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BOUNDEDNESS CONDITIONS FOR THE ANISOTROPIC NORM OF STOCHASTIC SYSTEMS WITH MULTIPLICATIVE NOISE*

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Abstract

The aim of this paper is to provide conditions for the boundedness of the anisotropic norm of discrete–time linear stochastic systems with multiplicative noise. It is proved that these conditions can be expressed in terms of the existence of a stabilising solution of a specific Riccati equation satisfying some additional constraints.

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keywords: anisotropic norm, stochastic systems, multiplicative noise, optimal estimation

1 Introduction

The signal filtering problem received much attention over the last seven decades, starting with the early formulation and developments due to E. Hopf and N. Wiener in the 1940's. Two decades later, the well-known results of Kalman and Bucy ([10], [11]) have been successfully implemented in

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many applications including aerospace, signal processing, geophysics, etc. and they have strongly influenced the research in this area. Some surveys on linear filtering and estimation can be found for instance in [9] and in [20]. Many papers devoted to this topics investigate the filtering performances with respect to the uncertainty modelling errors of the system which state is estimated. This interest is motivated by the fact that the filter performance deteriorates in the presence of modelling errors. Some of these papers consider the problem of robust filtering when the system is subject to parametric uncertainty (see *e.g.* [3], [6], [12], and the references therein). There are applications in which the system parameters are corrupted with random perturbations leading to stochastic models with *multiplicative noise*. Such stochastic systems have been intensively studied over the last few decades (see [23] for early references), many of the recent theoretical developments including optimal control and filtering results ([4], [6], [15]). Another important issue arising in filtering applications is related to the input of the systems generating the filtered signals. Besides developments based on Kalman filtering, also known as H_2-type filtering since the exogenous input signals are assumed white noises, alternative approaches have been proposed where deterministic bounded energy inputs are considered. Such formulations and developments have been performed in the framework of the H_∞-norm minimisation ([7], [20]). Many practical applications require a compromise between the H_2 and the H_∞ filtering since the H_2 norm minimisation of the estimation error may not be suitable when the considered signals are strongly coloured (e.g. periodic signals), and that H_∞-optimization may poorly perform when these signals are weakly coloured (e.g. white noise), (see e.g. [1] and [16]). An promising alternative to accomplish such compromise is to use the so-called *a-anisotropic norm* (see e.g. [5], [13], [22]) since it offers and intermediate topology between the H_2 and H_∞ norms. More precisely, if the coloured signal is generated by an m-dimensional exogenous input, the a-anisotropic norm $|||F|||_a$ of a stable system F has the property $1/\sqrt{m}\|F\|_2 \leq |||F|||_a \leq \|F\|_\infty$ (see, for instance [13]).

In [22] a Bounded Real Lemma type result is proved for the anisotropic norm of discrete-time deterministic systems. It is shown that the boundedness norm condition implies to solve a nonconvex optimization in which frequency representation of the filtered signal plays a crucial role.

The aim of the present paper is to determine boundedness conditions for the anisotropic norm of *stochastic systems with multiplicative noise*. By contrast with the above mentioned papers, all the developments of this paper use time representations of the signals and the obtained results provide a generalisation of the ones derived the absence of the multiplicative noise and for the case when the system is subject to state-dependent noise [21].

Notation. Throughout the paper the superscript 'T' stands for matrix transposition, $\mathbf{R}^n$ denotes the n dimensional Euclidean space, $\mathbf{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$ ($P\geq 0$), for $P\in\mathbf{R}^{n\times n}$ means that P is symmetric and positive definite (positive semidefinite). The trace of a matrix Z is denoted by $Tr\{Z\}$, and $|v|$ denotes the Euclidian norm of an n-dimensional vector v.

2 Preliminaries and Problem Statement

Consider the stochastic system with multiplicative noise

$$\begin{array}{rcl} x(t+1) & = & \left(A_0+\sum_{i=1}^{r}\xi_i(t)A_i\right)x(t)+\left(B_0+\sum_{i=1}^{r}\xi_i(t)B_i\right)w(t) \\ y(t) & = & Cx(t)+Dw(t),\ t=0,1,... \end{array} \quad (1)$$

where $\xi(t)=(\xi_1(t),...,\xi_r(t))^T$ is a sequence of independent random vectors $\xi:\Omega\to\mathbf{R}^r$ on a probability space$(\Omega,\mathcal{F},\mathcal{P})$. It is assumed that $\{\xi(t)\}_{t\geq 0}$ satisfies the conditions $E\left[\xi(t)\right]=0$ and $E\left[\xi(t)\xi^T(t)\right]=I_r$, $t=0,1,\dots$. The matrices of the state space model (1) have the dimensions $A_i\in\mathbf{R}^{n\times n}$, $B_i\in\mathbf{R}^{n\times m}$, $i=0,1....,r$, $C\in\mathbf{R}^{p\times n}$, $D\in\mathbf{R}^{p\times m}$.

It is assumed that the input $w(t)$ are random variables generated by a linear stochastic filter with multiplicative noise G

$$\begin{array}{rcl} \tilde{x}(t+1) & = & \left(A_{f_0}+\sum_{i=1}^{r}\xi_i(t)A_{f_i}\right)\tilde{x}(t)+\left(B_{f_0}+\sum_{i=1}^{r}\xi_i(t)B_{f_i}\right)v(t) \\ w(t) & = & C_f\tilde{x}(t)+D_fv(t),\ t=0,1,... \end{array} \quad (2)$$

where the order n_f and the matrices $A_{f_i}\in\mathbf{R}^{n_f\times n_f}$, $B_{f_i}\in\mathbf{R}^{n_f\times m}$, $i=0,1....,r$, $C_f\in\mathbf{R}^{m\times n_f}$, $D\in\mathbf{R}^{m\times m}$ are not prefixed and $v(t)\in\mathbf{R}^m$ are white noise vectors with the properties $E\left[v(t)\right]=0$ and $E\left[v(t)v^T(t)\right]=I_m$, $t=0,1,\dots$. It is assumed that $\{\xi(t)\}_{t\geq 0}$ and $\{v(t)\}_{t\geq 0}$ are independent stochastic processes.

Definition 1 *A stochastic system with multiplicative noise of form (1) with* $B_i=0$, $i=0,1,...,r$ *is called exponentially stable in mean square (ESMS) if there exist* $\beta\geq 1$ *and* $\rho\in(0,1)$ *such that* $E\left[|\Phi(t,s)x(0)|^2\right]\leq\beta\rho^{(t-s)}|x(0)|^2$ *for all* $t\geq s\geq 0$, $x(0)\in\mathbf{R}^n$*, where* $\Phi(t,s)$ *denotes the fundamental matrix solution of (1).*

Throughout the paper it will be assumed that both systems (1) and (2) are ESMS.

Definition 2 *The H_2-type norm of the ESMS system (1) is defined as*

$$\|F\|_2 = \left[\lim_{\ell \to \infty} \frac{1}{\ell} \sum_{t=0}^{\ell} E\left[y^T(t) y(t) \right] \right]^{\frac{1}{2}}.$$

The next result provides a method to compute the H_2 norm of the stochastic system (1) (see e.g. [4]).

Lemma 1 *The H_2 type norm of the ESMS system (1) is given by $\|F\|_2 = \left(Tr\left(\sum_{i=0}^{r} B_i^T X B_i + D^T D \right) \right)^{\frac{1}{2}}$ where $X \geq 0$ is the solution of the Lyapunov equation $X = \sum_{i=0}^{r} A_i^T X A_i + C^T C$.*

Let $L^2(\mathbf{Z} \times \Omega, \mathbf{R}^m)$ the space of all sequences $w = \{w(t)\}_{t \in \mathbf{Z}_+}$ of m-dimensional vectors with $\|w\|^2 := \sum_{t=-\infty}^{\infty} E|w(t)|^2 < \infty$ and by $\tilde{L}^2(\mathbf{Z}_+ \times \Omega, \mathbf{R}^m)$ the space of all $w \in L^2(\mathbf{Z}_+ \times \Omega, \mathbf{R}^m)$ such that $w(t)$ are measurable with respect to $\mathcal{F}_t$ for every $t \in \mathbf{Z}_+$, $\mathcal{F}_t \subset \mathcal{F}$ denoting a family of σ-algebras associated to the vectors $\xi(t)$. In [14] it is proved that if the system (1) is ESMS, one may define the linear bounded input-output operator

$$(Fw)(t) : \tilde{L}^2(\mathbf{Z}_+ \times \Omega, \mathbf{R}^m) \to \tilde{L}^2(\mathbf{Z}_+ \times \Omega, \mathbf{R}^p)$$

by

$$(Fw)(t) = Cx(t) + Dw(t), \ t \in \mathbf{Z}_+,$$

$x(t)$ being the solution of (1) with zero initial condition. Denoting by $\|F\|_\infty$ the norm of the above operator, one can prove the following Bounded Real Lemma type result for stochastic systems of form (1) with respect to the H_∞ norm [14].

Lemma 2 *The ESMS system (1) has the norm $\|F\|_\infty < \gamma$ for a certain $\gamma > 0$ if and only if the Riccati equation*

$$P = \sum_{i=0}^{r} A_i^T P A_i + \left(\sum_{i=0}^{r} A_i^T P B_i + C^T D \right) \left(\gamma^2 I - \sum_{i=0}^{r} B_i^T P B_i - D^T D \right)^{-1} \times \left(\sum_{i=0}^{r} A_i^T P B_i + C^T D \right)^T + C^T C$$

has a stabilizing solution $P \geq 0$ such that $\gamma^2 I - \sum_{i=0}^{r} B_i^T P B_i - D^T D > 0$.

It is recalled that a symmetric solution P of the above Riccati equation is called a *stabilising solution* if the stochastic system

$$x(t+1) = \left(A_0 + B_0 K + \sum_{i=1}^{r} \xi_i(t) \left(A_i + B_i K \right) \right) x(t)$$

is ESMS, where by definition

$$K := \left(\gamma^2 I - \sum_{i=0}^{r} B_i^T P B_i - D^T D\right)^{-1} \left(\sum_{i=0}^{r} A_i^T P B_i + C^T D\right)^T.$$

Given an ESMS filter of form (2), the *mean anisotropy* of the random variable $w(t)$ generated by G is defined as

$$\bar{A}(G) = -\frac{1}{2} \ln \det \left(\frac{mE\left[\tilde{w}(0)\tilde{w}^T(0)\right]}{\|G\|_2^2} \right) \tag{3}$$

where $\tilde{w}(0) = w(0) - E\left[w(0)|\left(w(k)\right)_{k<0}\right]$ denotes the prediction error of $w(0)$ based on $w(k)$, $k < 0$ (see details in [5]). Then the *a-anisotropic norm* of F is defined as ([5])

$$|||F|||_a = \sup_{G \in \mathcal{G}_a} \frac{\|FG\|_2}{\|G\|_2}, \tag{4}$$

where $\mathcal{G}_a$ denotes the set of all stochastic systems of form (2) with $\bar{A}(G) < a$.

3 Main result

Theorem 1 *The stochastic system with multiplicative noise (1) has the a-anisotropic norm less than a given* $\gamma > 0$ *if there exists* $q \in \left(0, \min\left(\gamma^{-2}, \|F\|_\infty^{-2}\right)\right)$ *such that the Riccati equation*

$$\begin{aligned} X = \textstyle\sum_{i=0}^{r} A_i^T X A_i + \left(\sum_{i=0}^{r} A_i^T X B_i + C^T D\right) \\ \times \left(\frac{1}{q} I - \sum_{i=0}^{r} B_i^T X B_i - D^T D\right)^{-1} \left(\sum_{i=0}^{r} A_i^T X B_i + C^T D\right)^T + C^T C \end{aligned} \tag{5}$$

has a stabilizing solution $X \geq 0$ *satisfying the following conditions*

$$\Psi_q := \frac{1}{q} I - \sum_{i=0}^{r} B_i^T X B_i - D^T D > 0 \tag{6}$$

and

$$\det\left(\frac{1}{q} - \gamma^2\right) \Psi_q^{-1} \leq e^{-2a}. \tag{7}$$

Proof. Using the Definition 1 of the H_2-type norm it follows that the condition $\sup_{G\in\mathcal{G}_a}\frac{\|FG\|_2}{\|G\|_2}<\gamma$ is equivalent with the condition

$$\lim_{\ell\to\infty}\frac{1}{\ell}\sum_{t=0}^{\ell}E\left[|y(t)|^2-\gamma^2|w(t)|^2\right]<0 \tag{8}$$

for all $w(t)$ generated by filters $G\in\mathcal{G}_a$.

For the sake of simplicity writing, the following notations will be introduced:

$$\begin{array}{rcl}\mathcal{A}(t) & := & A_0+\sum_{i=1}^{r}\xi(t)A_i\\ \mathcal{B}(t) & := & B_0+\sum_{i=1}^{r}\xi(t)B_i\,.\end{array}$$

Using (1) it follows that

$$\begin{gathered}x^T(t+1)Xx(t+1)-x^T(t)Xx(t)=[\mathcal{A}(t)x(t)+\\ \mathcal{B}(t)w(t)]^TX[\mathcal{A}(t)x(t)+\mathcal{B}(t)w(t)]-x^T(t)x(t)-y^T(t)y(t)+x^T(t)C^TCx(t)\\ +x^T(t)C^TDw(t)+w^T(t)D^TCx(t)+w^T(t)D^TDw(t)\end{gathered}$$

where we added the zero term $y^T(t)y(t)-(Cx(t)+Dw(t))^T(Cx(t)+Dw(t))$. Collecting terms we readily obtain

$$\begin{gathered}y^T(t)y(t)=x^T(t)[\mathcal{A}(t)^TX\mathcal{A}(t)-X+C^TC]x(t)+w^T(t)[D^TD+\\ \mathcal{B}(t)^TX\mathcal{B}(t)]w(t)+w^T(t)[D^TC+\mathcal{B}(t)^TX\mathcal{A}(t)]x(t)\\ +x^T(t)[C^TD+\mathcal{A}(t)^TX\mathcal{B}(t)]w(t)+x^T(t)Xx(t)-x^T(t+1)Xx(t+1)\,.\end{gathered}$$

Noting that the properties of the random sequence $\{\xi(t)\}_{t\geq 0}$ imply $E\{\mathcal{A}^TX\mathcal{A}\}=\sum_{i=0}^{r}A_i^TXA_i$, $E\{\mathcal{B}^TX\mathcal{B}\}=\sum_{i=0}^{r}B_i^TXB_i$ and $E\{\mathcal{A}^TX\mathcal{B}\}=\sum_{i=0}^{r}A_i^TXB_i$, it follows from the above equation that

$$\begin{gathered}E\{y^T(t)y(t)\}=E\{x^T(t)[\textstyle\sum_{i=0}^{r}A_i^TXA_i-X+C^TC]x(t)\\ +w^T(t)[D^TD+\textstyle\sum_{i=0}^{r}B_i^TXB_i]w(t)\\ +w^T(t)[D^TC+\textstyle\sum_{i=0}^{r}B_i^TXA_i]x(t)+x^T(t)[C^TD+\textstyle\sum_{i=0}^{r}A_i^TXB_i]w(t)\\ +x^T(t)Xx(t)-x^T(t+1)Xx(t+1)\}\,.\end{gathered}$$

Substituting from (5) into the first bracket in the above equation, one obtains

$$\begin{gathered}E\left[|y(t)|^2\right]=E\Big[x^T(t)Xx(t)-x^T(t+1)Xx(t+1)\\ +x^T\left(\textstyle\sum_{i=0}^{r}A_i^TXB_i\right)w(t)+w^T(t)\left(\textstyle\sum_{i=0}^{r}B_i^TXA_i\right)x(t)\\ -x^T(t)\left(\textstyle\sum_{i=0}^{r}A_i^TXB_i+C^TD\right)\left(\frac{1}{q}I-\textstyle\sum_{i=0}^{r}B_i^TXB_i-D^TD\right)^{-1}\\ \times\left(\textstyle\sum_{i=0}^{r}A_i^TXB_i+C^TD\right)^Tx(t)+w^T(t)\left(\textstyle\sum_{i=0}^{r}B_i^TXB_i\right)w(t)\\ +x^T(t)C^TDw(t)+w^T(t)D^TCx(t)+w^T(t)D^TDw(t)\Big]\,.\end{gathered} \tag{9}$$

Define

$$\begin{aligned}
\mathcal{P}(t) := \Big[& w^T(t) - x^T(t)\left(\textstyle\sum_{i=0}^r A_i^T X B_i + C^T D\right) \\
& \times \left(\tfrac{1}{q} I - \textstyle\sum_{i=0}^r B_i^T X B_i - D^T D\right)^{-1} \\
& - v^T(t)\left(\tfrac{1}{q} I - \textstyle\sum_{i=0}^r B_i^T X B_i - D^T D\right)^{-\frac{1}{2}}\Big] \\
& \times \left(\tfrac{1}{q} I - \textstyle\sum_{i=0}^r B_i^T X B_i - D^T D\right) \\
\times \Big[& w(t) - \left(\tfrac{1}{q} I - \textstyle\sum_{i=0}^r B_i^T X B_i - D^T D\right)^{-1}\left(\textstyle\sum_{i=0}^r B_i^T X A_i + D^T C\right) x(t) \\
& - \left(\tfrac{1}{q} I - \textstyle\sum_{i=0}^r B_i^T X B_i - D^T D\right)^{-\frac{1}{2}} v(t)\Big] .
\end{aligned} \tag{10}$$

Then, using the properties of $\{v(t)\}_{t\geq 0}$ it follows that

$$\begin{aligned}
E[\mathcal{P}(t)] = E\Big[& w^T(t)\left(\tfrac{1}{q} I - \textstyle\sum_{i=0}^r B_i^T X B_i - D^T D\right) w(t) \\
& - w^T(t)\left(\textstyle\sum_{i=0}^r B_i^T X A_i + D^T C\right) x(t) \\
& - x^T(t)\left(\textstyle\sum_{i=0}^r A_i^T X B_i + C^T D\right) w(t) \\
& + x^T(t)\left(\textstyle\sum_{i=0}^r A_i^T X B_i + C^T D\right)\left(\tfrac{1}{q} I - \textstyle\sum_{i=0}^r B_i^T X B_i - D^T D\right)^{-1} \\
& \times \left(\textstyle\sum_{i=0}^r B_i^T X A_i + D^T C\right) x(t) \\
& - 2 Tr\, D_f \left(\tfrac{1}{q} I - \textstyle\sum_{i=0}^r B_i^T X B_i - D^T D\right)^{\frac{1}{2}}\Big] + m .
\end{aligned} \tag{11}$$

Taking into account (9) and (11) one obtains

$$\begin{aligned}
& E\left[|y(t)|^2 - \gamma^2 |w(t)|^2\right] = E\Big[x^T(t) X x(t) - x^T(t+1) X x(t+1) - \mathcal{P}(t) \\
& - 2 Tr\, D_f \left(\tfrac{1}{q} I - \textstyle\sum_{i=0}^r B_i^T X B_i - D^T D\right)^{\frac{1}{2}} + m + \left(\tfrac{1}{q} - \gamma^2\right) w^T(t) w(t)\Big] .
\end{aligned} \tag{12}$$

Since the systems (1) and(2) are ESMS

$$\lim_{\ell\to\infty} \frac{1}{\ell} E\left[x^T(0) X x(0) - x^T(\ell) X x(\ell)\right] = 0,$$

and then one directly obtains that

$$\begin{aligned}
& \lim_{\ell\to\infty} \tfrac{1}{\ell} \textstyle\sum_{t=0}^{\ell} E\left[|y(t)|^2 - \gamma^2 |w(t)|^2\right] \\
& = \lim_{\ell\to\infty} \tfrac{1}{\ell} E\left[-\textstyle\sum_{t=0}^{\ell} \mathcal{P}(t) + \textstyle\sum_{t=0}^{\ell}\left(\tfrac{1}{q} - \gamma^2\right) w^T(t) w(t)\right] \\
& - 2 Tr\, D_f \left(\tfrac{1}{q} I - \textstyle\sum_{i=0}^r B_i^T X B_i - D^T D\right)^{\frac{1}{2}} + m.
\end{aligned} \tag{13}$$

From (10) it follows that $\mathcal{P}(t) \geq 0$ and $\mathcal{P}(t) = 0$ for

$$w(t) = \left(\tfrac{1}{q}I - \textstyle\sum_{i=0}^{r} B_i^T X B_i - D^T D\right)^{-1} \left(\textstyle\sum_{i=0}^{r} B_i^T X A_i + D^T C\right) x(t) \\ + \left(\tfrac{1}{q}I - \textstyle\sum_{i=0}^{r} B_i^T X B_i - D^T D\right)^{-\frac{1}{2}} v(t). \tag{14}$$

The above condition is fulfilled for a filter G having the state $\tilde{x}(t)$ equal to the state $x(t)$ of F and if the following conditions are accomplished

$$\begin{aligned} C_f &= \left(\tfrac{1}{q}I - \textstyle\sum_{i=0}^{r} B_i^T X B_i - D^T D\right)^{-1} \left(\textstyle\sum_{i=0}^{r} B_i^T X A_i + D^T C\right) \\ D_f &= \left(\tfrac{1}{q}I - \textstyle\sum_{i=0}^{r} B_i^T X B_i - D^T D\right)^{-\frac{1}{2}}. \end{aligned} \tag{15}$$

For $w(t)$ given by (14) the first equation (1) becomes

$$x(t+1) = \left(A_0 + \textstyle\sum_{i=1}^{r} \xi_i(t) A_i\right) x(t) + \left(B_0 + \textstyle\sum_{i=1}^{r} \xi_i(t) B_i\right) \\ \times \left(\tfrac{1}{q}I - \textstyle\sum_{i=0}^{r} B_i^T X B_i - D^T D\right)^{-1} \left(\textstyle\sum_{i=0}^{r} B_i^T X A_i + D^T C\right) x(t) \\ + \left(B_0 + \textstyle\sum_{i=1}^{r} \xi_i(t) B_i\right) \left(\tfrac{1}{q}I - \textstyle\sum_{i=0}^{r} B_i^T X B_i - D^T D\right)^{-\frac{1}{2}} v(t). \tag{16}$$

Since $\tilde{x}(t)$ equals $x(t), t = 0, 1, ...$ from the above equation one obtains

$$\begin{aligned} A_{f_i} &= A_i + B_i \left(\tfrac{1}{q}I - \textstyle\sum_{j=0}^{r} B_j^T X B_j - D^T D\right)^{-1} \\ &\quad \times \left(\textstyle\sum_{j=0}^{r} B_j^T X A_j + D^T C\right) \\ B_{f_i} &= B_i \left(\tfrac{1}{q}I - \textstyle\sum_{j=0}^{r} B_j^T X B_j - D^T D\right)^{-\frac{1}{2}}, i = 0, 1, ..., r. \end{aligned} \tag{17}$$

Since X is the stabilising solution of the Riccati equation (5) it follows that the filter with A_{f_i}, $i = 0, 1, ..., r$ given above is ESMS.

Based on the expression (15) of D_f and since $\tilde{x}(t) = x(t)$, from the second equation (2) it follows that

$$E\left[\tilde{w}(0)\tilde{w}^T(0)\right] = \left(\frac{1}{q}I - \sum_{i=0}^{r} B_i^T X B_i - D^T D\right)^{-1}. \tag{18}$$

In the following it will be shown that under the assumption (7) from the statement, for all ESMS filters $G \in \mathcal{G}_a$ having $D_f = \Psi_q^{-\frac{1}{2}}$ the following condition is accomplished

$$-m + \left(\frac{1}{q} - \gamma^2\right) \|G\|_2^2 < 0. \tag{19}$$

Indeed, since $G \in \mathcal{G}_a$ and since $D_f = \Psi_q^{-\frac{1}{2}}$ it follows that

$$\det \frac{m\Psi_q^{-1}}{\|G\|_2^2} > e^{-2a}. \tag{20}$$

Taking into account (7) and the above inequality it follows that

$$\det \frac{m\Psi_q^{-1}}{\|G\|_2^2} > \det \left(\frac{1}{q} - \gamma^2\right) \Psi_q^{-1}$$

from which one directly obtains (19). Using the inequality (19), equations (13), (11), the equation for D_f in (15) and Definition 2, it follows that $||FG||_2/||G||_2 < \gamma$.

Let us consider now the more general case for a certain filter $G \in \mathcal{G}_a$, satisfying therefore the condition

$$-\frac{1}{2} \ln \det \frac{mD_f D_f^T}{\|G\|_2^2} \leq a\,. \tag{21}$$

From the above condition and from the assumption (7) it follows that

$$\det \left(\frac{1}{q} - \gamma^2\right) \Psi_q^{-1} < \det \frac{mD_f D_f^T}{\|G\|_2^2}. \tag{22}$$

Using the general property $\det(A) \leq (Tr(A)/m)^m$, from the above inequality one obtains

$$Tr\left(D_f \Psi_q^{\frac{1}{2}}\right) > \left(\frac{1}{q} - \gamma^2\right)^{\frac{1}{2}} m^{\frac{1}{2}} \|G\|_2 \tag{23}$$

and thus

$$\begin{aligned}
&\left(\tfrac{1}{q} - \gamma^2\right) \|G\|_2^2 - 2Tr\left(D_f \Psi_q^{\frac{1}{2}}\right) + m \\
&< \left(\tfrac{1}{q} - \gamma^2\right) \|G\|_2^2 - 2\left(\tfrac{1}{q} - \gamma^2\right)^{\frac{1}{2}} m^{\frac{1}{2}} \|G\|_2 + m \\
&= \left(\left(\tfrac{1}{q} - \gamma^2\right)^{\frac{1}{2}} \|G\|_2 - m^{\frac{1}{2}}\right)^2
\end{aligned} \tag{24}$$

From the above inequality it follows that if

$$\left(\frac{1}{q} - \gamma^2\right) \|G\|_2^2 = m \tag{25}$$

the left hand side of it is negative and therefore from (12) it follows that $\|FG\|_2/\|G\|_2 < \gamma$. The condition (25) implies that

$$\frac{1}{q} - \gamma^2 = \frac{m}{\|G\|_2^2}.$$

Substituting the above expression into (7) one obtains the condition

$$-\frac{1}{2} \ln \det \frac{m\Psi_q^{-1}}{\|G\|_2^2} \geq a\,. \tag{26}$$

Comparing (26) with the definition of the mean anisotropy one concludes that if for a filter $G \in \mathcal{G}_a$ there exists another filter $\hat{G}$ with $\bar{A}\left(\hat{G}\right) \geq a$ such that $\|\hat{G}\|_2 = \|G\|_2$ and having $\hat{D}_f = \Psi_q^{-\frac{1}{2}}$ for a certain q satisfying the assumptions of Theorem 1, then $\|FG\|_2/\|G\|_2 < \gamma$. A similar conclusion is derived in the deterministic framework in [13]. Such a $\hat{G}$ always may be found. Indeed since $\Psi_q^{-1} \to 0$ for $q \to 0$, it follows that the Riccati equation (5) has a stabilising solution and the condition (26) is fulfilled for a small enough $q > 0$. Then for any $G \in \mathcal{G}_a$, based on Lemma 1 one can easily determine $\hat{A}_{f_i}, \hat{B}_{f_i}, i = 0, ..., r$ and $\hat{C}_f$ such that $\|\hat{G}\|_2 = \|G\|_2$.

Using the inequality (19), (13), (11), the equation for D_f in (15) and Definition 2, it follows that $||FG||_2/||G||_2 < \gamma$. Let us finally notice that according with the Lemma 2, it follows that a necessary condition for the existence of a stabilizing solution of the Riccati equation (5) is $1/q \geq \|F\|_\infty^2$, from which it follows that $q \leq \|F\|_\infty^{-2}$. Thus the proof is complete.

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