n)_n$ in $C([\,a, b\,]; X)$, with u_m a C^0-solution on $[\,a, b\,]$ of the problem $u_m'(t) \in Au_m(t) + f_m(t)$, $\quad m = 1, 2, \ldots$ satisfying:

$$\begin{cases} \lim\limits_n f_n = f & \text{weakly in} \quad L^1(a, b; X), \\ \lim\limits_n u_n = u & \text{strongly in} \quad C([\,a, b\,]; X), \end{cases}$$

it follows that u is a C^0 solution on $[\,a, b\,]$ of the limit problem $u'(t) \in Au(t) + f(t)$.

Remark 2 If the topological dual of X is uniformly convex and A generates a compact semigroup, then A is of complete continuous type. See Vrabie [45, Corollary 2.3.1, p. 49]. An m-dissipative operator of complete continuous type in a nonreflexive Banach space (and, by consequence, whose dual is not uniformly convex) is the nonlinear diffusion operator $\Delta\varphi$ in $L^1(\Omega)$. See the example below.

Example 1 Let Δ be the Laplace operator in the sense of distributions over Ω. Let $\varphi : D(\varphi) \subseteq \mathbf{R} \hookrightarrow \mathbf{R}$, let $u : \Omega \to D(\varphi)$ and let us denote by

$$\mathcal{S}_\varphi(u) = \{v \in L^1(\Omega);\ v(x) \in \varphi(u(x)), \quad \text{a.e. for } x \in \Omega\}.$$

We recall that $\varphi : D(\varphi) \subseteq \mathbf{R} \hookrightarrow \mathbf{R}$ is said to be *maximal monotone* if $-\varphi$ is m-dissipative.

The (i) part in Theorem 6 below is due to Brezis, Strauss [13], the (ii) part to Badii, Díaz, Tesei [9] and the (iii) part to Cârjă, Necula, Vrabie [19].

Theorem 6 *Let Ω be a nonempty, bounded and open subset in $\mathbf{R}^d$ with C^1 boundary Σ and let $\varphi : D(\varphi) \subseteq \mathbf{R} \hookrightarrow \mathbf{R}$ be maximal monotone with $0 \in \varphi(0)$.*

(*i*) *Then the operator* $\Delta\varphi : D(\Delta\varphi) \subseteq L^1(\Omega) \hookrightarrow L^1(\Omega)$*, defined by*

$$\begin{cases} D(\Delta\varphi) = \{u \in L^1(\Omega);\ \exists v \in \mathcal{S}_\varphi(u) \cap W_0^{1,1}(\Omega),\ \Delta v \in L^1(\Omega)\} \\ \Delta\varphi(u) = \{\Delta v;\ v \in \mathcal{S}_\varphi(u) \cap W_0^{1,1}(\Omega)\} \cap L^1(\Omega) \quad \text{for } u \in D(\Delta\varphi), \end{cases}$$

is m*-dissipative on* $L^1(\Omega)$.

(*ii*) *If, in addition,* $\varphi : \mathbf{R} \to \mathbf{R}$ *is continuous on* $\mathbf{R}$ *and* C^1 *on* $\mathbf{R} \setminus \{0\}$ *and there exist two constants* $C > 0$ *and* $\alpha > 0$ *if* $d \leq 2$ *and* $\alpha > (d-2)/d$ *if* $d \geq 3$ *such that*

$$\varphi'(r) \geq C|r|^{\alpha-1}$$

for each $r \in \mathbf{R} \setminus \{0\}$*, then* $\Delta\varphi$ *generates a compact semigroup.*

(*iii*) *In the hypotheses of* (*ii*)*,* $\Delta\varphi$ *is of complete continuous type.*

For the proof of (*i*) see Barbu [11, Theorem 3.5, p. 115], for the proof of (*ii*) see Vrabie [45, Theorem 2.7.1, p. 70] and for proof of the (*iii*) – which rests heavily on slight extension of a continuity result established in Díaz, Vrabie [26, Corollary 3.1, p. 527] which, in turn, follows from a compactness result due to Díaz, Vrabie [25] –, see Cârjă, Necula, Vrabie [19, Theorem 1.7.9, p. 22].

3 An auxiliary lemma

We begin by considering the problem

$$\begin{cases} u'(t) \in Au(t) + f(t), & t \in \mathbf{R}_+, \\ u(t) = g(u)(t), & t \in [-\tau, 0]. \end{cases} \tag{6}$$

Lemma 1 *Let us assume that* A *is* m*-dissipative,* $0 \in D(A)$*,* $0 \in A0$ *and there exists* $\omega > 0$ *such that* $A + \omega I$ *is dissipative, too. Let us assume, in addition, that there exists* $a > 0$ *such that* $g : C_b([-\tau, +\infty); \overline{D(A)}) \to C([-\tau, 0]; \overline{D(A)})$ *satisfies*

$$\|g(v) - g(\widetilde{v})\|_{C_b([-\tau,0];\overline{D(A)})} \leq \|v - \widetilde{v}\|_{C_b([a,+\infty);\overline{D(A)})}, \tag{7}$$

for each $v, \widetilde{v} \in C_b([-\tau, +\infty); \overline{D(A)})$ *and has affine growth, i.e. satisfies* (2)*. Then, for each* $f \in L^\infty(\mathbf{R}_+; X) \cap L^1(\mathbf{R}_+; X)$*,* (6) *has a unique* C^0*-solution* $u \in C_b([-\tau, +\infty); \overline{D(A)})$.

Remark 3 If $g : C_b([-\tau, +\infty); \overline{D(A)}) \to C([-\tau, 0]; \overline{D(A)})$ satisfies (7), then g depends only on the restriction $v_{|[a,+\infty)}$ of v to $[a, +\infty)$.

We can now pass to the proof of Lemma 1.

Proof. Let us observe first that, for each $v \in C_b([-\tau, +\infty); \overline{D(A)})$, the initial value problem for the delay equation

$$\begin{cases} u'(t) \in Au(t) + f(t), & t \in \mathbf{R}_+, \\ u(t) = g(v)(t), & t \in [-\tau, 0] \end{cases} \tag{8}$$

has a unique C^0-solution $u : [-\tau, +\infty) \to \overline{D(A)}$. Clearly, u is bounded on $[-\tau, 0]$ because it is continuous. Next, recalling that $0 \in A0$, from Theorem 4 we conclude that

$$\|u(t)\| \le e^{-\omega t}\|u(0)\| + \int_0^t e^{-\omega(t-\theta)}\|f(\theta)\| d\theta$$

$$\le \|u(0)\| + \frac{1}{\omega}\|f\|_{L^\infty(\mathbf{R}_+;X)},$$

for each $t \ge 0$. Finally, since u is bounded on both $[-\tau, 0]$ and $[0, +\infty)$, it follows that $u \in C_b([-\tau, +\infty); \overline{D(A)})$.

Now let us observe that, in view of Remark 3, $g(v)(t) = g(\widetilde{v})(t)$ for each $t \in [-\tau, 0]$ whenever v and $\widetilde{v}$ coincide on $[a, +\infty)$ and so, g depends only on the restriction of v on $[a, +\infty)$ To conclude the proof, it suffices to show that the operator

$$Q : C_b([a, +\infty); \overline{D(A)}) \to C_b([a, +\infty); \overline{D(A)}),$$

defined by

$$Q(v) := u_{|[a,+\infty)},$$

where u is the unique C^0-solution of the problem (8), is a strict contraction. Hence by the Banach Fixed Point Theorem, Q has a unique fixed point $v = u_{|[a,+\infty)}$ and

$$u(t) = \begin{cases} u(t), & t \in \mathbf{R}_+ \\ g(v)(t), & t \in [-\tau, 0], \end{cases}$$

is the unique C^0-solution of (6).

To this end, let $v, \widetilde{v} \in C_b([a, +\infty); \overline{D(A)})$ and $t \in [a, +\infty)$ be arbitrary. We have

$$\|Q(v)(t) - Q(\widetilde{v})(t)\| \le e^{-\omega t}\|Q(v)(0) - Q(\widetilde{v})(0)\|$$

$$\le e^{-\omega a}\|g(v)(0) - g(\widetilde{v})(0)\| \le e^{-\omega a}\|v - \widetilde{v}\|_{C_b([\,a,+\infty);X)}.$$

To complete the proof, we have merely to observe that

$$\|Q(v) - Q(\widetilde{v})\|_{C_b([\,a,+\infty);X)} \le e^{-\omega a}\|v - \widetilde{v}\|_{C_b([\,a,+\infty);X)}$$

for each $v, \widetilde{v} \in C_b([\,a, +\infty); \overline{D(A)})$. □

4 The general frame and basic assumptions

In the sequel we shall denote by $z : [\,-\tau, +\infty) \to \overline{D(A)}$ the unique C^0-solution of the unperturbed problem

$$\begin{cases} z'(t) \in Az(t), & t \in \mathbf{R}_+, \\ z(t) = g(z)(t), & t \in [\,-\tau, 0\,]. \end{cases} \tag{9}$$

which, in view of Lemma 1, belongs to $C_b([\,-\tau, +\infty); \overline{D(A)})$.

The assumptions we need in that follows are listed below.

(H_A) $A : D(A) \subseteq X \hookrightarrow X$ is an operator with the properties:

- (A_1) A is m-dissipative, there exists $\omega > 0$ such that $A + \omega I$ is dissipative too, $0 \in D(A)$, $0 \in A0$ and $\overline{D(A)}$ is convex;
- (A_2) the semigroup generated by A on $\overline{D(A)}$ is compact;
- (A_3) A is of complete continuous type. See Definition 4.

(H_F) $F : \mathbf{R}_+ \times C([\,-\tau, 0\,]; \overline{D(A)}) \hookrightarrow X$ is a nonempty, convex and weakly compact valued, almost strongly-weakly u.s.c. multifunction. See Definition 2.

(H_I) There exists $r > 0$ such that for each $t \in \mathbf{R}_+$, each $v \in C([\,-\tau, 0\,]; \overline{D(A)})$, with $\|v - z_t\|_{C([\,-\tau,0\,];X)} = r$ and $f \in F(t, v)$, we have $[\,v(0) - z(t), f\,]_+ \le 0$, where z is the unique C^0-solution of the unperturbed problem (9).

(H_I') There exists $r > 0$ such that for each $t \in \mathbf{R}_+$, each $v \in C([\,-\tau, 0\,]; \overline{D(A)})$ with $\|v(0) - z(t)\| > r$ and $f \in F(t, v)$, we have $[\,v(0) - z(t), f\,]_+ \le 0$, where z is the unique C^0-solution of the unperturbed problem (9).

(H_B) There exists $\ell \in L^\infty(\mathbf{R}_+; \mathbf{R}_+) \cap L^1(\mathbf{R}_+; \mathbf{R}_+)$ such that for almost every $t \in \mathbf{R}_+$ and for each $v \in C([\,-\tau, 0\,]; \overline{D(A)})$ satisfying $\|v(0) - z(t)\| \le r$, where $r > 0$ is given by (H_I), and each $f \in F(t, v)$, we have

$$\|f\| \le \ell(t).$$

(H_B') There exists $\ell \in L^\infty(\mathbf{R}_+;\mathbf{R}_+) \cap L^1(\mathbf{R}_+;\mathbf{R}_+)$ such that

$$\|f\| \le \ell(t)$$

for each $v \in C([-\tau,0];\overline{D(A)})$, each $f \in F(t,v)$ and a.e. for $t \in \mathbf{R}_+$.

(H_g) $g : C_b([-\tau,+\infty);\overline{D(A)}) \to C([-\tau,0];\overline{D(A)})$ satisfies:

(g_1) g has affine growth, i.e. there exists $m_0 \ge 0$ such that for each u in $C_b([-\tau,+\infty);\overline{D(A)})$, g satisfies (2) ;

(g_2) there exists $a > 0$ such that for each $u, v \in C_b([-\tau,+\infty);\overline{D(A)})$, we have

$$\|g(u) - g(v)\|_{C([-\tau,0];X)} \le \|u - v\|_{C_b([a,+\infty);X)} ;$$

(g_4) g is continuous from $\widetilde{C}_b([-\tau,+\infty);\overline{D(A)})$ to $C([-\tau,0];\overline{D(A)})$.

Remark 4 The hypothesis (H_I) ensures the invariance of a certain moving set with respect to the C^0-solutions of the problem

$$\begin{cases} u'(t) \in Au(t) + f(t), & t \in \mathbf{R}_+, \\ u(t) = g(v)(t), & t \in [-\tau,0]. \end{cases}$$

Namely, if a C^0-solution u of the problem above satisfies the initial constraint $u(t) - z(t) \in D(0,r)$ for each $t \in [-\tau,0]$, where z is the unique C^0-solution of (9), then (H_I) implies that u satisfies the very same constraint for all t belonging to domain of existence of u.

If $\|g(u)\|_{C([-\tau,0];X)} \le \|u\|_{C_b([0,+\infty);X)}$ for each $u \in C_b([-\tau,+\infty);X)$, case in which we will say that g *has linear growth*, we have $g(0) = 0$ and, accordingly, the unique C^0-solution z of (9) is identically 0. So, in this case, the invariance condition is nothing but a variant of the condition (H_3) in Vrabie [47].

Conditions $(g_1) \sim (g_2)$ and (g_4) are satisfied by all functions g of the general form specified in Remark 5 below.

Remark 5 Let $0 \le \tau < T$. If the function g is defined as

(i) $g(u)(t) = u(T + t)$, $t \in [-\tau,0]$ (T-periodicity condition);

(ii) $g(u)(t) = -u(T + t)$, $t \in [-\tau,0]$ (T-antiperiodicity condition);

(*iii*) $g(u)(t) = \displaystyle\int_{\tau}^{+\infty} k(\theta)u(t+\theta)\,d\theta$, $t \in [-\tau, 0]$, where $k \in L^1([\tau, +\infty); \mathbf{R})$ and $\displaystyle\int_{\tau}^{+\infty} |k(\theta)|\,d\theta = 1$ (mean condition);

(*iv*) $g(u)(t) = \displaystyle\sum_{i=1}^{n} \alpha_i u(t + t_i)$ for each $t \in [-\tau, 0]$, where $\displaystyle\sum_{i=1}^{n} |\alpha_i| \leq 1$ and $\tau < t_1 < t_2 < \cdots < t_n = T$ are arbitrary, but fixed (multi-point discrete mean condition);

then g satisfies (g_1) with $m_0 = 0$ and (g_2) with $a = T - \tau > 0$. A more general case is that in which the support of the measure μ is in $(\tau, +\infty)$ and the function is g given by

$$g(u)(t) = \int_{\tau}^{+\infty} \mathcal{N}(u(t+\theta))\,d\mu(\theta) + \psi(t), \tag{10}$$

for each $u \in C_b([-\tau, +\infty); \overline{D(A)})$ and $t \in [-\tau, 0]$. Here $\mathcal{N} : X \to X$ is a (possible nonlinear) nonexpansive operator with $\mathcal{N}(0) = 0$ and μ is a σ-finite and complete measure on $[\tau, +\infty)$, for which there exists $b > \tau$ such that supp $\mu = [b, +\infty)$, $\mu([b, +\infty)) = 1$ and $\psi \in C([-\tau, 0]; X)$ is such that $g(u)(t) \in \overline{D(A)}$ for each $t \in [-\tau, 0]$. Obviously, in this case, the constant $a > 0$ in (g_2) is exactly $b - \tau$.

Remark 6 From (g_2), (g_4) and Remark 3, we conclude that, for each convergent sequence $(u_k)_k$ in $\widetilde{C}_b([a, +\infty); \overline{D(A)})$ to some limit u we have $\lim_k g(u_k) = g(u)$ in $C([-\tau, 0]; X)$.

5 The main result

We may now proceed to the statement of the main result in this paper.

Theorem 7 *If* (H_A), (H_F), (H_I), (H_B) *and* (H_g) *are satisfied, then* (1) *has at least one* C^0*-solution,* $u \in C_b([-\tau, +\infty); \overline{D(A)})$ *satisfying* $u(t) - z(t) \in D(0, r)$ *for each* $t \in \mathbf{R}_+$*, where* z *is the unique* C^0*-solution of* (9) *and* $r > 0$ *is given by* (H_I).

We will prove our Theorem 7 with the help of:

Theorem 8 *If* (H_A), (H_F), (H_I'), (H_B') *and* (H_g) *are satisfied, then* (1) *has at least one* C^0*-solution,* $u \in C_b([-\tau, +\infty); \overline{D(A)})$ *and* $u(t) - z(t) \in D(0, r)$ *for each* $t \in \mathbf{R}_+$*, where* z *is the unique* C^0*-solution of* (9) *and* $r > 0$ *is given by* (H_I').

The proof of Theorem 8 is divided into four steps.

The first step. We begin by showing that, for each $\varepsilon \in (0,1)$ and $f \in L^1(\mathbf{R}_+; X)$, the problem

$$\begin{cases} u'(t) \in Au(t) - \varepsilon[\,u(t) - z(t)\,] + f(t), & t \in \mathbf{R}_+, \\ u(t) = g(u)(t), & t \in [\,-\tau, 0\,], \end{cases} \tag{11}$$

has a unique C^0-solution $u_\varepsilon^f \in C_b([\,-\tau, +\infty); \overline{D(A)})$.

The second step. We show that for each fixed $\varepsilon \in (0,1)$, the operator $f \mapsto u_\varepsilon^f$, which associates to f the unique C^0-solution u_ε^f of the problem (11), is compact from $L^\infty(\mathbf{R}_+; X) \cap L^1(\mathbf{R}_+; X)$ to $\widetilde{C}_b([\,-\tau, +\infty); \overline{D(A)})$.

The third step. As F is almost strongly-weakly u.s.c. – see Definition 1 –, it follows that, for the very same $\varepsilon > 0$, there exists $E_\varepsilon \subseteq \mathbf{R}_+$ whose Lebesgue measure $\lambda(E_\varepsilon) \le \varepsilon$ and such that $F_{|(\mathbf{R}_+ \setminus E_\varepsilon) \times C([\,-\tau,0\,]; \overline{D(A)})}$ is strongly-weakly u.s.c., we construct an approximation for F as follows. Let

$$D(F) = \mathbf{R}_+ \times C([\,-\tau, 0\,]; \overline{D(A)}),$$
$$D_\varepsilon(F) = (\mathbf{R}_+ \setminus E_\varepsilon) \times C([\,-\tau, 0\,]; \overline{D(A)})$$

and let us define the multifunction $F_\varepsilon : \mathbf{R}_+ \times C([\,-\tau, 0\,]; \overline{D(A)}) \hookrightarrow X$, by

$$F_\varepsilon(t, v) = \begin{cases} F(t, v), & (t, v) \in D_\varepsilon(F), \\ \{0\}, & (t, v) \in D(F) \setminus D_\varepsilon(F). \end{cases} \tag{12}$$

Further, we prove that the multifunction $f \mapsto \operatorname{Sel} F_\varepsilon(\cdot, u^f_{\varepsilon\,(\cdot)})$, where

$$\operatorname{Sel} F_\varepsilon(\cdot, u^f_{\varepsilon\,(\cdot)}) = \{h \in L^1(\mathbf{R}_+; X);\ h(t) \in F_\varepsilon(t, u^f_{\varepsilon\,t}) \text{ a.e. } t \in \mathbf{R}_+\},$$

maps some nonempty, convex and weakly compact set $\mathcal{K} \subseteq L^1(\mathbf{R}_+; X)$ into itself and has weakly×weakly sequentially closed graph. Then, we are in the hypotheses of Theorem 3, wherefrom it follows that this mapping has at least one fixed point which, by means of $f \mapsto u_\varepsilon^f$, produces a C^0-solution for the approximate problem

$$\begin{cases} u'(t) \in Au(t) - \varepsilon[\,u(t) - z(t)\,] + f(t), & t \in \mathbf{R}_+, \\ f(t) \in F_\varepsilon(t, u_t), & t \in \mathbf{R}_+, \\ u(t) = g(u)(t), & t \in [\,-\tau, 0\,], \end{cases} \tag{13}$$

where F_ε is defined by (12).

The fourth step. For each $\varepsilon \in (0,1)$, we fix a C^0-solution u_ε of the problem (13), and we show that there exists a sequence $\varepsilon_n \downarrow 0$ such that $(u_{\varepsilon_n})_n$ converges in $\widetilde{C}_b([\,0, +\infty); \overline{D(A)})$ to a C^0-solution of the problem (1).

6 Proofs of Theorems 7 and 8

We begin with the proofs of the four steps outlined above which are labeled here as four lemmas.

Lemma 2 *Let us assume that* (A_1) *in* (H_A)*, and* $(g_1) \sim (g_2)$ *in* (H_g) *are satisfied. Then, for each* $\varepsilon > 0$ *and each* $f \in L^\infty(\mathbf{R}_+; X) \cap L^1(\mathbf{R}_+; X)$*, the problem* (11) *has a unique* C^0*-solution* $u_\varepsilon^f : [\,-\tau, +\infty) \to X$ *which belongs to* $C_b([\,-\tau, +\infty); \overline{D(A)})$*. Moreover,* u_ε^f *satisfies*

$$\|u_\varepsilon^f - z\|_{C_b([\,-\tau,+\infty);X)} \leq \frac{1}{\varepsilon}\|f\|_{L^\infty(\mathbf{R}_+;X)}, \tag{14}$$

where z *is the unique* C^0*-solution of the problem* (9)*.*

Proof. First, let us observe that the problem (11) has the form

$$\begin{cases} u'(t) \in A_\varepsilon u(t) + f_\varepsilon(t), & t \in \mathbf{R}_+, \\ u(t) = g(u)(t), & t \in [\,-\tau, 0\,], \end{cases} \tag{15}$$

where $A_\varepsilon = A - \varepsilon I$ and $f_\varepsilon(t) = f(t) + \varepsilon z(t)$ for $t \in \mathbf{R}_+$. Clearly, $A_\varepsilon + \varepsilon I$ is m-dissipative, $0 \in \overline{D(A_\varepsilon)}$ and $0 \in A_\varepsilon 0$. Since $z \in C_b([\,0, +\infty); \overline{D(A)})$ we have $f_\varepsilon \in L^\infty(\mathbf{R}_+; X) \cap L^1(\mathbf{R}_+; X)$ and so Lemma 1 applies with $\omega = \varepsilon$ and this implies the existence and uniqueness of solution $u_\varepsilon^f \in C_b([\,-\tau, +\infty); \overline{D(A)})$.

Next, using the very same operator $A_\varepsilon = A - \varepsilon I$, we rewrite the unperturbed problem (9) as

$$\begin{cases} z'(t) \in A_\varepsilon z(t) + h_\varepsilon(t), & t \in \mathbf{R}_+, \\ z(t) = g(z)(t), & t \in [\,-\tau, 0\,], \end{cases} \tag{16}$$

with $h_\varepsilon(t) = \varepsilon z(t)$, for $t \in \mathbf{R}_+$. Then, for each $t \in (0, +\infty)$, the unique C^0-solution u_ε^f of (15) and the unique solution z of (16) satisfy

$$\|u_\varepsilon^f(t) - z(t)\| \leq e^{-\varepsilon t}\|u_\varepsilon^f(0) - z(0)\| + \int_0^t e^{-\varepsilon(t-s)}\|f(s)\|\,ds$$

$$\leq e^{-\varepsilon t}\|u_\varepsilon^f - z\|_{C_b([\,a,+\infty);X)} + \frac{1 - e^{-\varepsilon t}}{\varepsilon}\|f\|_{L^\infty(\mathbf{R}_+;X)},$$

for each $t \in (0, +\infty)$.

Clearly, there exists a sequence (α_n) in $(0, a)$ such that

$$\lim_n \|u_\varepsilon^f - z\|_{C_b([\,\alpha_n,+\infty);X)} = \|u_\varepsilon^f - z\|_{C_b([\,0,+\infty);X)}. \tag{17}$$

From the last inequality it follows that, for every $n \in \mathbf{N}$, we have

$$\|u_\varepsilon^f(t) - z(t)\| \le e^{-\varepsilon\alpha_n}\|u_\varepsilon^f - z\|_{C_b([\,\alpha_n,+\infty);X)} + \frac{1-e^{-\varepsilon\alpha_n}}{\varepsilon}\|f\|_{L^\infty(\mathbf{R}_+;X)} \tag{18}$$

for each $t \in [\alpha_n, +\infty)$, and so

$$\|u_\varepsilon^f - z\|_{C_b([\,\alpha_n,+\infty);X)} \le \frac{1}{\varepsilon}\|f\|_{L^\infty(\mathbf{R}_+;X)},$$

for every $n \in \mathbf{N}$. From (17), it readily follows that

$$\|u_\varepsilon^f - z\|_{C_b([\,0,+\infty);X)} \le \frac{1}{\varepsilon}\|f\|_{L^\infty(\mathbf{R}_+;X)}.$$

Next, if $t \in [\,-\tau, 0\,]$, from (g_2) in (H_g), we get

$$\|u_\varepsilon^f(t) - z(t)\| = \|g(u_\varepsilon^f)(t) - g(z)(t)\|$$

$$\le \|u_\varepsilon^f - z\|_{C_b([\,a,+\infty);X)} \le \|u_\varepsilon^f - z\|_{C_b([\,0,+\infty);X)}$$

and thus (14) holds true, and this completes the proof. □

Lemma 3 *Let us assume that* (A_1), (A_2) *in* (H_A) *and* (H_g) *are satisfied, let* $\varepsilon > 0$ *be fixed and let* $\ell \in L^\infty(\mathbf{R}_+;\mathbf{R}_+) \cap L^1(\mathbf{R}_+;\mathbf{R}_+)$. *Then the operator* $f \mapsto u_\varepsilon^f$, *where* u_ε^f *is the unique solution of the problem* (11) *corresponding to* f, *maps the set*

$$\mathcal{F} = \{f \in L^\infty([\,0,+\infty);X) \cap L^1(\mathbf{R}_+;X);\ \|f(t)\| \le \ell(t) \quad a.e.\ for\ t \in \mathbf{R}_+\},$$

into a relatively compact set in $\widetilde{C}_b([\,-\tau,+\infty);\overline{D(A)})$.

Proof. By (14), $\{u_\varepsilon^f;\ f \in \mathcal{F}\}$ is bounded in $C_b([\,0,+\infty);\overline{D(A)})$ and thus $\{u_\varepsilon^f(0);\ f \in \mathcal{F}\}$ is bounded in $\overline{D(A)}$. Since $\mathcal{F}$ is uniformly integrable in $L^1(0,k;X)$ for $k = 1, 2, \ldots$ – see Definition 3 –, from (A_2) and Theorem 5, we conclude that, for every $k = 1, 2, \ldots$, and $\sigma \in (0,k)$, $\{u_\varepsilon^f; f \in \mathcal{F}\}$ is relatively compact in $C([\,\sigma,k\,];\overline{D(A)})$. Thanks to (g_2), (g_4) in (H_g) and to Remark 6, we deduce that the set $\{g(u_\varepsilon^f);\ f \in \mathcal{F}\}$ is relatively compact in $C([\,-\tau,0\,];\overline{D(A)})$, and therefore $\left\{g(u_\varepsilon^f)(0);\ f \in \mathcal{F}\right\} = \{u_\varepsilon^f(0);\ f \in \mathcal{F}\}$ is relatively compact in $\overline{D(A)}$. Again, from (g_1) and the second part of Theorem 5, it follows that the set $\{u_\varepsilon^f;\ f \in \mathcal{F})\}$ is relatively compact in $\widetilde{C}_b([\,-\tau,+\infty);\overline{D(A)})$. The proof is complete. □

Lemma 4 *Let us assume that* (H_A), (H_F), (H'_B) *and* (H_g) *are satisfied. Then, for each* $\varepsilon > 0$, *the problem* (13) *has at least one solution* u_ε.

Since the proof follows the very same lines as those in the proof of Lemma 4.3 in Vrabie [47], we do not give details.

□

Lemma 5 *If* (H_A), (H_F), (H'_I), (H'_B) *and* (H_g) *are satisfied, then, for each* $\varepsilon \in (0,1)$, *each* C^0*-solution* u_ε *of the problem* (13) *satisfies*

$$\|u_\varepsilon - z\|_{C_b([0,+\infty);X)} \leq r, \tag{19}$$

where $r > 0$ *is given by* (H'_I).

Proof. Let us observe that, if $0 \leq t < \widetilde{t}$, we have

$$\|u_\varepsilon(\widetilde{t}) - z(\widetilde{t})\| \leq \|u_\varepsilon(t) - z(t)\| \tag{20}$$

$$+ \int_t^{\widetilde{t}} [\, u_\varepsilon(s) - z(s), f(s)\,]_+ \, ds - \varepsilon \int_t^{\widetilde{t}} \|u_\varepsilon(s) - z(s)\| \, ds.$$

Let us assume by contradiction that there exists $t \in \mathbf{R}_+$ such that

$$\|u_\varepsilon(t) - z(t)\| > r.$$

We distinguish between two cases.

Case 1. There exists $t_m \in \mathbf{R}_+$ such that

$$r < \|u_\varepsilon - z\|_{C_b([0,+\infty);X)} = \|u_\varepsilon(t_m) - z(t_m)\|. \tag{21}$$

If $t_m = 0$, then

$$r < \|u_\varepsilon - z\|_{C_b([0,+\infty);X)} = \|u_\varepsilon(0) - z(0)\| = \|g(u_\varepsilon)(0) - g(z)(0)\|$$

$$\leq \|u_\varepsilon - z\|_{C_b([a,+\infty);X)} \leq \|u_\varepsilon - z\|_{C_b([0,+\infty);X)}$$

and so

$$\|u_\varepsilon - z\|_{C_b([0,+\infty);X)} = \|u_\varepsilon - z\|_{C_b([a,+\infty);X)}.$$

Therefore, we can always confine ourselves to analyze the case when, in (21), either $t_m \in (0,+\infty)$ or there is no $t_m \in (0,+\infty)$ satisfying the equality in (21).

So, if there exists $t_m \in (0,+\infty)$ such that (21) holds true, then the mapping

$$t \mapsto \|u_\varepsilon(t) - z(t)\|$$

cannot be constant on $(0,t_m)$. Indeed, if we assume that

$$\|u_\varepsilon(s) - z(s)\| = \|u_\varepsilon(t_m) - z(t_m)\|$$

for each $s \in (0,t_m)$, then, taking $t \in (0,t_m)$ and $\widetilde{t} = t_m$ in (20) and using (H_I') with $v(0) = u_{\varepsilon s}(0) = u_\varepsilon(s)$, we get

$$r < r - \varepsilon(t_m - t)r < r \tag{22}$$

which is impossible. Consequently, there exists $t_0 \in (0,t_m)$ such that

$$r < \|u_\varepsilon(t_0)-z(t_0)\| < \|u_\varepsilon(s)-z(s)\| \le \|u_\varepsilon(t_m)-z(t_m)\| = \|u_\varepsilon-z\|_{C_b([\,0,+\infty);X)}$$

for each $s \in (t_0,t_m)$. Since

$$\|u_\varepsilon(s) - z(s)\| \le \|u_{\varepsilon s} - z_s\|_{C([\,-\tau,0\,];X)},$$

for each $s \in \mathbf{R}_+$, we have

$$r < \|u_{\varepsilon s} - z_s\|_{C([\,-\tau,0\,];X)}$$

for each $s \in (t_0,t_m)$ and then, using again (20) and (H_I'), we get

$$r < \|u_\varepsilon(t_m) - z(t_m)\| \le \|u_\varepsilon(t_0) - z(t_0)\| - \varepsilon(t_m - t_0)r$$

which implies the very same contradiction as before, i.e. (22).

It remains only to analyze

Case 2. There is no $t_m \in \mathbf{R}_+$ such that (21) holds true. Then, there exists at least one sequence $(t_k)_k$ such that

$$\begin{cases} \lim\limits_k t_k = +\infty, \\ \lim\limits_k \|u_\varepsilon(t_k) - z(t_k)\| = \|u_\varepsilon - z\|_{C_b([\,0,+\infty);X)}. \end{cases}$$

If there exists $\widetilde{t} \in \mathbf{R}_+$ such that $\|u_\varepsilon(\widetilde{t})-z(\widetilde{t})\| = r$, then $\|u_\varepsilon(t)-z(t)\| \le r$ for each $t \in [\,\widetilde{t},+\infty)$. Indeed, if we assume the contrary, there would exists $[\,t,\widetilde{t}\,] \subseteq [\,0,+\infty)$ such that

$$\|u_\varepsilon(t) - z(t)\| = r$$

and

$$r < \|u_\varepsilon(s) - z(s)\|$$

for each $s \in (t, \widetilde{t}\,]$. Then, using once again (20) and (H_I'), we get

$$r < \|u_\varepsilon(\widetilde{t}) - z(\widetilde{t})\| \leq \|u_\varepsilon(t) - z(t)\| - \varepsilon(\widetilde{t} - t)r$$

$$\leq r - \varepsilon(\widetilde{t} - t)r$$

leading to (22) which is impossible.

So, when both

$$r < \|u_\varepsilon - z\|_{C_b([\,0,+\infty);X)}$$

and

$$\|u_\varepsilon(t) - z(t)\| < \|u_\varepsilon - z\|_{C_b([\,0,+\infty);X)}$$

hold true for each $t \in \mathbf{R}_+$, we necessarily have

$$\|u_\varepsilon(t) - z(t)\| > r$$

for each $t \in \mathbf{R}_+$. If this is the case, let us remark that we may assume with no loss of generality, by extracting a subsequence if necessary, that

$$t_{k+1} - t_k \geq 1$$

for $k = 0, 1, 2, \ldots$. Then, by (3) and (H_I'), we have

$$r < \|u_\varepsilon(t_{k+1}) - z(t_{k+1})\|$$

$$\leq \|u_\varepsilon(t_k) - z(t_k)\| + \int_{t_k}^{t_{k+1}} [\, u_\varepsilon(s) - z(s), f(s) - \varepsilon(u_\varepsilon(s) - z(s))\,]_+ \, ds$$

$$\leq \|u_\varepsilon(t_k) - z(t_k)\| - \varepsilon \int_{t_k}^{t_{k+1}} \|u_\varepsilon(s) - z(s)\| \, ds$$

$$\leq \|u_\varepsilon(t_k) - z(t_k)\| - \varepsilon(t_{k+1} - t_k)r \leq \|u_\varepsilon(t_k) - z(t_k)\| - \varepsilon r$$

for each $k \in \mathbf{N}$. Passing to the limit for $k \to +\infty$ in the inequalities

$$\|u_\varepsilon(t_{k+1}) - z(t_{k+1})\| \leq \|u_\varepsilon(t_k) - z(t_k)\| - \varepsilon r, \; k = 1, 2, \ldots$$

we get

$$\|u_\varepsilon - z\|_{C_b([\,0,+\infty);X)} \leq \|u_\varepsilon - z\|_{C_b([\,0,+\infty);X)} - \varepsilon r.$$

But, in view of Lemma 2, $\|u_\varepsilon - z\|_{C_b([\,0,+\infty);X)}$ is finite and thus we get a contradiction. This contradiction can be eliminated only if **Case 2** cannot hold. Thus, both **Case 1** and **Case 2** are impossible. In turn, this is a contradiction too, because at least one of these two cases should hold true. So, the initial supposition, that $\|u_\varepsilon - z\|_{C_b([0,+\infty);X)} > r$, is necessarily false. It then follows that (19) holds true and this completes the proof. $\square$

Now, we can pass to the proof of Theorem 8.

Proof. Let $(\varepsilon_n)_n$ be a sequence with $\varepsilon_n \downarrow 0$, let $(u_n)_n$ be the sequence of the C^0-solutions of the problem (13) corresponding to $\varepsilon = \varepsilon_n$ for $n \in \mathbf{N}$, and let $(f_n)_n$ be such that

$$\begin{cases} u_n'(t) \in Au_n(t) - \varepsilon_n[\,u_n(t) - z(t)\,] + f_n(t), & t \in \mathbf{R}_+, \\ f_n(t) \in F_{\varepsilon_n}(t, u_{nt}), & t \in \mathbf{R}_+, \\ u_n(t) = g(u_n)(t), & t \in [\,-\tau, 0\,]. \end{cases}$$

In view of Remark 1, we may assume without loss of generality that $E_{\varepsilon_{n+1}} \subset E_{\varepsilon_n}$ for $n = 0, 1, \ldots$. This means that

$$F_{\varepsilon_n}(t, v) = F_{\varepsilon_{n+1}}(t, v) \tag{23}$$

for each $t \in \mathbf{R}_+ \setminus E_{\varepsilon_n}$ and $v \in C([\,-\tau, 0\,]; \overline{D(A)})$.

From (H_B'), we deduce that, for $k = 1, 2, \ldots$, the set $\{f_n;\ n \in \mathbf{N}\}$ is uniformly integrable in $L^1(0, k; X)$. Then, from Lemma 5, (A_2) in (H_A) and Theorem 5, it follows that, for $k = 1, 2, \ldots$, and each $\sigma \in (0, k)$, the set $\{u_n;\ n \in \mathbf{N}\}$ is relatively compact in $C([\,\sigma, k\,]; \overline{D(A)})$. In view of (g_4) in (H_g), we deduce that the set $\{u_n;\ n \in \mathbf{N}\}$ is relatively compact in $C([\,-\tau, 0\,]; \overline{D(A)})$. In particular, the set

$$\{u_n(0) = g(u_n)(0);\quad n \in \mathbf{N}\}$$

is relatively compact in $\overline{D(A)}$. From the second part of Theorem 5, we conclude that $\{u_n;\ n \in \mathbf{N}\}$ is relatively compact in $C([\,0, k\,]; \overline{D(A)})$ for $k = 1, 2, \ldots$ and thus in $C([\,-\tau, k\,]; \overline{D(A)})$. So, $\{u_n;\ n \in \mathbf{N}\}$ is relatively compact in $\widetilde{C}_b([\,-\tau, +\infty); \overline{D(A)})$. Accordingly, for each $k = 1, 2, \ldots$,

$$C_k = \overline{\{u_n(t);\quad n \in \mathbf{N},\quad t \in [\,0, k\,]\}}$$

is compact in $\overline{D(A)}$. Let $\gamma \in (0, 1)$ be arbitrary, let E_γ be the Lebesgue measurable set in $[\,0, +\infty)$ given by Definition 2 and, for each $k = 1, 2, \ldots$, let us define the set

$$D_{\gamma,k} = \overline{\bigcup_{n \in \mathbf{N}} \{(t, u_{\varepsilon_n t});\quad t \in [\,0, k\,] \setminus E_\gamma\}}.$$

Clearly, $D_{\gamma,k}$ is compact in $\mathbf{R}_+ \times C([-\tau,0];\overline{D(A)})$. Next, for each $\gamma \in (0,1)$ and each $k = 1,2,\ldots$, let us define

$$C_{\gamma,k} = F_\gamma(D_{\gamma,k}) = F(D_{\gamma,k}) \cup \{0\}$$

which is weakly compact since $D_{\gamma,k}$ is compact and $F_{|D_{\gamma,k}}$ is strongly-weakly u.s.c. See Lemma 2.6.1, p. 47 in Cârjă, Necula, Vrabie [19]. Further, the family $\mathcal{F} = \{f_{\varepsilon_n};\ n = 0,1,\ldots\} \subseteq L^1(\mathbf{R}_+;X)$ satisfies the hypotheses of Theorem 4.1 in Vrabie [46]. So, on a subsequence at least, we have

$$\begin{cases} \lim_n f_n = f \quad \text{weakly in} \quad L^1(\mathbf{R}_+;X), \\ \lim_n u_n = u \quad \text{in} \quad \widetilde{C}_b([-\tau,+\infty);\overline{D(A)}), \\ \lim_n u_{nt} = u_t \quad \text{in} \quad C([-\tau,0];\overline{D(A)}) \quad \text{for each } t \in \mathbf{R}_+. \end{cases}$$

From Lemma 2.6.2, p. 47 in Cârjă, Necula, Vrabie [19] combined with (23), we get

$$f(t) \in F_{\varepsilon_n}(t,u_t)$$

for each $n \in \mathbf{R}$ and a.e. $t \in \mathbf{R}_+ \setminus E_{\varepsilon_n}$. Since $\lim_n \lambda(E_{\varepsilon_n}) = 0$, it follows that

$$f(t) \in F(t,u_t)$$

a.e. $t \in \mathbf{R}_+$. But A is of complete continuous type, wherefrom it follows that u is a C^0-solution of the problem (1) corresponding to the selection f of $t \mapsto F(t,u_t)$. Finally, it suffices to observe that, from (19) in Lemma 5, it follows that $u(t) - z(t) \in D(0,r)$ for each $t \in \mathbf{R}_+$. □

We can now proceed to the proof of Theorem 7.

Proof. Let $r > 0$ be given by (H_I) and let us define the set

$$\mathcal{K}_r = \{(t,v) \in \mathbf{R}_+ \times C([-\tau,0];\overline{D(A)});\ \|v(0) - z(t)\| \le r\}.$$

Clearly, $\mathcal{K}_r$ is nonempty and closed in $\mathbf{R}_+ \times C([-\tau,0];X)$, In addition, since by (A_1) in (H_A), $\overline{D(A)}$ is convex, it follows that for each $t \in \mathbf{R}_+$, the cross-section of $\mathcal{K}_r$ at t, i.e.

$$\mathcal{K}_r(t) = \{v \in C([-\tau,0];\overline{D(A)});\ (t,v) \in \mathcal{K}_r\}$$

is convex. Let $\pi : \mathbf{R}_+ \times C([-\tau,0];\overline{D(A)}) \to \mathbf{R}_+ \times C([-\tau,0];X)$ be defined by

$$\pi(t,v) = \begin{cases} (t,v) & \text{if } \|v(0) - z(t)\| \le r, \\ \left(t, \dfrac{r}{\|v - z_t\|_{C([-\tau,0];X)}}(v - z_t) + z_t\right) & \text{if } \|v(0) - z(t)\| > r. \end{cases}$$

We observe that π is continuous, π restricted to $\mathcal{K}_r$ is the identity operator and π maps $\mathbf{R}_+ \times C([\,-\tau,0\,];\overline{D(A)})$ into $\mathcal{K}_r$. The first two properties mentioned are obvious. To prove the fact that π maps $\mathbf{R}_+ \times C([\,-\tau,0\,];\overline{D(A)})$ into $\mathcal{K}_r$, we have merely to observe that if $\|v(0)-z(t)\| > r$ then, inasmuch as $\overline{D(A)}$ is convex and $v, z_t \in C([\,-\tau,0\,];\overline{D(A)}$, it follows that their convex combination

$$\frac{r}{\|v-z_t\|_{C([\,-\tau,0\,];X)}}(v-z_t)+z_t-z_t \in C([\,-\tau,0\,];\overline{D(A)}).$$

Moreover

$$\left\|\frac{r}{\|v-z_t\|_{C([\,-\tau,0\,];X)}}(v-z_t)+z_t-z_t\right\|_{C([\,-\tau,0\,];X)} = r$$

and so, in this case, $\pi(t,v) \in \mathcal{K}_r$. If $\|v(0)-z(t)\| \leq r$, then $\pi(t,v) = (t,v)$ and thus, π maps $\mathbf{R}_+ \times C([\,-\tau,0\,];\overline{D(A)})$ into $\mathcal{K}_r$.

Then, we can define the multifunction $F_\pi : \mathbf{R}_+ \times C([\,-\tau,0\,];\overline{D(A)}) \hookrightarrow X$ by

$$F_\pi(t,v) = F(\pi(t,v)),$$

for each $(t,v) \in \mathbf{R}_+ \times C([\,-\tau,0\,];\overline{D(A)})$. As π is continuous, it follows that F_π satisfies (H_F). Moreover, one can easily verify that it satisfies (H'_B). Moreover, since

$$\pi(\mathbf{R}_+ \times C([\,-\tau,0\,];\overline{D(A)})) \subseteq \mathcal{K}_r,$$

we conclude that F_π satisfies (H'_I) too. Indeed, let $(t,v) \in \mathbf{R}_+ \times C([\,-\tau,0\,];\overline{D(A)})$ be arbitrary and satisfying

$$\|v(0)-z(t)\| > r \tag{24}$$

and let $f \in F(\pi(t,v))$.

From the definition of π, it follows that the projection P_2 of $\pi(t,v)$ on the second component, i.e.

$$P_2(\pi(t,v)) = \begin{cases} v & \text{if } \|v(0)-z(t)\| \leq r, \\ \dfrac{r}{\|v-z_t\|_{C([\,-\tau,0\,];X)}}(v-z_t)+z_t & \text{if } \|v(0)-z(t)\| > r. \end{cases}$$

satisfies:

$$\|P_2(\pi(t,v))-z_t\|_{C([\,-\tau,0\,];X)} = \begin{cases} r & \text{if } \|v(0)-z(t)\| > r, \\ \|v-z_t\|_{C([\,-\tau,0\,];X)} & \text{if } \|v(0)-z(t)\| \leq r. \end{cases}$$

Therefore, if (t, v) satisfies (24), it follows that

$$\|P_2(\pi(t, v)) - z_t\|_{C([\,-\tau,0\,];X)} = r.$$

So, by (H_I), we have

$$[\,v(0) - z(t), f\,]_+ = [\,P_2(\pi(t, v))(0) - z(t), f\,]_+ \le 0$$

which proves that F_π satisfies (H_I').

Hence, by virtue of Theorem 8, the problem

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in \mathbf{R}_+, \\ f(t) \in F_\pi(t, u_t), & t \in \mathbf{R}_+, \\ u(t) = g(u)(t), & t \in [\,-\tau, 0\,] \end{cases}$$

has at least one C^0-solution $u \in C_b([\,-\tau, +\infty); \overline{D(A)})$.

By (19), we have $\|u_t(0) - z(t)\| \le r$ for each $t \in \mathbf{R}_+$. So, $(t, u_t) \in \mathcal{K}_r$, which shows that

$$F_\pi(t, u_t) = F(t, u_t)$$

for each $t \in \mathbf{R}_+$. Thus u is a C^0-solution of (1) and this completes the proof of Theorem 7. □

7 Nonlinear diffusion in $L^1(\Omega)$

Let Ω be a nonempty, bounded and open subset in $\mathbf{R}^d$, $d \ge 1$, with C^1 boundary Σ, let $\varphi : D(\varphi) \subseteq \mathbf{R} \hookrightarrow \mathbf{R}$ be maximal monotone with $0 \in \varphi(0)$ and let $\omega > 0$. Let us consider the porous medium equation subjected to nonlocal initial conditions

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) \in \Delta\varphi(u(t, x)) - \omega u(t, x) + f(t, x), & \text{in } Q_+, \\ f(t, x) \in F\left(t, u(t), \displaystyle\int_{-\tau}^{0} u(t + s, x)\, ds\right), & \text{in } Q_+, \\ \varphi(u(t, x)) = 0, & \text{on } \Sigma_+, \\ u(t, x) = \displaystyle\int_{\tau}^{+\infty} \mathcal{N}(u(\theta + t))(x)\, d\mu(\theta) + \psi(t)(x), & \text{in } Q_\tau. \end{cases} \tag{25}$$

Let us consider the auxiliary problem

$$\begin{cases} \dfrac{\partial z}{\partial t}(t,x) \in \Delta\varphi(z(t,x)) - \omega z(t,x), & \text{in } Q_+, \\ \varphi(z(t,x)) = 0, & \text{on } \Sigma_+, \\ z(t,x) = \displaystyle\int_\tau^{+\infty} \mathcal{N}(z(\theta+t))(x)\,d\mu(\theta) + \psi(t)(x), & \text{in } Q_\tau \end{cases} \tag{26}$$

and let us denote by $z \in C_b([-\tau,+\infty);L^1(\Omega))$ the unique C^0-solution of (26).

Before passing to the statement of the main existence result concerning (25), we need to introduce some notation and to explain the exact definition of F.

Let $f_i : \mathbf{R}_+ \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be two functions with $f_1(t,u,v) \leq f_2(t,u,v)$ for each $(t,u,v) \in \mathbf{R}_+ \times \mathbf{R} \times \mathbf{R}$ and let

$$F : \mathbf{R}_+ \times C([-\tau,0];L^1(\Omega)) \hookrightarrow L^1(\Omega)$$

be given by

$$F := F_0 + F_1,$$

where

$$F_0(t,v) = \Big\{ f \in L^1(\Omega); f(x) \in [\,\widetilde{f}_1(t,v)(x), \widetilde{f}_1(t,v)(x)\,], \quad \text{a.e. for } x \in \Omega \Big\}$$

and

$$F_1(t,v)(x) := \{\sigma(t)h(x)\}$$

for each $(t,v) \in \mathbf{R}_+ \times C([-\tau,0];L^1(\Omega))$. Here

$$\widetilde{f}_i : \mathbf{R}_+ \times \Omega \times C([-\tau,0];L^1(\Omega)) \to \mathbf{R},\ i = 1,2,$$

are defined as:

$$\begin{cases} \widetilde{f}_1(t,x,v) := f_1\left(t, v(0)(x), \displaystyle\int_{-\tau}^0 v(s)(x)\,ds\right) \\ \widetilde{f}_2(t,x,v) := f_2\left(t, v(0)(x), \displaystyle\int_{-\tau}^0 v(s)(x)\,ds\right) \end{cases} \tag{27}$$

for each $(t,v) \in \mathbf{R}_+ \times C([-\tau,0];L^1(\Omega))$, a.e. in Ω, $h \in L^1(\Omega)$ is a fixed element satisfying $\|h\|_{L^1(\Omega)} \neq 0$ and $\sigma \in L^1(\mathbf{R}_+;\mathbf{R})$.

Theorem 9 *Let Ω be a nonempty, bounded and open subset in $\mathbf{R}^d$ with C^1 boundary Σ, let $\omega > 0$ and let $\varphi : \mathbf{R} \to \mathbf{R}$ be continuous on $\mathbf{R}$ and C^1 on $\mathbf{R} \setminus \{0\}$ with $\varphi(0) = 0$ and for which there exist two constants $C > 0$ and $\alpha > 0$ if $d \leq 2$ and $\alpha > (d-2)/d$ if $d \geq 3$ such that*

$$\varphi'(r) \geq C|r|^{\alpha-1}$$

for each $r \in \mathbf{R} \setminus \{0\}$. Let $f_i : \mathbf{R}_+ \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be two given functions, $h \in L^1(\Omega)$, $\|h\|_{L^1(\Omega)} > 0$, $\sigma \in L^1(\mathbf{R}_+; \mathbf{R})$ and let F be defined as above.

Let $\mathcal{N} : L^1(\Omega) \to L^1(\Omega)$, $\psi \in C([-\tau, 0]; L^1(\Omega))$ and let μ be a σ-finite and complete measure on $[\tau, +\infty)$. Let us assume that:

(σ_1) *$\|\sigma(t)\| \leq 1$ for each $t \in \mathbf{R}_+$;*

(F_1) *$f_1(t, u, v) \leq f_2(t, u, v)$ for each $(t, u, v) \in \mathbf{R}_+ \times \mathbf{R} \times \mathbf{R}$;*

(F_2) *f_1 is l.s.c. and f_2 is u.s.c. and, for each $(t, u, v), (t, u, w) \in \mathbf{R}_+ \times \mathbf{R} \times \mathbf{R}$ with $v \leq w$, we have*

$$\begin{cases} f_1(t, u, v) \leq f_1(t, u, w), \\ f_2(t, u, v) \geq f_2(t, u, w); \end{cases}$$

(F_3) *there exists $c > 0$ such that, for every $(t, x, v) \in D(f_1, f_2)$ with*

$$\|v(0)(\cdot) - z(t, \cdot)\|_{L^1(\Omega)} \leq c^{-1}\|h\|_{L^1(\Omega)}$$

we have

$$sign\,[\,v(0)(x) - z(t, x)\,]f_0(x) \leq -c|v(0)(x) - z(t, x)|$$

for each $f_0(x) \in [\,f_1(t, x, v), f_2(t, x, v)\,]$, z being the unique C^0-solution of the problem (26)*;*

(F_4) *there exists a nonnegative function $\widetilde{\ell} \in L^1(\mathbf{R}_+; \mathbf{R}) \cap L^\infty(\mathbf{R}_+; \mathbf{R})$ such that*

$$|f_i(t, u, v)| \leq \widetilde{\ell}(t)$$

for $i = 1, 2$ and for each $(t, u, v) \in \mathbf{R}_+ \times \mathbf{R} \times \mathbf{R}$;

(F_5) *for each $t \in \mathbf{R}_+$ and each $v \in C([-\tau, 0]; L^1(\Omega))$, we have*

$$f_i(t, z(t, x), v) = 0$$

for $i = 1, 2$ and a.e. for $x \in \Omega$;

(μ_1) *there exists* $b > \tau$ *such that* $supp\,\mu \subseteq [\,b, +\infty)\,;$

(μ_2) $\mu([\,b, \infty)) = 1\,;$

($\mathcal{N}_1$) $\|\mathcal{N}(u) - \mathcal{N}(v)\|_{L^1(\Omega)} \leq \|u - v\|_{L^1(\Omega)}$ *for each* $u, v \in L^1(\Omega)\,;$

($\mathcal{N}_2$) $\mathcal{N}(0) = 0.$

Then, the problem (25) *has at least one* C^0*-solution* $u \in C_b([\,-\tau, +\infty); L^1(\Omega))$ *satisfying*

$$\|u - z\|_{C_b(\mathbf{R}_+; L^1(\Omega))} \leq c^{-1}\|h\|_{L^1(\Omega)}.$$

Remark 7 Condition (F_5) is satisfied, for instance, if

$$f_i(t, u, v) = \psi(t, u) \cdot \overline{f}_i(t, u, v),$$

where ψ is positive, continuous and bounded and $\psi(t, z(t, x)) = 0$, while $\overline{f}_i$ satisfy $(F_1) \sim (F_4)$, $i = 1, 2$. In the particular case in which $\psi \equiv 0$, it follows that $z \equiv 0$ and so, (F_5) reduces to

$$f_i(t, 0, v) = 0$$

for each $(t, v) \in \mathbf{R}_+ \times \mathbf{R}$.

Proof. Let $X = L^1(\Omega)$ and let us define $A : D(A) \subseteq L^1(\Omega) \to L^1(\Omega)$, by

$$Au := \Delta\varphi(u) - \omega u$$

for each $u \in D(A)$, where

$$D(A) = \left\{u \in L^1(\Omega);\ \varphi(u) \in W_0^{1,1}(\Omega),\ \Delta\varphi(u) \in L^1(\Omega)\right\}.$$

As $\varphi(0) = 0$, $C_0^\infty(\Omega)$ is dense in $D(A)$ and so $\overline{D(A)} = L^1(\Omega)$.

Theorem 6 implies that A is m-dissipative and $A + \omega I$ is dissipative in $L^1(\Omega)$, $A0 = 0$, A generates a compact semigroup and is of complete continuous type on $\overline{D(A)} = L^1(\Omega)$. Hence, A satisfies (H_A). Let F be defined as above and

$$g : C_b([\,-\tau, +\infty); L^1(\Omega)) \to C([\,-\tau, 0\,]; L^1(\Omega))$$

be defined by

$$g(u)(t)(x) = \int_\tau^{+\infty} \mathcal{N}(u(t + \theta))(x)\, d\mu(\theta) + \psi(t)(x)$$

for each $u \in C_b([\,-\tau,+\infty);L^1(\Omega))$, each $t \in [\,-\tau,0\,]$ and a.e. for $x \in \Omega$.

From (σ_1), (F_1), (F_2), (F_4) and Lemma 5.1 in Vrabie [47], using a similar arguments as in the proof of the corresponding part in the preceding section, we conclude that F satisfies (H_F). From (F_2) and (F_3), we conclude that F satisfies (H_I) and (H_B) with

$$r = c^{-1}\|h\|_{L^1(\Omega)}.$$

Indeed, we will show that for each $(t,v) \in \mathbf{R}_+ \times C([\,-\tau,0\,];L^1(\Omega))$, with

$$\|v(0)(\cdot) - z(t,\cdot)\|_{L^1(\Omega)} = r,$$

and every $f \in F(t,v)$, we have

$$[\,v(0)(\cdot) - z(t,\cdot), f\,]_+ \leq 0.$$

Let us observe that in our case, i.e. $X = L^1(\Omega)$, we have

$$[\,v(0)(\cdot) - z(t,\cdot), f\,]_+ = \int_{\{y\in\Omega;\,v(0)(y)-z(t,y)>0\}} f(x)\,dx$$

$$-\int_{\{y\in\Omega;\,v(0)(y)-z(t,y)<0\}} f(x)\,dx + \int_{\{y\in\Omega;\,v(0)(y)-z(t,y)=0\}} |f(x)|\,dx.$$

Let $f \in F(t,v)$. Clearly f is of the form $f = f_0 + h$, where $f_0 \in L^1(\Omega)$ satisfies $f_1(t,x,v) \leq f_0(x) \leq f_2(t,x,v)$ a.e. for $x \in \Omega$. From the definition of $[\,\cdot,\cdot\,]_+$ in $L^1(\Omega)$, we deduce

$$[\,v(0)(\cdot) - z(t,\cdot), f\,]_+$$

$$\leq \int_{\{y\in\Omega;\,v(0)(y)-z(t,y)>0\}} f_0(x)\,dx - \int_{\{y\in\Omega;\,v(0)(y)-z(t,y)<0\}} f_0(x)\,dx$$

$$+\int_{\{y\in\Omega;\,v(0)(y)-z(t,y)=0\}} |f_0(x)|\,dx + \int_{\{y\in\Omega;\,v(0)(y)-z(t,y)>0\}} \alpha(t)h(x)\,dx$$

$$-\int_{\{y\in\Omega;\,v(0)(y)-z(t,y)<0\}} h(x)\,dx + \int_{\{y\in\Omega;\,v(0)(y)-z(t,y)=0\}} |\alpha(t)|\cdot|h(x)|\,dx.$$

Next, taking into account that, from (F_5), we have $f_0(x) = 0$ a.e. for those $x \in \Omega$ for which $v(0)(x) = z(t,x)$, the last inequality, conjunction with (F_4), yields

$$[\,v(0)(\cdot)-z(t,\cdot), f\,]_+ \leq \int_\Omega \operatorname{sign}[\,v(0)(x)-z(t,x)\,]f_0(x)\,dx + \int_\Omega |\alpha(t)|\cdot|h(x)|\,dx$$

$$\leq -c\int_{\Omega}|v(0)(x)-z(t,x)|\,dx+\int_{\Omega}|h(x)|\,dx\leq 0.$$

So, F satisfies (H_I). As (H_4) follows from (F_3), we deduce that F satisfies (H_B). Since the proof of (H_g) is very simple, we do not enter into details. So, we are in the hypotheses of Theorem 7 wherefrom the conclusion.

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