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Methods and Algorithms for Approximating the Gamma Function and Related Functions. A survey. Part II: Completely monotonic functions*

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Abstract

In this survey we present our recent results on analysis of gamma function and related functions. The results obtained are in the theory of asymptotic analysis, approximation of gamma and polygamma functions, or in the theory of completely monotonic functions. In the second part of this survey we show how the theory of completely monotonic functions can be used to establish sharp bounds for gamma and related functions.

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1 Introduction and Motivation

By a completely monotonic function on an interval I we mean a function $z : I \to \mathbb{R}$ which admits derivatives of any order and satisifies the following

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inequalities for every $x \in I$ and integer $n \geq 0$:

$$(-1)^n z^{(n)}(x) \geq 0.$$

The definition and further properties of other classes of completely monotonic functions (including (almost) completely monotonic, (almost) logarithmically completely monotonic, strongly completely monotonic, completely monotonic of nth order) can be found for example in [6], [7], [15], [16], [29], or [34].

Completely monotonic functions are of great help in the problem of approximating the function z itself as well the derivatives $z^{(n)}$. More precisely, if we take into account that the derivative of $z^{(n)}$ keep constant sign and consequently the function $z^{(n)}$ is monotone, $z^{(n)}(x)$ lies between $z^{(n)}(a)$ and $z^{(n)}(b)$, as x runs between a and b.

Moreover, completely monotone functions involving gamma function provide sharp bounds for gamma and polygamma functions.

A tool for proving the complete monotonicity of a function is Bernstein-Widder-Hausdorff theorem (see, *e.g.,* [35, p. 161]) which states that a function is completely monotonic on $(0, \infty)$ if and only if the following integral representation is valid for every $x > 0$:

$$z(x) = \int_0^\infty e^{-xt} d\mu(t). \tag{1}$$

Here μ is a non-negative measure on $[0, \infty)$ such that the integral converges for all $x > 0$.

The Euler gamma function is defined by the following formula for every real $x > 0$:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt,$$

while the logarithmic derivative of Γ is called digamma (or psi) function,

$$\psi(x) = \frac{d}{dx}(\ln \Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

Further derivatives ψ', ψ'', ψ''', ... are called tri-, tetra-, penta-gamma function, and in general, $\psi^{(n)}$ with $n = 1, 2, 3, ...$ are polygamma functions.

In order to prove the complete monotonicity of a function involving gamma and polygamma functions on $(0, \infty)$ using (1), the following integral representations are of main help:

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-tx} dt \quad (x > 0,\ r > 1) \tag{2}$$

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}} \right) dt \quad (x > 0)$$

and

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1-e^{-t}} e^{-tx} dt \quad (x > 0,\ n \geq 1). \tag{3}$$

For further details, please see [1].

Usually to an approximation formula of the gamma function of type

$$\Gamma(x+1) \sim \omega(x) \tag{4}$$

in the sense that the ratio $\Gamma(x+1)/\omega(x)$ tends to 1, as x approaches infinity, the following function is attached:

$$F(x) = \ln \frac{\Gamma(x+1)}{\omega(x)}. \tag{5}$$

If F (sometimes $-F$) is completely monotonic then important results related to approximation formula (4) can be established. Let us assume for example that F is completely monotonic on $[1, \infty)$, possible on $(0, \infty)$. As $F' < 0$, the function F is strictly decreasing on $[1, \infty)$. Thus $F(\infty) < F(x) \leq F(1)$, which can be rearranged in the form of the following double inequality valid for every $x \in [1, \infty)$:

$$\alpha \cdot \omega(x) < \Gamma(x+1) < \beta \cdot \omega(x).$$

Here the constants $\alpha = \exp F(1)$ and $\beta = \exp F(\infty) = 1$ are the best possible.

Furthermore, we can exploit the monotonicity of F' to obtain sharp bounds for the digamma function. Assuming that ω is derivable, we get

$$F'(x) = \psi(x+1) - \frac{\omega'(x)}{\omega(x)}.$$

But $F'' > 0$, so F' is strictly increasing on $[1, \infty)$, which can be written as $F'(1) \leq F'(x) < F'(\infty)$. The following sharp inequalities hold true for every real $x \in [1, \infty)$:

$$\alpha' + \frac{\omega'(x)}{\omega(x)} \leq \psi(x+1) < \beta' + \frac{\omega'(x)}{\omega(x)},$$

where $\alpha' = F'(1)$ and $\beta' = F'(\infty)$.

These are the first illustration of our method for establishing sharp bounds for gamma and digamma functions related to approximation formula (4). In a similar manner inequalities for polygamma functions can be stated using the nth derivative of F.

In conclusion the study of the monotonicity of the function F associated to an approximation formula (4) is of great importance in the theory of approximation of gamma, polygamma and other related functions.

2 The Technique

In order to illustrate the technique, we present the results stated in [18]. Undoubtedly the most used formula for approximating large factorials is Stirling's formula

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^{x}.$$

A slightly better result was proposed by Burnside (see, *e.g.* [5]):

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x+1/2}{e}\right)^{x+\frac{1}{2}}. \tag{6}$$

It has been proved in [24] that the function

$$F(x) = \ln \frac{\Gamma(x+1)}{\sqrt{2\pi}\left(\frac{x+1/2}{e}\right)^{x+\frac{1}{2}}}.$$

associated to the Burnside formula is completely monotonic.

For sake of completness, we reproduce here a sketch of proof of the above result stated in [24]. As

$$F(x) = \ln\Gamma(x+1) - \ln\sqrt{2\pi} - \left(x+\frac{1}{2}\right)\ln\left(x+\frac{1}{2}\right) + x + \frac{1}{2},$$

we obtain

$$F'(x) = \psi(x+1) - \ln\left(x+\frac{1}{2}\right).$$

Using the recurrence formula

$$\psi(x+1) = \psi(x) + \frac{1}{x}$$

(see, *e.g.,* [1, p. 258]), we obtain

$$F'(x) = \psi(x) + \frac{1}{x} - \ln\left(x + \frac{1}{2}\right),$$

then

$$F''(x) = \psi''(x) - \frac{1}{x^2} - \frac{1}{x + \frac{1}{2}}.$$

Using (2) and (3), we deduce that

$$F''(x) = \int_0^\infty \frac{te^{-xt}}{1 - e^{-t}} dt - \int_0^\infty te^{-xt} dt - \int_0^\infty e^{-\left(x + \frac{1}{2}\right)t} dt,$$

or

$$F''(x) = \int_0^\infty \frac{e^{-(x+1)t}}{1 - e^{-t}} \varphi(t)\, dt,$$

where

$$\varphi(t) = t - e^{-\frac{1}{2}t}\left(e^t - 1\right).$$

The function φ is strictly decreasing, since $\varphi'(t) = -\frac{1}{2}e^{-\frac{1}{2}t}\left(e^{\frac{1}{2}t} - 1\right)^2 < 0$. For $t > 0$, we have $\varphi(t) < \varphi(0) = 0$. According to Bernstein- Widder-Hausdorff theorem, $-F''$ is strictly completely monotonic. using the definition, we obtain

$$(-1)^n \left(-F''\right)^{(n)} \geq 0,$$

for every integer $n \geq 0$. By replacing $(-F'')^{(n)}$ by $(-F)^{n+2}$, we deduce

$$(-1)^n F^{(n)} \geq 0, \tag{7}$$

for every integer $n \geq 2$. In order to finalize our proof, we have to show that (7) is valid also for $n = 1$ and $n = 0$.

In this sense, note that F' is strictly decreasing, since $F'' < 0$. But $\lim_{x\to\infty} F'(x) = 0$, so $F'(x) > 0$ and consequently, F is strictly increasing. Using the fact that $\lim_{x\to\infty} F(x) = 0$, we deduce that $F < 0$. This assures the veridicity our assertion that $-F$ is strictly completely monotonic.

As applications of the complete monotonicity of $-F$, the following sharp bounds for the gamma and digamma function were presented in [24] for every real $x \geq 1$:

$$\omega \cdot \sqrt{2\pi}\left(\frac{x + \frac{1}{2}}{e}\right)^{x+\frac{1}{2}} \leq \Gamma(x + 1) < \sqrt{2\pi}\left(\frac{x + \frac{1}{2}}{e}\right)^{x+\frac{1}{2}},$$

where the constant $\omega = \frac{2}{3\sqrt{3\pi}}e^{3/2} = 0.97323\cdots$ is best possible. For every real $x \geq 1$, it holds

$$\ln\left(x+\frac{1}{2}\right)-\frac{1}{x}<\psi\left(x\right)\leq\ln\left(x+\frac{1}{2}\right)-\frac{1}{x}+\zeta,$$

with best possible constant $\zeta = 1 - \ln\frac{3}{2} - \gamma = 0.01731\cdots$.

The same technique was used in [27] to prove the complete monotonicity of a class of functions related to the following inequalities

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}}<\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}<\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}},\quad n\geq1,$$

now called Kazarinoff's inequalities. Please see further details in [4], [8], [9], [12], [13]. Precisely, the function

$$F_a\left(x\right)=\ln\Gamma\left(x+1\right)-\ln\Gamma\left(x+\frac{1}{2}\right)-\frac{1}{2}\ln\left(x+a\right)$$

is completely monotonic when $a \in \left[0, \frac{1}{4}\right]$, while $-F_b$ is completely monotonic when $b \in \left[\frac{1}{2}, \infty\right)$. The following integral representation is valid

$$F_a''\left(x\right)=\int_0^\infty\frac{te^{-(x+1+a)t}}{1-e^{-t}}\varphi_a\left(t\right)dt,$$

where φ_a admits the following expansion in power series in t:

$$\varphi_a\left(t\right)=\sum_{k=2}^{\infty}w_kt^k,$$

with

$$w_k=a^k-\left(a+\frac{1}{2}\right)^k+\frac{1}{2}.$$

It is stated in [27, Lemma 2.1] that $w_k \geq 0$, if $a \in \left[0, \frac{1}{4}\right]$ and $w_k \leq 0$, if $a \in \left[\frac{1}{2}, \infty\right)$, so the previuos assertions on complete montonicity of functions F_a are now proved. As a consequence, the following inequalities hold true for every $x \geq 1$,

$$\sqrt{x+\frac{1}{4}}<\frac{\Gamma\left(x+1\right)}{\Gamma\left(x+\frac{1}{2}\right)}\leq\omega\sqrt{x+\frac{1}{4}},$$

and

$$\mu\sqrt{x+\frac{1}{2}}\leq\frac{\Gamma\left(x+1\right)}{\Gamma\left(x+\frac{1}{2}\right)}<\sqrt{x+\frac{1}{2}},$$

where the constants $\omega = \frac{4}{\sqrt{5\pi}} = 1.00930\cdots$ and $\mu = \frac{2\sqrt{2}}{\sqrt{3\pi}} = 0.92132\cdots$ are best possible.

It is studied in [19] the following class of approximations for every real parameter a :

$$\Gamma(x+1) \sim \nu_x(a) := \sqrt{2\pi e}e^{-a}\left(\frac{x+a}{e}\right)^{x+\frac{1}{2}}. \tag{8}$$

This class incorporates Stirling's formula $\Gamma(x+1) \sim \nu_x(0)$, Burnside's formula $\Gamma(x+1) \sim \nu_x\left(\frac{1}{2}\right)$, but also a recent formula discovered by Schuster [32]

$$\Gamma(x+1) \sim \sqrt{2\pi}e^{-\frac{1}{\sqrt{12}}}\left(\frac{x+\frac{1}{2}+\frac{1}{\sqrt{12}}}{e}\right)^{x+\frac{1}{2}},$$

which can be written as

$$\Gamma(x+1) \sim \nu_x\left(\frac{1}{2}+\frac{1}{\sqrt{12}}\right).$$

Schuster's formula demonstrates the preoccupation of the researchers to find increasingly better approximations of type (8). It is proven in [26] that the best approximations possible (8) are $\Gamma(x+1) \sim \nu_x(\omega)$ and $\Gamma(x+1) \sim \nu_x(\zeta)$, where

$$\omega = \frac{3-\sqrt{3}}{6}, \quad \zeta = \frac{3+\sqrt{3}}{6}.$$

The following result was presented in [18] relative to the functions associated to (8):

$$G_a(x) = \ln\frac{\Gamma(x+1)}{\sqrt{2\pi e}e^{-a}\left(\frac{x+a}{e}\right)^{x+\frac{1}{2}}}.$$

This function G_a is completely monotonic when $a \in [0, \omega]$, while $-G_b$ is completely monotonic when $b \in \left[\frac{1}{2}, \zeta\right]$. As a consequence of the complete monotonicity of G_ω and $-G_\zeta$, the following double inequalities are valid for every $x \geq 0$:

$$\sqrt{2\pi e}\cdot e^{-\omega}\left(\frac{x+\omega}{e}\right)^{x+1/2} < \Gamma(x+1) \leq \alpha\cdot\sqrt{2\pi e}\cdot e^{-\omega}\left(\frac{x+\omega}{e}\right)^{x+1/2},$$

where $\alpha = 1.07204\cdots$, and

$$\beta\cdot\sqrt{2\pi e}\cdot e^{-\zeta}\left(\frac{x+\zeta}{e}\right)^{x+1/2} < \Gamma(x+1) \leq \sqrt{2\pi e}\cdot e^{-\zeta}\left(\frac{x+\zeta}{e}\right)^{x+1/2},$$

where $\beta = 0.98850\cdots$. By exploiting the monotonicity of G'_ω and $-G'_\zeta$, the following sharp inequalities on digamma function were presented in [18, Theorem 2.2]:

$$\ln\left(x+\frac{3-\sqrt{3}}{6}\right)+\frac{\sqrt{3}}{6x+3-\sqrt{3}}-\tau\leq\psi\left(x\right)+\frac{1}{x}<$$
$$\ln\left(x+\frac{3-\sqrt{3}}{6}\right)+\frac{\sqrt{3}}{6x+3-\sqrt{3}}$$

and

$$\ln\left(x+\frac{3+\sqrt{3}}{6}\right)-\frac{\sqrt{3}}{6x+3+\sqrt{3}}<$$
$$\psi\left(x\right)+\frac{1}{x}\leq\ln\left(x+\frac{3+\sqrt{3}}{6}\right)-\frac{\sqrt{3}}{6x+3+\sqrt{3}}+\sigma,$$

where $\tau = 0.00724\cdots$ and $\sigma = 0.00269\cdots$.

Furthermore using the monotonicity of G''_ω and $-G''_\zeta$, the following sharp inequalities on trigamma function were estabilshed in [18, Theorem 2.3]:

$$\frac{6}{6x+3-\sqrt{3}}-\frac{6\sqrt{3}}{\left(6x+3-\sqrt{3}\right)^2}$$
$$<\psi'\left(x\right)-\frac{1}{x^2}\leq\frac{6}{6x+3-\sqrt{3}}-\frac{6\sqrt{3}}{\left(6x+3-\sqrt{3}\right)^2}+\lambda$$

and

$$\frac{6}{6x+3+\sqrt{3}}+\frac{6\sqrt{3}}{\left(6x+3+\sqrt{3}\right)^2}-\nu\leq\psi'\left(x\right)-\frac{1}{x^2}<$$
$$\frac{6}{6x+3+\sqrt{3}}+\frac{6\sqrt{3}}{\left(6x+3+\sqrt{3}\right)^2},$$

where $\lambda = 0.01612\cdots$ and $\nu = 0.00436\cdots$.

As another example we present the following class of lower and upper bounds for gamma function:

$$\frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\alpha}} \leq \Gamma(n+1) < \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\beta}}, \tag{9}$$

where α, β are any real numbers. Sandor and Debnath [31] found (9) with $\alpha = 0$, $\beta = 1$, while Batir [3] proposed better estimates using $\alpha = 1 - 2\pi e^{-2}$ and $\beta = 1/6$.

Motivated by the fact that the double inequality (9) can be rearranged as

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(\frac{n}{n-\alpha}\right)^{1/2} \leq n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(\frac{n}{n-\beta}\right)^{1/2}, \tag{10}$$

Mortici [21] introduced the class of approximations

$$\Gamma(n+1) \sim \mu_n(a,b) := \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(\frac{n+a}{n+b}\right)^{1/2}. \tag{11}$$

which enclose the previous formulas by Sandor and Debnath and Batir. It is proven that the most accurate approximation (11) is obtained in $a = 1/12$, $b = -1/12$ case. The corresponding approximation is better than those arising in (9)-(10). The next comparison table shows the superiority of (11) over

$$\Gamma(n+1) \sim \kappa_n := \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(\frac{n}{n-\frac{1}{6}}\right)^{1/2},$$

which is the best approximation among (9)-(10).

n	$\ln(\kappa_n/\Gamma(n+1))$	$\ln(\mu_n/\Gamma(n+1))$
25	1.13×10^{-5}	1.90×10^{-7}
50	2.80×10^{-6}	2.37×10^{-8}
100	6.97×10^{-7}	2.97×10^{-9}
1000	6.94×10^{-9}	2.97×10^{-12}

It is considered in [21] the function associated to approximation formula (11):

$$G(x) = \ln\frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(\frac{x+\frac{1}{12}}{x-\frac{1}{12}}\right)^{1/2}}$$

and it has been proved that $-G$ is completely monotonic. As a direct consequence of this fact, the following sharp inequalities are valid for every real

$x \geq 1:$

$$\omega \cdot \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(\frac{x+\frac{1}{12}}{x-\frac{1}{12}}\right)^{1/2} < \Gamma(x+1) < \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(\frac{x+\frac{1}{12}}{x-\frac{1}{12}}\right)^{1/2},$$

where $\omega = e\sqrt{\frac{11}{26\pi}} = 0.99754\cdots$, and

$$\frac{1}{2\left(x+\frac{1}{12}\right)} - \frac{1}{2\left(x-\frac{1}{12}\right)} < \psi(x) - \left(\ln x - \frac{1}{2x}\right) \leq \frac{1}{2\left(x+\frac{1}{12}\right)} - \frac{1}{2\left(x-\frac{1}{12}\right)} + \tau,$$

with $\tau = -\gamma + \frac{167}{286} = 0.00670\cdots$.

3 Further completely monotone functions

One of the first estimate for the remainder λ_n in the Stirling formula

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\lambda_n}$$

was presented by Robbins [30], who proved

$$\frac{1}{12n+1} < \lambda_n < \frac{1}{12n}.$$

Increasingly better estimates were found by Maria [14], Nanjundiah, [28], or Shi et al [33]. Representations of the form

$$\Gamma(x+1) = \sqrt{2\pi}\left(\frac{x}{e}\right)^x e^{\theta(x)/12x}$$

were introduced in the recent past. Shi et al [33] proved that $\theta(x)$ is monotonically increasing on $[1,\infty)$. This result was extended by Mortici [22], who proved that θ decreases monotonically on $(0,\beta)$ and increases monotonically on (β,∞), where $\beta = 0.34142...$ is the solution of the equation

$$\ln\Gamma(x+1) + x\psi(x+1) - \ln\sqrt{2\pi} - 2x\ln x + x = 0.$$

Moreover θ is strictly convex on $(0,\infty)$ and the function $-x^{-1}\theta'''$ is completely monotonic on $(0,\infty)$.

It has been studied in [20] the remainder w of the Burnside formula (6)

$$\Gamma(x+1) = \sqrt{2\pi}\left(\frac{x+1/2}{e}\right)^{x+1/2} e^{w(x)}$$

and stated that $-w$ is completely monotonic, in particular w is concave.

Kečkić and Vasić [10] presented the following double inequality

$$\frac{x^{x-1}e^y}{y^{y-1}e^x} \leq \frac{\Gamma(x)}{\Gamma(y)} \leq \frac{x^{x-\frac{1}{2}}e^y}{y^{y-\frac{1}{2}}e^x}, \tag{12}$$

for all $x \geq y > 1$, which can be rewritten as

$$\frac{e^x\Gamma(x)}{x^{x-1/2}} \leq \frac{e^y\Gamma(y)}{y^{y-1/2}} \quad \text{and} \quad \frac{e^y\Gamma(y)}{y^{y-1}} \leq \frac{e^x\Gamma(x)}{x^{x-1}}.$$

This becomes equivalent to the fact that the function

$$f(x) = x + \ln\Gamma(x) - \left(x - \frac{1}{2}\right)\ln x - \ln\sqrt{2\pi}$$

is decreasing and the function

$$g(x) = x + \ln\Gamma(x) - (x-1)\ln x$$

is increasing. It is proved in [23] that the functions f and g' are completely monotonic on $(0,\infty)$. As a direct consequence, Kečkić-Vasić inequality (12) follows and it holds also for every $x \geq y > 0$. By using the monotonicity of f' and g', there are established the following sharp inequalities for every real $x \geq 1$:

$$\ln x - \frac{1}{2x} - \tau \leq \psi(x) < \ln x - \frac{1}{2x},$$

where the constant $\tau = \gamma - \frac{1}{2} = 0.07721\cdots$ is the best possible, and for every real $x \geq 1$:

$$\ln x - \frac{1}{x} < \psi(x) \leq \ln x - \frac{1}{x} + \sigma,$$

where the constant $\sigma = -\gamma + 1 = 0.42278\cdots$ is the best possible.

In 1965, Minc and Sathre [17] have given one of the first estimates of the expression $\phi(r) = (r!)^{1/r}$ and the ratio $\phi(r+1)/\phi(r)$ for every real $r \geq 1$:

$$1 < \frac{\phi(r+1)}{\phi(r)} < 1 + \frac{1}{r}. \tag{13}$$

Inequalities involving the function $\phi(r)$ are of interest in some branches of pure and applied mathematics and they have important applications in the theory of $(0,1)$-matrices.

Mortici [25] improved (13) in the sense of the following inequality for every $x \geq 1$:

$$\frac{\Gamma(x+2)^{1/(x+1)}}{\Gamma(x+1)^{1/x}} \geq \frac{(4x+4)^{1/(x+1)}}{(4x)^{1/x}} \left(1+\frac{1}{x}\right) > 1.$$

The corresponding function

$$h(x) = x(x+1) \ln \frac{x\Gamma(x+1)^{1/(x+1)}}{(x+1)\Gamma(x)^{1/x}}$$

is considered and the complete monotonicity on $(1,\infty)$ of h' is established. In particular h' is positive, so h is strictly increasing. In consequence, for every $x \geq 1$, we have $h(1) \leq h(x)$. As $h(1) = -\ln 4$, we obtain

$$-\ln 4 \leq x(x+1) \ln \frac{x\Gamma(x+1)^{1/(x+1)}}{(x+1)\Gamma(x)^{1/x}},$$

or

$$\frac{\Gamma(x+1)^{1/(x+1)}}{\Gamma(x)^{1/x}} \geq 4^{\frac{-1}{x(x+1)}} \left(1+\frac{1}{x}\right) > 1,$$

where the constant 4 is best possible. The obtained approximation formula

$$\frac{\Gamma(x+2)^{1/(x+1)}}{\Gamma(x+1)^{1/x}} \sim \frac{(4x+4)^{1/(x+1)}}{(4x)^{1/x}} \left(1+\frac{1}{x}\right),$$

is much better than Minc-Sathre. See [25].

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