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DYNAMICS AND CONTROL OF AN INTEGRO-DIFFERENTIAL SYSTEM OF GEOGRAPHICAL ECONOMICS*

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Dedicated to the memory of Prof. Dr. Viorel Arnăutu

Abstract

In this paper we consider the impact of induced environmental pollution on the qualitative behavior and control of a system of geographical economics. Our underlying mathematical model extends other results in the literature along different directions. A general class of production functions is considered, including, in addition to the classical Cobb-Douglas production function, convex-concave production functions. The dynamics of the pollution is modelled via a diffusion equation coupled, via an integral source, with the geographically distributed production. Reciprocally, we suppose that the (negative) influence of pollution may be modeled as a negative feedback acting on the production function, and therefore on capital accumulation. We analyze the qualitative behavior of the coupled system, and

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then propose an optimal control problem for the above model. In order to solve the system of partial differential equations which describes the optimality conditions, we implement a Forward-Backward Sweep algorithm. Numerical simulations are reported which illustrate the behavior of the system and its optimal control.

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1 Introduction

Different from standard macroeconomic models and environmental economics, recent literature tends to develop a global theory combining these two branches of literature (see [12]). In this paper, following this new trend, we analyze the negative impact of induced pollution on the qualitative behavior, and on the control of a mathematical model of geographical economics. We suppose that the (negative) influence of pollution may be modeled as a negative feedback to the production function and therefore on capital accumulation.

The first studies in geographical economics date back to Beckman [9] and Puu [25], who consider regional problems based simply on flow equations. These works led to the development of a notion of geographical economics that uses general equilibrium models to analyze the peculiarities of local and global markets, as well as the mobility of production factors (see [20], [22], [23]). More recently, this geographical approach has been introduced in economic growth models to study the connections between accumulation and diffusion of capital on economic dynamics (see [10], [11], [14], [16]). The Solow model [30] with a continuous spatial dimension has been extensively studied. Camacho and Zou [14] analyze problems of convergence across regions when capital is mobile, while Brito [11] considers the case in which both capital and labor are mobile. Capasso et al. [16] introduce technology diffusion in the same model, under the additional feature of a convex-concave production function. The Ramsey model [26] has been extended to a spatial dimension by Brito [11] and Boucekkine et al. [10], respectively in average and total utilitarianism versions. Other contributions which explore the spatial dimension in environmental and resource economics can be found in [5], [13], [31].

Before moving to the case analyzed in this paper, we wish to point out a key feature of our models (see also [16]) which concerns the extension of the production function to a larger class of functions, including both the classical Cobb-Douglas case, and convex-concave production functions. The neoclassical theory, developed firstly by Solow (1956), is founded, among others, on two main assumptions driving its main results, namely the fact that the production function exhibits decreasing marginal returns, and infinite marginal productivity for very small input levels (Inada conditions). This ensured that a unique non-trivial equilibrium exists, so that every economic system would converge in the long-run to such a capital level. However, such a model provide a good description of systems with an high level of economic development, and are not applicable to less developed countries (see [27], [28]). In fact, the presence of fixed costs is an importance hindrance to the development of poor countries and only when the production level can get sufficiently high to compensate for such costs, returns will become decreasing. In order to build a theory able to describe the evolution of both advanced and less favored countries, we have relaxed these assumptions while keeping the general framework unaltered; in this way we have shown that it is possible to predict the so called *poverty traps* [16]. In [17] related inverse problems have been faced.

A short announcement of the main results of our research has been presented in a letter [3]; here we offer all relevant mathematical analysis supporting the anticipated results, together with the outcomes of related numerical simulations.

In Section 2 we present the underlying mathematical model describing the strong coupling of the evolution equations for the production and the induced environmental pollution.

In Section 3 we analyze the qualitative behaviour of the system, for large times and for some relevant cases.

In Section 4, we perform a numerical simulation of the steady-state model using reasonable parameter values. The results illustrate that both k and p approach nontrivial and spatially heterogenous equilibria.

In Section 5, we formulate an optimal control problem and solve a particular case using the Forward-Backward Sweep method. We assume that there is a representative agent who wishes to maximize his inter-temporal utility subject to the dynamic constraints (2)-(4). If we denote by $c(x,t)k(x,t)$ the pointwise instantaneous "harvesting effort", the control problem reads as

$$\max_{c\in U} \int_0^T \int_\Omega e^{-\delta t} c(x,t)k(x,t)dx\ dt, \tag{1}$$

subject to the relevant dynamics; δ is a nonnegative real number. The general cost function using the CIES utility function would have been $\dfrac{[c(x,t)k(x,t)]^{1-\beta}-1}{1-\beta}$, where $\beta \in [0,1)$ is a positive parameter; we may anticipate that this general case can be treated in the same fashion as we have treated here the case when $\beta = 0$, as in (1).

2 The underlying dynamical model

Let $k(x,t)$ and $p(x,t)$ respectively denote the capital stock held by and the pollution stock faced by a representative household located at x at date t, in a habitat $\overline{\Omega}$ (where $\Omega \subset \mathrm{R}^{\mathbf{2}}$ is taken as a nonempty and bounded domain with a smooth boundary), and $t \geq 0$. We also assume that the initial capital and pollution distributions, $k(x,0) = k_0(x)$ and $p(x,0) = p_0(x)$, are known and satisfy

$$k_0,\ p_0 \in L^\infty(\Omega), \quad k_0(x) \geq k_{00} > 0, p_0(x) \geq 0 \quad \text{a.e. } x \in \Omega, \tag{H}$$

and there is no capital or pollution flow through the boundary of Ω, namely that the normal derivatives $\frac{\partial k}{\partial \nu}(x,t) = \frac{\partial p}{\partial \nu}(x,t) = 0$ at $x \in \partial\Omega$ and $t \geq 0$. We assume a continuous space structure of both physical capital and pollution, so that the model we are interested in is the following:

$$\begin{cases} \dfrac{\partial k}{\partial t}(x,t) = d_1 \Delta k(x,t) + \dfrac{sf(k(x,t))}{g(p)} - \delta_1 k(x,t) - c(x,t)k(x,t) \\ \dfrac{\partial p}{\partial t}(x,t) = d_2 \Delta p(x,t) + \theta \displaystyle\int_\Omega f(k(x',t))\varphi(x',x)dx' - \delta_2 p(x,t) \end{cases} \tag{2}$$

for $(x,t) \in Q_{0,\infty}$, where

$g : [0,+\infty) \to (0,+\infty)$ is continuously differentiable and increasing,
$g(0) = 1$ and $\lim_{r\to+\infty} g(r) = +\infty$,

subject to homogeneous Neumann boundary conditions

$$\frac{\partial k}{\partial \nu}(x,t) = \frac{\partial p}{\partial \nu}(x,t) = 0, \qquad (x,t) \in \Sigma_{0,\infty}, \tag{3}$$

and initial conditions

$$k(x,0) = k_0(x),\ p(x,0) = p_0(x), \qquad x \in \Omega. \tag{4}$$

The control variable $c(x,t)$ describes the level of consumption at the location x, at the time t ($c \in L^\infty(\Omega \times (0,+\infty))$, $0 \leq c(x,t) \leq L$ a.e.) and $d_1, d_2, s, \theta, \delta_1, \delta_2, L$ are positive parameters. Here $Q_{a,b} = \Omega \times (a,b)$ and $\Sigma_{a,b} = \partial\Omega \times (a,b)$.

In the above model (2) the symbol f denotes a production function; we assume it is of the following form

$$f(r) = \frac{\alpha_1 r^\gamma}{1 + \alpha_2 r^\gamma}, \tag{5}$$

where $\alpha_1 \in (0,+\infty)$, $\alpha_2 \in [0,+\infty)$, $\gamma \in (0,+\infty)$. The choice of $g(p) = 1+p^2$ appears suddenly in the literature. We shall use this assumption in our present paper as well.

For basic results concerning the solutions to reaction-diffusion systems without integral terms we refer to [29]. We wish to remark that we deal here with a reaction-diffusion system including an integral (nonlocal) feedback in the evolution equation of the pollution concentration.

Let us notice that for $\alpha_2 = 0$ and $\gamma \in (0,1]$, we get the well known Cobb-Douglas production function. On the other hand, for $\alpha_2 > 0$ and $\gamma > 1$, we get an S-shaped production function. Paper [28] is the first contribution in the economic literature dealing with non-concave or convex/concave production functions. From an economic perspective this kind of assumption is justified by empirical evidences from less developed countries. Finally, the kernel $\varphi(x',x)$ describes the way in which pollution spreads over space; it satisfies the following hypotheses: $\varphi \in L^\infty(\Omega \times \Omega)$, and $\varphi(x',x) \geq 0$, for a.e. $(x',x) \in \Omega \times \Omega$. For $\gamma \in [1,+\infty)$, via Banach's fixed point theorem and using the fact that f is continuously differentiable, it is possible to prove that there exists a unique and nonnegative solution to (2)-(4) on the whole positive time semi-axis. Whenever $\gamma \in (0,1)$, f is no longer differentiable at 0; however, since by (H) we have that $k_0(x) \geq k_{00} > 0$ a.e. $x \in \Omega$, comparison results for parabolic equations and the fixed point theorem imply, in this case too, the existence and uniqueness of a nonnegative solution to (2)-(4), on the whole positive time semi-axis.

3 Large-time behavior of the underlying dynamical system

3.1 The case $p \equiv 0$ and time-independent c

In this case system (2)-(4) reduces to

$$\begin{cases} \dfrac{\partial k}{\partial t} = d_1 \Delta k(x,t) + sf(k(x,t)) - \delta_1 k(x,t) - c(x)k(x,t), & (x,t) \in Q_{0,\infty} \\ \dfrac{\partial k}{\partial \nu}(x,t) = 0, & (x,t) \in \Sigma_{0,\infty} \\ k(x,0) = k_0(x), & x \in \Omega. \end{cases} \tag{6}$$

In the following we discuss the large time behavior of (6) under different hypotheses on the parameter values.

I) In the production function (5) we first assume that $\alpha_2 = 0$ and $\gamma \in (0,1)$. In this case we are assuming that the production function f takes a Cobb-Douglas form. Then for any space independent initial datum k_{01}, with $k_{01} > 0$ sufficiently small, we get

$$sf(k_{01}) - \delta_1 k_{01} - c(x)k_{01} > 0, \quad \text{a.e. } x \in \Omega;$$

i.e. $k_{01}(\cdot)$ is a (strict) lower solution to

$$\begin{cases} d_1 \Delta \widetilde{k}(x) + sf(\widetilde{k}(x)) - \delta_1 \widetilde{k}(x) - c(x)\widetilde{k}(x) = 0, & x \in \Omega \\ \dfrac{\partial \widetilde{k}}{\partial \nu}(x) = 0, & x \in \partial\Omega. \end{cases} \tag{7}$$

Hence, by using comparison results for parabolic equations (see e.g. [19]) we obtain that the solution k_1 to (6), subject to the initial datum k_{01}, is monotonically increasing in $t \in [0, +\infty)$, for almost any $x \in \Omega$.

On the other hand any space independent initial datum k_{02} with $k_{02} > 0$ sufficiently large, is a (strict) upper solution of (7), i.e.

$$sf(k_{02}) - \delta_1 k_{02} - c(x)k_{02} < 0, \quad \text{a.e. } x \in \Omega.$$

By the same arguments as above, we obtain that the solution k_2 to (6), subject to the initial datum k_{02}, is monotonically decreasing in $t \in [0, +\infty)$, for almost any $x \in \Omega$.

The monotonicity of k_1 and k_2 implies that $k_1(\cdot,t)\to\tilde{k}_1$, $k_2(\cdot,t)\to\tilde{k}_2$ in $L^q(\Omega)$, as $t\to+\infty$, for any $q\in[1,+\infty)$, where $\tilde{k}_1$ and $\tilde{k}_2$ are nonnegative solutions to (7). In addition this yields $0<\tilde{k}_1(x)\le\tilde{k}_2(x)$ a.e. $x\in\Omega$.

Actually, using a standard argument for parabolic equations (see [29]), let us prove that $\tilde{k}_1(x)=\tilde{k}_2(x)$, a.e. $x\in\Omega$. Since $\tilde{k}_1$ satisfies

$$\begin{cases} d_1\Delta\tilde{k}_1(x)+s\alpha_1\tilde{k}_1(x)^\gamma-\delta_1\tilde{k}_1(x)-c(x)\tilde{k}_1(x)=0, & x\in\Omega\\ \dfrac{\partial\tilde{k}_1}{\partial\nu}(x)=0, & x\in\partial\Omega,\end{cases}$$

multiplying by $\tilde{k}_2$ and integrating on Ω gives that

$$-d_1\int_\Omega\nabla\tilde{k}_1\nabla\tilde{k}_2dx+s\alpha_1\int_\Omega\tilde{k}_1^\gamma\tilde{k}_2dx=\int_\Omega(\delta_1+c)\tilde{k}_1\tilde{k}_2dx.$$

In the same manner we get that

$$-d_1\int_\Omega\nabla\tilde{k}_1\nabla\tilde{k}_2dx+s\alpha_1\int_\Omega\tilde{k}_1\tilde{k}_2^\gamma dx=\int_\Omega(\delta_1+c)\tilde{k}_1\tilde{k}_2dx.$$

We infer that

$$\int_\Omega\tilde{k}_1(x)^\gamma\tilde{k}_2(x)^\gamma(\tilde{k}_2(x)^{1-\gamma}-\tilde{k}_1(x)^{1-\gamma})dx=0$$

and taking into account that $\tilde{k}_1$ and $\tilde{k}_2$ are positive and $\tilde{k}_1(x)\le\tilde{k}_2(x)$ a.e. $x\in\Omega$, we conclude that $\tilde{k}_1\equiv\tilde{k}_2$. Let us denote by $\tilde{k}(x)$, a.e. $x\in\Omega$, the common function.

We may now notice that, for any k_0 satisfying (H) we can choose the space independent $k_{01}>0$ sufficiently small and $k_{02}>0$ sufficiently large and such that $k_{01}\le k_0(x)\le k_{02}$ for a.e. $x\in\Omega$. Again the comparison results in [19] imply that any solution k to (6) subject to the initial datum k_0 satisfies $\lim_{t\to+\infty}k(\cdot,t)=\tilde{k}$ in $L^2(\Omega)$. Regularity results for the solutions of parabolic equations imply that $\lim_{t\to+\infty}k(\cdot,t)=\tilde{k}$ in $L^\infty(\Omega)$ as well [4].

II) In the production function, see (5), we assume that $\alpha_2=0$ and $\gamma\in(1,+\infty)$. For any space independent $k_{01}>0$, with k_{01} sufficiently small, we get that for any $t\in(0,+\infty)$ sufficiently small :

$$\begin{aligned}&sf(k_1(x,t))-\delta_1k_1(x,t)-c(x)k_1(x,t)\\ =\ & s\alpha_1k_1(x,t)^\gamma-\delta_1k_1(x,t)-c(x)k_1(x,t)\ \le\ -\frac{\delta_1}{2}k_1(x,t)\end{aligned}$$

a.e. $x \in \Omega$, where k_1 is the solution to (6) corresponding to $k_0 := k_{01}$. The comparison result for parabolic equations implies that the mapping $t \mapsto k_1(x,t)$ is decreasing on $[0,+\infty)$ for almost any $x \in \Omega$ and consequently

$$\begin{aligned} & sf(k_1(x,t)) - \delta_1 k_1(x,t) - c(x)k_1(x,t) \\ = \; & s\alpha_1 k_1(x,t)^\gamma - \delta_1 k_1(x,t) - c(x)k_1(x,t) \;\leq\; -\frac{\delta_1}{2}k_1(x,t) \end{aligned}$$

for any $t \in [0,+\infty)$, a.e. $x \in \Omega$. We then deduce that

$$0 < k_1(x,t) \leq k_{11}(x,t)$$

a.e. $x \in \Omega$, for any $t \in [0,+\infty)$, where k_{11} is the solution to

$$\begin{cases} \dfrac{\partial k}{\partial t}(x,t) = d_1 \Delta k(x,t) - \dfrac{\delta_1}{2}k(x,t), & (x,t) \in Q_{0,\infty} \\ \dfrac{\partial k}{\partial \nu}(x,t) = 0, & (x,t) \in \Sigma_{0,\infty} \\ k(x,0) = k_{01}, & x \in \Omega. \end{cases} \tag{8}$$

Since the unique solution to (8) is $k_{11}(x,t) = k_{01}\exp\{-\frac{\delta_1 t}{2}\}$, $\forall t \geq 0$, a.e. $x \in \Omega$, we conclude that

$$k_1(\cdot,t) \to 0 \quad \text{in } L^\infty(\Omega),$$

as $t \to +\infty$, exponentially.

On the other hand for any space independent and sufficiently large $k_{02} > 0$, we get in the same manner as in (I1) that for any $t \in [0,+\infty)$:

$$\begin{aligned} & sf(k_2(x,t)) - \delta_1 k_2(x,t) - c(x)k_2(x,t) \\ & = k_2(x,t)[s\alpha_1 k_2(x,t)^{\gamma-1} - \delta_1 - c(x)] \geq \zeta k_2(x,t) \end{aligned}$$

a.e. $x \in \Omega$, where k_2 is the solution to (6) corresponding to $k_0 := k_{02}$ and ζ is a positive constant, and that the mapping $t \mapsto k_2(x,t)$ is increasing on $[0,+\infty)$ for almost any $x \in \Omega$.

Using again the comparison result for parabolic equations we get that

$$k_2(x,t) \geq k_{02}\exp\{\zeta t\},$$

a.e. $x \in \Omega$, for any $t \in [0,+\infty)$, and consequently

$$Ess\ inf_\Omega\ k_2(\cdot,t) \to +\infty,$$

as $t \to +\infty$, exponentially.

III) Let us now assume that $\alpha_2 = 0$ and $\gamma = 1$. Then system (6) becomes

$$\begin{cases} \dfrac{\partial k}{\partial t}(x,t) = d_1 \Delta k(x,t) + (s\alpha_1 - \delta_1 - c(x))k(x,t), & (x,t) \in Q_{0,\infty} \\ \dfrac{\partial k}{\partial \nu}(x,t) = 0, & (x,t) \in \Sigma_{0,\infty} \\ k(x,0) = k_0(x), & x \in \Omega. \end{cases}$$

We have now the following cases happening.

1. For any c satisfying $s\alpha_1 - \delta_1 - c(x) \leq -\zeta$ a.e. $(x,t) \in Q_{0,\infty}$, where ζ is a positive constant, then the comparison result for parabolic equations implies that
$$k(x,t) \leq \|k_0\|_{L^\infty(\Omega)} \exp\{-\zeta t\},$$
a.e. $x \in \Omega$, $\forall t \geq 0$, and so
$$k(\cdot,t) \to 0 \quad \text{in } L^\infty(\Omega),$$
as $t \to +\infty$, exponentially.
2. For any c satisfying $s\alpha_1 - \delta_1 - c(x) \geq \zeta$ a.e. $(x,t) \in Q_{0,\infty}$, where ζ is a positive constant, then the comparison result for parabolic equations implies that
$$k(x,t) \geq k_{00} \exp\{\zeta t\},$$
a.e. $x \in \Omega$, $\forall t \geq 0$, and so
$$Ess\ inf_\Omega\ k(\cdot,t) \to +\infty \quad \text{in } L^\infty(\Omega),$$
as $t \to +\infty$, exponentially.
3. For any c a constant satisfying $s\alpha_1 - \delta_1 - c = 0$, then
$$k(\cdot,t) \to \int_\Omega k_0(x)dx \quad \text{in } L^\infty(\Omega),$$
as $t \to +\infty$.

IV) We consider now the case $\alpha_2 > 0$, with $\gamma \in (1, +\infty)$, so that the production function f, as defined by (5), is $S-$shaped. Then there exists a positive constant η such that $\eta = \sup_{r>0} \dfrac{sf(r)}{r}$. We also have that $f'(r) > 0$, $\forall r > 0$. In addition, if $r \geq 0$ is small then $f(r) \approx \alpha_1 r^\gamma$. If $r > 0$ is large, then $f(r) \approx \frac{\alpha_1}{\alpha_2}$. By the same comparison techniques used above, it is possible to prove the following

1. If we also have that $\eta - \delta_1 - c(x) \leq -c_1 < 0$ a.e. $x \in \Omega$ (where c_1 is a positive constant), then $k(\cdot, t) \to 0$ in $L^\infty(\Omega)$, as $t \to +\infty$.
2. If we assume that $\eta - \delta_1 - c(x) \geq c_1 > 0$ a.e. $x \in \Omega$ (where c_1 is a positive constant), then for any initial datum k_0 with a sufficiently small norm (in $L^\infty(\Omega)$), one gets $k(\cdot, t) \to 0$ in $L^\infty(\Omega)$, as $t \to +\infty$. On the other hand for any initial datum k_0 such that $\dfrac{sf(k_{00})}{k_{00}} \geq \delta_1 + \|c\|_{L^\infty(\Omega)}$ a.e. $x \in \Omega$, we may conclude that $k(\cdot, t) \to \tilde{k}$ in $L^\infty(\Omega)$, as $t \to +\infty$, and $\tilde{k}$ is a nontrivial nonnegative solution to

$$\begin{cases} d_1 \Delta \tilde{k} + s \frac{\alpha_1 \tilde{k}(x)^\gamma}{1+\alpha_2 \tilde{k}(x)^\gamma} - \delta_1 \tilde{k} - c(x)\tilde{k} = 0, & x \in \Omega \\ \dfrac{\partial \tilde{k}}{\partial \nu}(x) = 0, & x \in \partial\Omega. \end{cases} \tag{9}$$

V) Assume now that $\alpha_2 > 0$ and $\gamma = 1$. Then the derivative of

$$G(x, r) := s\frac{\alpha_1 r}{1 + \alpha_2 r} - \delta_1 r - c(x)r$$

with respect to r is

$$\frac{\partial G}{\partial r}(x, r) = s\frac{\alpha_1}{(1 + \alpha_2 r)^2} - \delta_1 - c(x)$$

which is a decreasing function of r.

1. If $\dfrac{\partial G}{\partial r}(x, 0) = s\alpha_1 - \delta_1 - c(x) \leq -c_0 < 0$ a.e. $x \in \Omega$ (c_0 is a positive constant), then the solution k to (2) satisfies
$$k(\cdot, t) \to 0 \qquad \text{in } L^\infty(\Omega)$$
as $t \to +\infty$.
2. If $\dfrac{\partial G}{\partial r}(x, 0) \geq c_0 > 0$ a.e. $x \in \Omega$, then it follows as in the case (I1) that for any space independent and sufficiently small $k_{01} > 0$ we get that
$$k_1(\cdot, t) \to \tilde{k}_1 \quad \text{in } L^\infty(\Omega),$$
as $t \to +\infty$, where k_1 is the solution to (6) corresponding to $k_0 := k_{01}$, and $\tilde{k}_1$ is a positive solution to (7). In the

same manner as in case (I) it also follows that $\tilde{k}_1 = \tilde{k}_2 = \tilde{k}$, which is the unique nontrivial nonnegative solution to (7), where k_2 and $\tilde{k}_2$ are constructed as in (I). Using again the comparison result for parabolic equations we get that

$$k(\cdot, t) \to \tilde{k} \quad \text{in } L^\infty(\Omega),$$

as $t \to +\infty$, where k is the solution to (6).

VI) If $\alpha_2 > 0$ and $\gamma \in (0, 1)$, then for any space independent and sufficiently small $k_{01} > 0$ we get that the mapping $t \mapsto k_1(x, t)$ is increasing on $[0, +\infty)$, for almost any $x \in \Omega$ and that

$$k_1(\cdot, t) \to \tilde{k}_1 \quad \text{in } L^\infty(\Omega),$$

as $t \to +\infty$, where k_1 is the solution to (6) corresponding to $k_0 := k_{01}$ and $\tilde{k}_1$ is a solution to (7) satisfying in addition

$$0 < \tilde{k}_1(x) \le \tilde{k}_2(x)$$

a.e. $x \in \Omega$. Here k_2 and $\tilde{k}_2$ are constructed as in (I). As in the first case (I) we get that

$$\int_\Omega (\tilde{k}_2(x) f(\tilde{k}_1(x)) - \tilde{k}_1(x) f(\tilde{k}_2(x)))dx = 0$$

and since the function integrated here is nonnegative, we may conclude that

$$\tilde{k}_2(x) f(\tilde{k}_1(x)) - \tilde{k}_1(x) f(\tilde{k}_2(x)) = 0$$

a.e. $x \in \Omega$ and consequently that $\tilde{k}_1 = \tilde{k}_2 = \tilde{k}$ and this is the unique nontrivial nonnegative solution to (7).

Repeating the argument in case (I) we may finally infer that for any k_0 satisfying (H) we get

$$k(\cdot, t) \to \tilde{k} \quad \text{in } L^\infty(\Omega),$$

as $t \to +\infty$.

The above discussion shows that, under the hypotheses $\alpha_2 > 0$ and $\gamma > 1$, which correspond to the case of an S-shaped production function, a saddle behavior emerges; i.e. for sufficiently small initial datum k_0, the production

$k(x,t)$ diminishes to 0, as $t \to +\infty$, while when k_0 is sufficiently large the production tends to a certain nontrivial steady state. As a conclusion, let us notice that these results could also be obtained by assuming a general function $f \in C^1([0,+\infty))$ such that $f'(0)=0$, $f'(r)>0$ for any $r>0$ and $\lim_{r\to+\infty} f(r) = \tau \in (0,+\infty)$ (see e.g. [15]). To investigate systems with nonlinear diffusion we have to combine the techniques in this paper with those in [18].

3.2 The general case with pollution diffusion

We are dealing here with the case when $\alpha_2 > 0$ and $\gamma > 1$. Assume that $0 \le c(x,t) \le L$ a.e. $(x,t) \in Q_{0,\infty}$. Let (k,p) be the solution to (2)-(4). Comparison results for parabolic equations imply that $k(x,t) \le \overline{k}_2(x,t)$ a.e. $(x,t) \in Q_{0,\infty}$, where $\overline{k}_2$ is the solution to (6) corresponding to $c \equiv 0$ and $p \equiv 0$. By using comparison results for parabolic equations including integral terms (see [2]), we get that $0 \le p(x,t) \le \overline{p}_2(x,t)$ a.e. $(x,t) \in Q_{0,\infty}$, where $\overline{p}_2$ is the solution to

$$\frac{\partial \overline{p}_2}{\partial t}(x,t) = d_2 \Delta \overline{p}_2(x,t) + \theta \int_\Omega f(\overline{k}_2(x',t))\varphi(x',x)dx' - \delta_2 \overline{p}_2(x,t),\ (x,t) \in Q_{0,\infty},$$

subject to boundary and initial conditions as in (3) and (4).

If k_{00} is sufficiently large, then $\overline{k}_2(t) \to \tilde{k}_2$ in $L^\infty(\Omega)$, as $t \to +\infty$, where $\tilde{k}_2$ is the maximal nonnegative solution to (5) corresponding to $c \equiv 0$. This implies that $\overline{p}_2(\cdot,t) \to \tilde{p}_2$ in $L^\infty(\Omega)$, as $t \to +\infty$, where $\tilde{p}_2$ is the solution to

$$\begin{cases} d_2 \Delta \tilde{p}_2(x) + \theta \displaystyle\int_\Omega f(\tilde{k}_2(x'))\varphi(x',x)dx' - \delta_2 \tilde{p}_2(x) = 0, & x \in \Omega \\ \dfrac{\partial \tilde{p}_2}{\partial \nu}(x) = 0, & x \in \partial\Omega. \end{cases}$$

Now for any $\varepsilon > 0$, there exists $t(\varepsilon) > 0$ such that $k(x,t) \ge k^*_\varepsilon(x,t)$ a.e. $x \in \Omega$, for all $t \ge t(\varepsilon)$, where k^*_ε is the solution to

$$\begin{cases} \dfrac{\partial k^*_\varepsilon}{\partial t} = d_1 \Delta k^*_\varepsilon + \dfrac{s f(k^*_\varepsilon(x,t))}{g(\tilde{p}_2(x)+\varepsilon)} - \delta_1 k^*_\varepsilon(x,t) - Lk(x,t), & (x,t) \in Q_{t(\varepsilon),\infty} \\ \dfrac{\partial k^*_\varepsilon}{\partial \nu}(x,t) = 0, & (x,t) \in \Sigma_{t(\varepsilon),\infty}, \end{cases}$$

satisfying $k^*_\varepsilon(x,t(\varepsilon)) = k(x,t(\varepsilon))$ a.e. $x \in \Omega$. In conclusion, if $\frac{\eta}{g(\tilde{p}_2(x))} - \delta_1 - L \ge \mu > 0$ a.e. $x \in \Omega$, then for any k_{00} sufficiently large we get the existence a sustainable economy, characterized by the persistence of k and the boundedness of the level of pollution. Moreover, for k_0 sufficiently small, the production $k(\cdot,t)$ tends to 0 in $L^\infty(\Omega)$, as $t \to +\infty$, which corresponds to a collapsing economy.

4 Numerical simulations

In the following simulations we use the parameter values

$$\begin{cases} \delta_1 = 0.05,\ \delta_2 = 0.01,\ s = 0.25,\ d_1 = 0.01,\ d_2 = 0.01,\ \theta = 0.1,\ c \equiv 0 \\ \varphi(x', x) = \frac{1}{\sqrt{\pi\varepsilon}} e^{-\frac{|x-x'|^2}{\varepsilon}} \psi(x),\ \varepsilon = 0.001,\ \psi(x) = x^2 \\ k(x,0) = e^{-x^2} \text{ and } p(x,0) = e^x,\ \Omega = [a,b] = [-1,1],\ T = 600 \\ \alpha_1 = 100,\ \alpha_2 = 100,\ \gamma = 4 \\ g(p) = 1 + p^2. \end{cases} \tag{10}$$

The above choices of parameter values are explained as follows:

- $\delta_1 = 0.05$ can be found in [6] and they describe the physical capital share and the depreciation rate of physical capital, respectively.
- $\delta_2 = 0.01$ represents the environmental ability to absorb pollution. The growth of CO2 emissions tripled between 2000 and 2004, growing by more than 3 percent per year according to a new study published in Proceedings of the National Academy of Sciences USA. Since the air quality is decreasing, it is ural to suppose that the environmental ability to absorb pollution is less than 3 per cent.
- d_1 and d_2 determine the diffusivity. We set them both equal to 0.01.
- s is an efficiency parameter that we assume to be greater than 0.2 (see [21]).
- θ is a normalization factor.
- In $\varphi(x', x)$, the expression $\frac{1}{\sqrt{\pi\varepsilon}} e^{-\frac{|x-x'|^2}{\varepsilon}}$ is a classical Gaussian kernel, and the function $\psi(x)$ allows from some place-dependent behaviour in the kernel.
- $\alpha_1 = \alpha_2 = 100$ and $\gamma = 4$ are set to values so that f is S-shaped.
- $T = 600$ is the length of the time interval.

The solution surfaces are plotted in Figure 1.

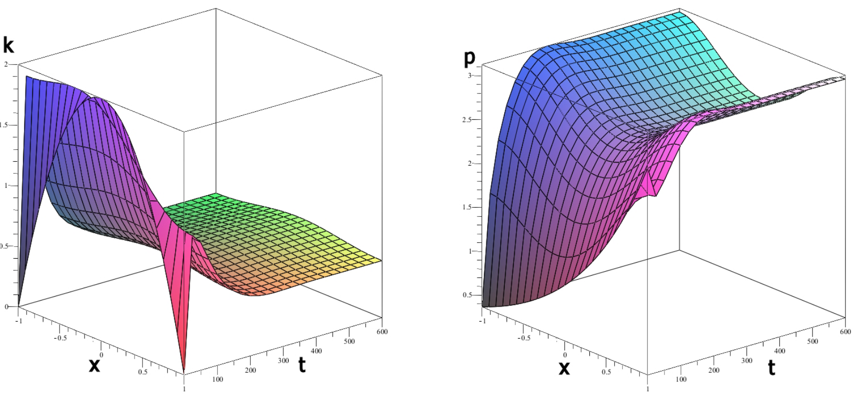

Figure 1: Solution surfaces for capital and pollution in the case of an S-shaped production function, kernel, and parameter values as in (10).

5 An optimal harvesting problem

Here we will consider only the case $\beta = 0$, $\alpha_2 > 0$, $\gamma > 1$. The results in the previous section imply that for any $c \in L^\infty(Q_{0,\infty})$, $0 \le c(x,t) \le L$ a.e. in Q, then (k^c, p^c), the solution to (2)-(4) corresponding to c satisfies $0 \le k^c(x,t) \le \overline{k}_2(x,t)$ a.e. $(x,t) \in Q$. Since $\overline{k}_2 \in L^\infty(Q_{0,\infty})$, there exists $M \ge 0$ such that $0 \le \overline{k}_2(x,t) \le M$ a.e. $(x,t) \in Q_{0,\infty}$. We may conclude that

$$0 \le \int_0^\infty \int_\Omega e^{-\delta t} c(x,t) k^c(x,t) dx\ dt \le LM meas(\Omega) \frac{1}{\delta}$$

and

$$0 \le \int_T^\infty \int_\Omega e^{-\delta t} c(x,t) k^c(x,t) dx\ dt \le LM meas(\Omega) \frac{e^{-\delta T}}{\delta}.$$

This means that instead of investigating the control problem formulated in Section 2 we could treat the following approximating optimal control problem with a finite horizon time (it is an optimal harvesting problem):

$$(OH) \qquad \max_{c \in U} \int_0^T \int_\Omega e^{-\delta t} c(x,t) k^c(x,t) dx\ dt,$$

where $T > 0$ is fixed (and large), and $U = \{v \in L^\infty(Q_{0,T});\ 0 \le v(x,t) \le L \text{ a.e. in } Q_{0,T}\}$ is the set of controls, and (k^c, p^c) is the solution to (2)-(4) corresponding to $g(p) = 1 + p^2$, and $Q_{0,T}$ and $\Sigma_{0,T}$ (instead of $Q_{0,\infty}$ and $\Sigma_{0,\infty}$, respectively).

Since this is a standard optimal control problem, the existence of at least one optimal control c^* can be proven following [1, 7, 8]. In addition, the following result holds.

Theorem 1 *If (k^*, p^*) is the optimal state corresponding to c^*, and if (q_1, q_2) is the solution to the following problem*

$$\left\{\begin{array}{l} \dfrac{\partial q_1}{\partial t}(x,t) = -d_1 \Delta q_1(x,t) - \left(s\dfrac{f'(k^*(x,t))}{1+p^*(x,t)^2} - \delta_1 \right) q_1(x,t) \\ \qquad -\theta f'(k^*(x,t)) \int_\Omega q_2(x',t)\varphi(x,x')dx' + c^*(x,t)(e^{-\delta t} + q_1(x,t)) \\ \dfrac{\partial q_2}{\partial t}(x,t) = -d_2 \Delta q_2(x,t) + \dfrac{2sf(k^*(x,t))p^*(x,t)}{(1+p^*(x,t)^2)^2} q_1(x,t) + \delta_2 q_2(x,t), \end{array}\right. \tag{11}$$

for $(x,t) \in Q_{0,T}$, subject to homogeneous Neumann boundary conditions and final conditions

$$q_1(x,T) = 0, \ q_2(x,T) = 0, \qquad x \in \Omega, \tag{12}$$

then

$$c^*(x,t) = \left\{ \begin{array}{ll} 0, & if \ e^{-\delta t} + q_1(x,t) < 0 \\ L, & if \ e^{-\delta t} + q_1(x,t) > 0. \end{array}\right. \tag{13}$$

Equations (11)-(13) provide the necessary optimality conditions for (OH). It is obvious that taking into account (13) we may rewrite (11) as

$$\left\{\begin{array}{l} \dfrac{\partial q_1}{\partial t}(x,t) = -d_1 \Delta q_1(x,t) - \left(s\dfrac{f'(k^*(x,t))}{1+p^*(x,t)^2} - \delta_1 \right) q_1(x,t) \\ \qquad -\theta f'(k^*(x,t)) \int_\Omega q_2(x',t)\varphi(x,x')dx' + L(e^{-\delta t} + q_1(x,t))^+, \\ \dfrac{\partial q_2}{\partial t}(x,t) = -d_2 \Delta q_2(x,t) + \dfrac{2sf(k^*(x,t))p^*(x,t)}{(1+p^*(x,t)^2)^2} q_1(x,t) + \delta_2 q_2(x,t), \end{array}\right. \tag{14}$$

for $(x,t) \in Q_{0,T}$, subject to homogeneous Neumann boundary conditions, and final conditions (12). Using the theorem given before, we can derive a gradient type algorithm (see [1]) to approximate the optimal control c^*.

We must solve Equations (2) and (14) subject to homogeneous Neumann boundary conditions, the initial conditions at $t = 0$ for $k(x,t)$ and $p(x,t)$, and the final conditions at $t = T$ for $q_1(x,t)$ and $q_2(x,t)$. One solution approach is to reverse time in Equations (14) via the change of variable $\tau = T - t$, turning the problem for q_1 and q_2 into forward problem with zero initial conditions. Starting with the solutions $k^0(x,t)$ and $p^0(x,t)$ corresponding to $c^0(x,t) = 0$, we use an iterative procedure.

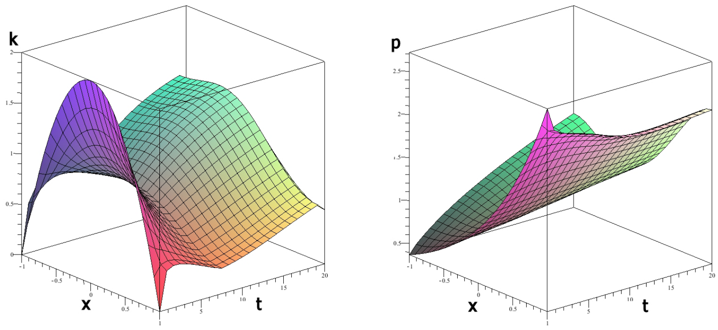

Figure 2: Approximations of the optimal states k^* and p^*

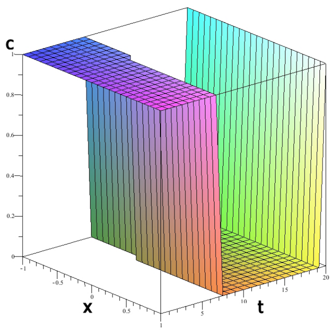

Figure 3: Approximation of the optimal control c^*.

The iterative algorithm is very intuitive, and efficient. It is generally referred to as the Forward-Backward Sweep method [24]. A numerical simulation of the above procedure is provided in the following example.

Example: We solve the optimal control problem using the same parameters as in Section 4, but for $L = 1$, and $T = 20$, and in addition setting $\delta = 0.1$ in the objective function. In Figure 2 we plot the optimal states k^* and p^*. We also include an approximation of the level sets of c^* in Figure 3.

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