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OPTIMIZING THE POSITION OF THE SUPPORT OF THE CONTROL FOR SOME OPTIMAL HARVESTING PROBLEMS*

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Dedicated to the memory of Prof. Dr. Viorel Arnăutu

Abstract

In this paper we investigate the optimal position of the support of the control for some optimal harvesting problems. First we refer to a logistic model with diffusion. We remind the existence result of an optimal control and the necessary optimality conditions for the related optimal harvesting problem. Then we obtain an iterative method to improve the position of the support of the optimal harvesting effort in order to maximize the harvest (for a simplified model without logistic term). Numerical tests illustrating the effectiveness of the theoretical results are given.

MSC: 92D25; 49K20; 65M06; 35Q92

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1 Introduction

Since R. A. Fisher introduced in [12] a mathematical model of spatially structured population, a related flourishing literature was developed (e.g.

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[14], [15], [16],) which combines diffusive motion of individuals with nonlinearities arising from their growth and competition process. For models related to dynamics of population we refer to [3]. In this paper we recall the results obtained in [8] for a logistic model with diffusion. We consider a related optimal harvesting problem. We want to find firstly the magnitude of the control that acts on a certain subdomain and to study the position of the subdomain where the control acts in order to optimize the cost (for basic results and methods in the optimal shape design theory we refer to [13]).

We consider the following Fisher's model corresponding to a biological population that is free to move in an isolated habitat $\Omega \subset \mathbf{R}^N,\ N \in \{2,3\}$:

$$\begin{cases} \partial_t y(x,t) - d\Delta y(x,t) = a(x)y(x,t) - k(x)y^2(x,t), & (x,t) \in Q_T, \\ \partial_\nu y(x,t) = 0, & (x,t) \in \Sigma_T, \\ y(x,0) = y_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where Ω is a domain with a sufficiently smooth boundary $\partial\Omega$, $Q_T := \Omega \times (0,T)$, $\Sigma_T := \partial\Omega \times (0,T)$, $T > 0$, $y = y(x,t)$ is the population density at $(x,t) \in \overline{\Omega} \times [0,T]$ and $y_0(x)$ is the initial population density. The logistic term, $k(x)y^2(x,t)$, describes a local intraspecific competition for resources. Here d is the diffusion coefficient and $a(x)$ indicates the natural growth rate of the population. We have prescribed homogeneous Neumann conditions on the boundary $\partial\Omega$, corresponding to the case of isolated populations. This is an extended model of the one in Section 5.2 from [4], because the population coefficients become functions of x. We start with the following hypotheses:

(H1) $a \in L^\infty(\Omega), \quad d \in (0,+\infty);$

(H2) $y_0 \in L^\infty(\Omega), \quad y_0(x) \geq 0$ a.e. $x \in \Omega$ with $\|y_0\|_{L^\infty(\Omega)} > 0;$

(H3) $k \in L^\infty(\Omega), \quad k(x) \geq 0$ a.e. $x \in \Omega$.

The optimal harvesting problem is

$$Maximize \int_0^T \int_\omega u(x,t)y^u(x,t)dx\ dt, \tag{2}$$

subject to $u \in K$, where $K = \{w \in L^\infty(\omega \times (0,T));\ 0 \leq w(x,t) \leq L \text{ a.e.}\}$, $L > 0$. $u(x,t)$ represents the harvesting effort at $(x,t) \in \omega \times [0,T]$, where $\omega \subset \Omega$ is a nonempty domain with sufficiently smooth boundary $\partial\omega$

and such that $\Omega \setminus \overline{\omega}$ is a domain. Here y^u is the solution of the problem

$$\begin{cases} \partial_t y - d\Delta y = a(x)y - k(x)y^2 - \chi_\omega(x)u(x,t)y(x,t), & (x,t) \in Q_T, \\ \partial_\nu y(x,t) = 0, & (x,t) \in \Sigma_T, \\ y(x,0) = y_0(x), & x \in \Omega, \end{cases} \tag{3}$$

(χ_ω is the characteristic function of ω).
We intend to use the necessary optimality conditions to find the position of ω in Ω (in the set of all of its translations) which gives the maximum value for the harvest. So we have two maximizing problems: firstly, for a fixed ω we find the harvesting effort which gives the maximum harvest; secondly, using this optimal effort (control) we investigate the best position of ω in order to maximize the harvest.

In fact, our problem of optimal harvesting takes the following form:

$$Maximize_{\omega \in \mathcal{O}} Maximize_{u \in K} \int_0^T \int_\omega u(x,t)y^u(x,t)dx\ dt, \tag{4}$$

where $\mathcal{O}$ denotes the set of all translations of ω in Ω.

The paper is structured as follows: in the second section we recall the necessary optimality conditions for our boundary value problem with logistic term and we find the derivative of the optimal cost value with respect to translations of ω in Ω for the linear problem (see [8]). In section 3, we use these results to develop a conceptual iterative algorithm suitable for improving the position of the support of the control. In the last section numerical test are included to sustain the theoretical results.

2 An iterative method to improve the position of the support of the harvesting effort. The case $k \equiv 0$ (the model without logistic term)

First, we refer to the model with logistic term. The existence result of an optimal control for the problem (2) follows the lines in [4].

Theorem 1 *Problem (2) admits at least one optimal control.*

Let us denote by $p = p(x,t)$ the adjoint state, i.e. p satisfies

$$\begin{cases} \partial_t p + d\Delta p = -a(x)p + 2k(x)y^{u^*}p + \chi_\omega(x)u^*(1+p), & (x,t) \in Q_T, \\ \partial_\nu p(x,t) = 0, & (x,t) \in \Sigma_T, \\ p(x,T) = 0, & x \in \Omega, \end{cases} \tag{5}$$

where (u^*, y^{u^*}) is an optimal pair for (2). For the construction of the adjoint problems in optimal control theory we refer to [11]. We have

Theorem 2 *If (u^*, y^{u^*}) is an optimal pair for problem (2) and if p is the solution of problem (5), then we have:*

$$u^*(x,t) = \begin{cases} 0, & 1+p(x,t)<0 \\ L, & 1+p(x,t)>0 \end{cases} \quad a.e.\ (x,t) \in \omega \times (0,T). \tag{6}$$

Proof. The existence and uniqueness of the adjoint state p can be proved via Banach's fixed point theorem.
Let $v \in L^\infty(\omega \times (0,T))$, arbitrary but fixed, such that $u^* + \varepsilon v \in K$ for sufficiently small $\varepsilon > 0$.
From the optimality of u^* we get that

$$\int_0^T \int_\omega u^*(x,t) \frac{y^{u^*+\varepsilon v}(x,t) - y^{u^*}(x,t)}{\varepsilon} dxdt + \int_0^T \int_\omega v(x,t) y^{u^*+\varepsilon v}(x,t) dxdt \leq 0, \tag{7}$$

for sufficiently small $\varepsilon > 0$.
In order to continue the proof of the theorem, we need the following convergence result (see [4]).

Lemma 1 *One has*

$$y^{u^*+\varepsilon v} \to y^{u^*} \quad in \quad L^\infty(Q_T)$$

and

$$\frac{y^{u^*+\varepsilon v} - y^{u^*}}{\varepsilon} \to f \quad in \quad L^\infty(Q_T),$$

as $\varepsilon \to 0+$, where $f = f(x,t)$ is the solution to

$$\begin{cases} \partial_t f - d\Delta f = a(x)f - 2k(x)y^{u^*}f - \chi_\omega(x)u^*f - \chi_\omega(x)vy^{u^*}, & (x,t) \in Q_T, \\ \partial_\nu f(x,t) = 0, & (x,t) \in \Sigma_T, \\ f(x,0) = 0. & x \in \Omega. \end{cases} \tag{8}$$

Returning to the proof of the theorem, passing to the limit in relation (7) and taking into consideration the results above, we obtain that:

$$\int_0^T \int_\omega u^*(x,t) f(x,t) dxdt + \int_0^T \int_\omega v(x,t) y^{u^*}(x,t) dxdt \leq 0. \tag{9}$$

We multiply the parabolic equation in (8) by p and integrate on Q_T. We get that:

$$\int_\Omega [p(x,T)f(x,T) - p(x,0)f(x,0)]dx - \int_0^T\int_\Omega f\partial_t p dxdt - \int_0^T\int_\Omega df\Delta p dxdt =$$

$$= \int_0^T a(x)pf dxdt - \int_0^T\int_\Omega 2k(x)y^{u^*} fpdxdt - \int_0^T\int_\omega u^* pf dxdt - \int_0^T\int_\omega pvy^{u^*} dxdt.$$

We using the fact that p is the solution of the problem (5) and we obtain that

$$-\int_0^T\int_\omega fu^*\, dxdt = -\int_0^T\int_\omega pvy^{u^*}\, dxdt. \tag{10}$$

From (9) and (10) we get that

$$\int_0^T\int_\omega v(x,t)y^{u^*}(x,t)(1+p(x,t))dxdt \le 0, \quad \text{for any } v \in L^\infty(\omega\times(0,T)),$$

such that $u^*+\varepsilon v \in K$, for sufficiently small $\varepsilon > 0$ (we have used the positivity of y^{u^*} in Q_T). So, the optimal control satisfies (6).

Next we remind an iterative method to improve the position of the support of the harvesting effort obtained in [8] for the model without logistic term. So, in the follows we will ignore the logistic process, i.e., we will take the case $k \equiv 0$. Let us consider $\omega_0 \subset \Omega$, where $\omega_0 \subset \Omega$ is a nonempty domain with sufficiently smooth boundary $\partial\omega_0$ and such that $\Omega \setminus \overline{\omega}_0$ is a domain. We denote by $\mathcal{O}$ the set

$$\mathcal{O} = \{\omega_0 + V \subset \Omega;\ V \in \mathbf{R}^N\}.$$

For any arbitrary but fixed $\omega \in \mathcal{O}$, we denote by (u^*_ω, y^*_ω) an optimal pair for problem (2). The optimal control problem to be investigated is:

$$Maximize_{\omega\in\mathcal{O}} \int_0^T\int_\omega u^*_\omega(x,t)y^*_\omega(x,t)dxdt, \tag{11}$$

where $y^*_\omega = y^*_\omega(x,t)$ is the solution to problem

$$\begin{cases} \partial_t y(x,t) - d\Delta y(x,t) = a(x)y(x,t) - \chi_\omega(x)u^*_\omega(x,t)y(x,t), & (x,t)\in Q_T, \\ \partial_\nu y(x,t) = 0, & (x,t)\in\Sigma_T, \\ y(x,0) = y_0(x), & x\in\Omega \end{cases} \tag{12}$$

In this case, the adjoint system is

$$\begin{cases} \partial_t p + d\Delta p = -a(x)p + \chi_\omega(x)u^*_\omega(1+p), & (x,t) \in Q_T, \\ \partial_\nu p(x,t) = 0, & (x,t) \in \Sigma_T, \\ p(x,T) = 0, & x \in \Omega \end{cases} \tag{13}$$

and the optimal control is given by

$$u^*_\omega(x,t) = \begin{cases} 0, & 1 + p_\omega(x,t) < 0 \\ L, & 1 + p_\omega(x,t) > 0 \end{cases} \tag{14}$$

a.e. $(x,t) \in \omega \times (0,T)$, where $p_\omega = p_\omega(x,t)$ is the solution to (13).
By (13) and (14) we get that p_ω is the solution to

$$\begin{cases} \partial_t p + d\Delta p = -a(x)p + \chi_\omega(x)L(1+p)^+, & (x,t) \in Q_T, \\ \partial_\nu p(x,t) = 0, & (x,t) \in \Sigma_T, \\ p(x,T) = 0, & x \in \Omega. \end{cases} \tag{15}$$

Multiplying (12) by p_ω and multiplying (13) by y^*_ω, and both integrating on Q_T we obtain:

$$\int_0^T \int_\omega u^*_\omega(x,t)y^*_\omega(x,t)dxdt = -\int_\Omega y_0(x)p_\omega(x,0)dx.$$

In conclusion our problem of optimal harvesting becomes a problem of minimizing another functional with respect to the positions of ω.
Let us denote

$$J^\omega = \int_\Omega y_0(x)p_\omega(x,0)dx,$$

where p_ω is the solution to (15).
Hence the minimization problem to be investigated is

$$Minimize_{\omega\in\mathcal{O}} J^\omega. \tag{16}$$

For every $V \in \mathbf{R}^n$, consider the derivative of J^ω with respect to translations. Actually

$$J^{\omega+\varepsilon V} - J^\omega = \int_\Omega (p_{\omega+\varepsilon V}(x,0) - p_\omega(x,0))y_0(x)dx$$

and multiplying with $\frac{1}{\varepsilon}$ we have

$$\frac{1}{\varepsilon}[J^{\omega+\varepsilon V} - J^\omega] = \int_\Omega \frac{p_{\omega+\varepsilon V}(x,0) - p_\omega(x,0)}{\varepsilon} y_0(x)dx$$

For $\varepsilon \to 0+$ we obtain that

$$dJ^{\omega}(V) = \int_{\Omega} z(x,0)y_0(x)dx,$$

where $z = z(x,t)$ is the solution of the following boundary value problem:

$$\begin{cases} \partial_t z + d\Delta z = -a(x)z + L\chi_\omega z\partial h(1+p_\omega(x,t)) + Lm_{p_\omega(t)}, & (x,t) \in Q_T, \\ \partial_\nu z(x,t) = 0, & (x,t) \in \Sigma_T, \\ z(x,T) = 0, & x \in \Omega \end{cases} \tag{17}$$

Here $h(r) = r^+$,

$$\partial h(r) = \begin{cases} 1, & r > 0 \\ I, & r = 0 \\ 0, & r < 0 \end{cases}$$

where $I = [0,1]$, and

$$m_{p_\omega(t)}(\varphi) = \int_{\partial\omega} (1+p_\omega(x,t))^+ \varphi(x) V \cdot \nu(x) d\sigma, \text{ for any } \varphi \in H^1(\Omega)$$

where $\nu(x)$ is the outward normal versor at x to $\partial\omega$, outward with respect to $\Omega \setminus \omega$. We need to evaluate the form of the directional derivative for our functional. We recall the following result obtained in [8]:

Theorem 3 *For any $\omega \in \mathcal{O}$ and for any $V \in \mathbf{R}^N$,*

$$dJ^{\omega}(V) = -LV \cdot \int_0^T \int_{\partial\omega} (1+p_\omega(x,t))^+ g_\omega(x,t)\nu(x)d\sigma dt,$$

where p_ω is the solution for (15) and $g_\omega = g_\omega(x,t)$ is the solution for the following boundary value problem:

$$\begin{cases} \partial_t g - d\Delta g = a(x)g - L\chi_\omega g\partial h(1+p_\omega(x,t)), & (x,t) \in Q_T, \\ \partial_\nu g(x,t) = 0, & (x,t) \in \Sigma_T, \\ g(x,0) = y_0(x), & x \in \Omega \end{cases} \tag{18}$$

(For basic properties of the solution to such a problem we refer to [10]).
Proof. We multiply equation (17) with g_ω and we integrate on Q_T. This yields

$$\int_0^T \int_{\Omega} g_\omega(\partial_t z + d\Delta z + a(x)z)dxdt = L\int_0^T \int_{\omega} z(x,t)\xi(x,t)g_\omega(x,t)dxdt +$$

$$+L\int_0^T\int_{\partial\omega}(1+p_\omega)^+g_\omega(x,t)V\cdot\nu(x)d\sigma dt$$

where $\xi(x,t)\in\partial h(1+p_\omega(x,t))$ a.e. $(x,t)\in\omega\times(0,T)$.
Integrating by parts and using the fact that $z(x,T)=0$ and $g_\omega(x,0)=y_0(x)$ we obtain that

$$-\int_\Omega y_0(x)z(x,0)dx-\int_0^T\int_\Omega z[\partial_t g_\omega-d\Delta g_\omega-a(x)g_\omega]dxdt$$

$$=L\int_0^T\int_\omega z\xi(x,t)g_\omega(x,t)dxdt+L\int_0^T\int_{\partial\omega}(1+p_\omega)^+g_\omega(x,t)V\cdot\nu(x)d\sigma dt$$

and from (18) we get that:

$$-\int_\Omega y_0(x)z(x,0)dx=L\int_0^T\int_{\partial\omega}(1+p_\omega)^+g_\omega(x,t)V\cdot\nu(x)d\sigma dt.$$

The directional derivative of J^ω will be of the form

$$dJ^\omega(V)=-L\int_0^T\int_{\partial\omega}(1+p_\omega(x,t))^+g_\omega(x,t)V\cdot\nu(x)d\sigma dt$$

and we get the conclusion of the theorem.

3 A numerical algorithm

From Theorem 3 we derive the following conceptual iterative algorithm, based on a gradient method, to improve the position (translation) of $\omega\in\mathcal{O}$ in order to obtain a smaller value for J^ω.

Step 0: set $k:=0$, $J^{(0)}:=10^6$.
choose $\omega^{(0)}$ the initial positon of ω.

Step 1: compute $p^{(k+1)}$ the solution of the adjoint problem (15) corresponding to $\omega^{(k)}$.
compute $J^{(k+1)}=\int\limits_\Omega y_0(x)p^{(k+1)}(x,0)dx$.

Step 2: if $\left|J^{(k+1)}-J^{(k)}\right|<\varepsilon_1$ or $J^{(k+1)}\geq J^{(k)}$
then **STOP** ($\omega^{(k)}$ is the optimal position of ω)
else go to **Step 3**.

Step 3: compute $g^{(k+1)}$ the solution of problem (18) corresponding to

$\omega^{(k)}$ and $p^{(k+1)}$.

Step 4: compute

$$V := -\int_0^T \int_{\partial\omega^{(k)}} \left(1 + p^{(k+1)}(x,t)\right)^+ g^{(k+1)}(x,t)\nu(x)d\sigma dt$$

if $|V| < \varepsilon_2$
then **STOP** ($\omega^{(k)}$ is the optimal position of ω)
else go to **Step 5**.

Step 5: compute the new position of ω
$\omega^{(k+1)} := \rho V + \omega^{(k)}$;

Step 6: if $\omega^{(k+1)} = \omega^{(k)}$
then **STOP** ($\omega^{(k)}$ is the optimal position of ω)
else $k := k + 1$;
go to **Step 1**.

In Step 5, $\rho > 0$ is a given parameter (the gradient steplength), and $\varepsilon_1 > 0$ in Step 2 and $\varepsilon_2 > 0$ in Step 4 are prescribed convergence parameters.
The conceptual iterative algorithm, used to improve the position of ω in order to obtain a smaller value for J^ω, is a descent method. For more information about gradient (descent) methods, see [9], Section 2.3. The steplength ρ from Step 5 is variable from an iteration to the next one. To fit it we have used Armijo method (see [7] for more details).

4 Numerical tests

In order to simplify the discretization formulae for the numerical tests we have considered Ω and $\omega^{(0)}$ to be squares with the sides parallel with Ox_1 and Ox_2 axes (the space variable is $x = (x_1, x_2)$). Let $\Omega = (0,1) \times (0,1)$ and the length side of ω is equal with 0.2. We introduce equidistant discretization nodes for both axis corresponding to Ω. The interval $[0,T]$ is also discretized by equidistant nodes. The parabolic system from Step 1 is approximated by a finite difference method, descending with respect to time levels. An implicit scheme is used. The resulting algebraic linear system is solved by Gaussian elimination. The parabolic system from Step 3 is approximated also using a finite difference method, but ascending with respect to time levels. Integrals from Step 1 and Step 4 are numerical computed using Simpson's method corresponding to the discrete grid. In all following figures the square drawn with solid line represent the initial position of ω, and the

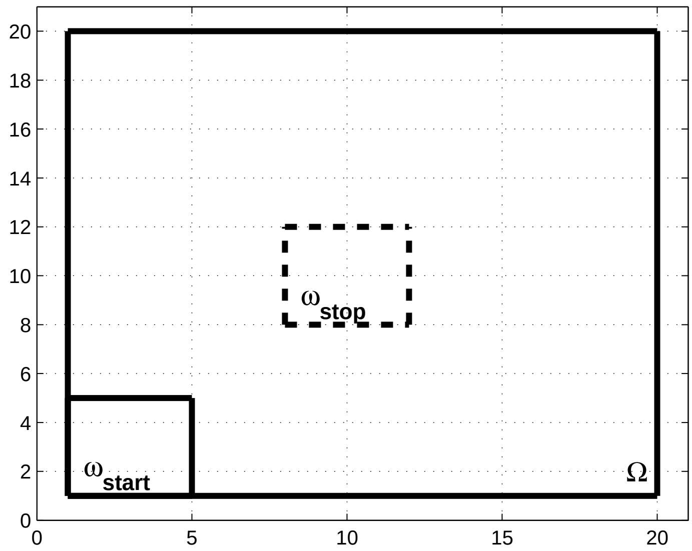

Figure 1. START/STOP position of ω

square drawn with dashed line is the improved position of ω.

Test 1. We consider the natural growth rate of the population $a(x_1, x_2) = 5, (x_1, x_2) \in \Omega$, $d = 1$, and final time $T = 1$. We take the space discretization step $\Delta x_1 = \Delta x_2 = 0.05$, and the time discretization step $\Delta t = 0.025$. The nodes along both axes Ox_1 and Ox_2 are numbered from 1 to 20. The left-down corner of Ω is numbered as $(1, 1)$ while the right-up corner is numbered as $(20, 20)$. For the convergence tests we consider $\varepsilon_1 = \varepsilon_2 = 0.001$. We start with $\omega^{(0)}$ which has the left-down corner at node $(1, 1)$ and the MATLAB program corresponding to the above algorithm gives after 5 iterations the optimal ω which has the left-down corner at node $(8, 8)$. The convergence was obtained by the test in Step 2. The initial and the optimal position of ω are shown in Figure 1. The corresponding graph of the optimal state $y(x_1, x_2, t)$ for $t = 1$ is given in Figure 2.

Test 2. The results obtained using the same input data from example 1, except the initial position of ω, are shown in Figure 3. We start with $\omega^{(0)}$ which has the left-down corner at node $(16, 10)$ and the MATLAB program corresponding to the above algorithm gives after 5 iterations the optimal

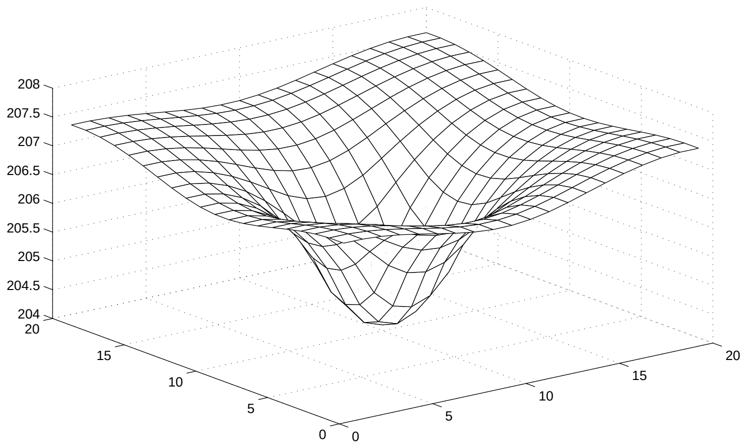

Figure 2. The optimal state y for $t = 1$

ω which has the left-down corner at the node $(7, 8)$. The convergence was obtained by the test in Step 4.

Test 3. We consider $a(x_1, x_2) = x_2 - x_1$, $(x_1, x_2) \in \Omega$, $d = 1$, and final time $T = 1$. We take the space discretization step $\Delta x_1 = \Delta x_2 = 0.05$, and the time discretization step $\Delta t = 0.05$ since the finite difference method used is implicit. We start with $\omega^{(0)}$ which has the left-down corner at node $(16, 16)$ and the MATLAB program corresponding to the above algorithm gives after 4 iterations the optimal ω which has the left-down corner at the node $(7, 9)$ (see Figure 4). The convergence was obtained by the test in Step 4.

Let us point out that the final position of ω given by the computer program is central with respect to Ω no matter the starting position $\omega^{(0)}$. This is in accordance with a more general theoretical result obtained in [2].

Test 4. For the natural growth rate of the population $a(x_1, x_2) = x_2 sin(x_1)$, $(x_1, x_2) \in \Omega$, the optimal position of ω is no more central with respect to Ω. The left-down corner of $\omega^{(0)}$ is $(4, 7)$ and the left-hand corner of the final ω is $(2, 4)$ and it is obtained after 4 iterations (see Figure 5). The convergence was obtained by the test in Step 4.

Let us point out that the algorithm is fast according to the number of iterations.

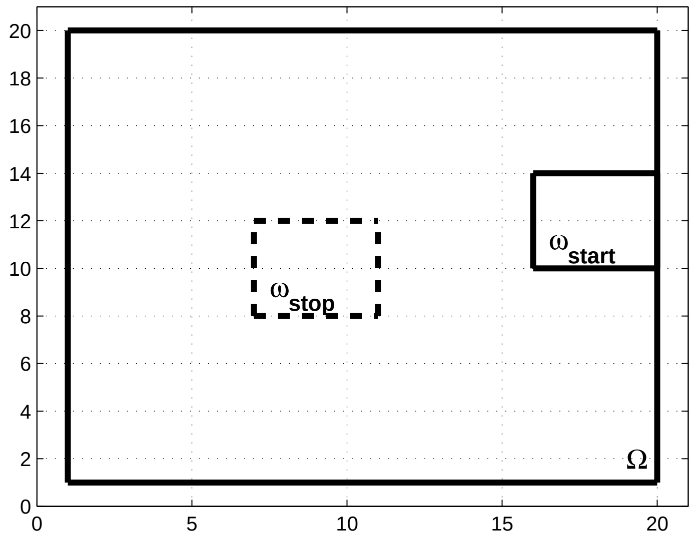

Figure 3. START/STOP position of ω

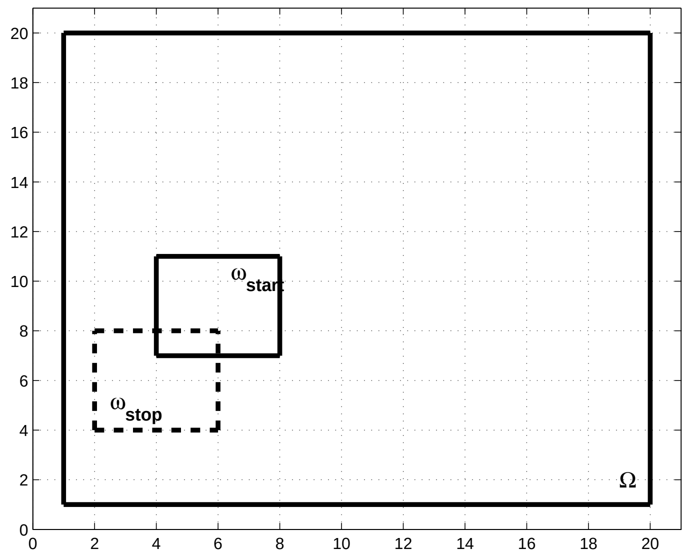

Figure 4. START/STOP position of ω

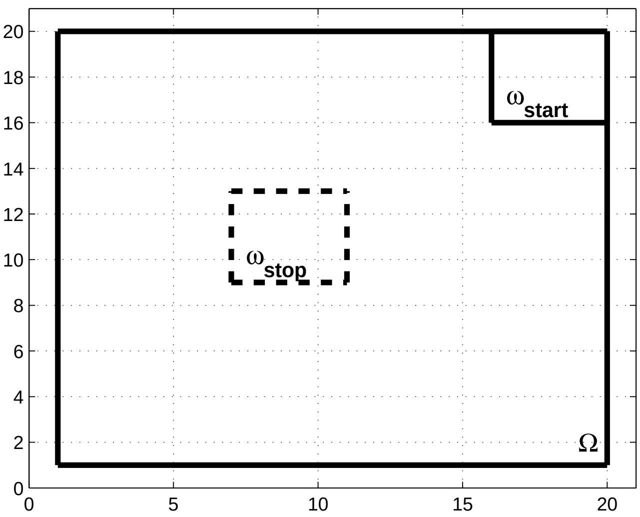

Figure 5. START/STOP position of ω

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