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Methods and Algorithms for Approximating the Gamma Function and Related Functions. A survey. Part I: Asymptotic Series*

Cristinel Mortici†

Abstract

In this survey we present our recent results on analysis of gamma function and related functions. The results obtained are in the theory of asymptotic analysis, approximation of gamma and polygamma functions, or in the theory of completely monotonic functions. The motivation of this first part is the work of C. Mortici [Product Approximations via Asymptotic Integration Amer. Math. Monthly 117 (2010) 434-441] where a simple strategy for constructing asymptotic series is presented. The classical asymptotic series associated to Stirling, Wallis, Glaisher-Kinkelin are rediscovered. In the second section we discuss some new inequalities related to Landau constants and we establish some asymptotic formulas.

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†cristinel.mortici@hotmail.com, Valahia University of Târgovişte, Bd. Unirii 18, 130082 Târgovişte, Romania and Academy of Romanian Scientists, Splaiul Independenţei 54, 050094 Bucharest, Romania

1 A method for constructing asymptotic series and applications

The problem of approximating the gamma function goes back to Laplace formula which is the continuous version of the Stirling formula. In 1916 Srinivasa Ramanujan (see [4]) proposed a formula which was later studied by E. A. Karatsuba in [13] and Alzer [2].

A method for improving some approximation formulas for large factorials is to consider the corresponding asymptotic series. It is presented in [16] an original approach to the asymptotic evaluation of sums and products. As for usual, to an approximation formula $f(n) \sim g(n)$, it is associated the series

$$f(n) \sim g(n) \exp\left(\sum_{k=1}^{\infty} \frac{a_k}{n^k}\right), \tag{1}$$

also called an asymptotic series. Such series have the advantage that in a truncated form, provides approximations to any accuracy n^{-k}.

The strategy in [16] is based on the idea that when series (1) is truncated at the mth term, the approximation obtained should be the most precise possible among all approximations

$$f(n) \sim g(n) \exp\left(\sum_{k=1}^{m} \frac{a'_k}{n^k}\right), \tag{2}$$

where $a'_1, a'_2, \ldots, a'_m$ are any real numbers.

The first task is to compare the accuracy of two approximation formulas. We do this by associate to an approximation formula $f(n) \sim g(n)$ the relative error sequence r_n by the relations

$$f(n) = g(n) \exp r_n\ , \qquad n \geq 1.$$

We consider $f(n) \sim g(n)$ as better as r_n converges to zero faster.

Now a new task appears, that is to measure the speed of convergence of the sequence r_n. The tool used is the following

Lemma 1 (Speed of Convergence Lemma). *If $(r_n)_{n\geq 1}$ is convergent to zero and*

$$\lim_{n\to\infty} n^k(r_n - r_{n+1}) = l\ , \qquad \text{then} \qquad \lim_{n\to\infty} n^{k-1} r_n = \frac{l}{k-1}, \quad (k \geq 2)\,.$$

In other words, r_n is of $n^{-(k-1)}$ speed of convergence, in case $r_n - r_{n+1}$ is of order n^{-k}.

We cite from Batir [3]: “This lemma, despite of its simple appearance, is a strong tool to accelerate and measure the speed of convergence of some sequences having limit zero, and has proved by C. Mortici in [16]”. As the reviewer of [16] asked, a detailed proof of Lemma 1 was presented for sake of completness.

We introduce the relative error sequence $(\lambda_n)_{n\geq 1}$ by

$$f(n) = g(n)\exp\left(\sum_{k=1}^{m}\frac{a_k}{n^k}\right)\exp\lambda_n\ , \quad n\geq 1.$$

In order to use Lemma 1, we write

$$\lambda_n-\lambda_{n+1}=\sum_{k=2}^{m+1}\frac{x_k-y_k}{n^k}+O\left(\frac{1}{n^{m+2}}\right),$$

where

$$\sum_{k=1}^{m}\frac{a_k}{n^k}-\sum_{k=1}^{m}\frac{a_k}{(n+1)^k}=\sum_{k=2}^{m+1}\frac{y_k}{n^k}+O\left(\frac{1}{n^{m+2}}\right)$$

with

$$a_1-\binom{k-1}{1}a_2+\cdots+(-1)^k\binom{k-1}{k-2}a_{k-1}=(-1)^k\,y_k,\quad 2\leq k\leq m+1$$

and assuming

$$\ln\frac{f(n)g(n+1)}{g(n)f(n+1)}=\sum_{k=2}^{\infty}\frac{x_k}{n^k}. \tag{3}$$

The following main result is stated in [16].

Theorem 1. *Suppose there is some k such that $2\leq k\leq m+1$ and $x_k\neq y_k$, and let $s=\min\{k\mid 2\leq k\leq m+1,\ x_k\neq y_k\}$. Then*

$$\lim_{n\to\infty}n^{s-1}\lambda_n=\frac{x_s-y_s}{s-1}\in\mathbb{R}\setminus\{0\},$$

and therefore the speed of convergence of $(\lambda_n)_{n\geq 1}$ is $n^{-(s-1)}$.

If $s\geq 3$, conditions $x_k=y_k$, for $2\leq k\leq s-1$, are equivalent to the triangular system

$$x_k=(-1)^k\left(a_1-\binom{k-1}{1}a_2+\cdots+(-1)^k\binom{k-1}{k-2}a_{k-1}\right), \tag{4}$$

which defines uniquely the best coefficients a_k, $1 \le k \le s-2$.

These theoretical results were applied in [16] to deduce the series associated with some approximation formulas: Stirling, Burnside, Glaisher-Kinkelin, Wallis. Standard construction of these series makes appeal to Bernoulli numbers and Euler-Maclaurin summation formula.

To the Glaisher-Kinkelin constant defined by

$$A = \lim_{n\to\infty} \frac{1^1 2^2 3^3 \cdots n^n}{n^{n^2/2+n/2+1/12} e^{-n^2/4}},$$

the following asymptotic series is considered

$$1^1 2^2 3^3 \cdots n^n \sim A \cdot n^{\frac{n^2+n}{2}+\frac{1}{12}} e^{-n^2/4} \exp\left(\sum_{k=1}^{\infty} \frac{a_k}{n^k}\right).$$

Here we have $f(n) = 1^1 2^2 3^3 \cdots n^n$ and $g(n) = A \cdot n^{\frac{n^2+n}{2}+\frac{1}{12}} e^{-n^2/4}$. The values x_k in (3) are

$$x_k = (-1)^k \left(\frac{1}{2k+2} - \frac{1}{2k+4} - \frac{1}{12k}\right),$$

and the solution of the triangular system (4) is $a_1 = 0$, $a_2 = 1/720$, $a_3 = 0, a_4 = -1/5040$, $a_5 = 0$, $a_6 = 1/10080$, $\ldots$. Hence

$$1^1 2^2 3^3 \cdots n^n \sim A \cdot n^{\frac{n^2+n}{2}+\frac{1}{12}} e^{-n^2/4} \exp\left(\frac{1}{720n^2} - \frac{1}{5040n^4} + \frac{1}{10\,080n^6} - \cdots\right).$$

The asymptotic series associated to Wallis formula is

$$\frac{\pi}{2} \sim \left(\prod_{j=1}^{n} \frac{4j^2}{4j^2-1}\right) \exp\left(\sum_{k=1}^{\infty} \frac{a_k}{n^k}\right).$$

With $f(n) = \frac{\pi}{2}$ and $g(n) = \prod_{j=1}^{n} \frac{4j^2}{4j^2-1}$ in (3), we get

$$x_k = \frac{(-1)^k}{k} \left(\frac{3^k+1}{2^k} - 2\right).$$

The solution of the triangular system (4) is $a_1 = 1/4$, $a_2 = -1/8$, $a_3 = 5/96$, $a_4 = -1/64$, $\ldots$. Hence

$$\frac{\pi}{2} \sim \left(\prod_{j=1}^{n} \frac{4j^2}{4j^2-1}\right) \exp\left(\frac{1}{4n} - \frac{1}{8n^2} + \frac{5}{96n^3} - \frac{1}{64n^4} + \cdots\right).$$

By using standard transforms on asymptotic series, it is obtained in [16] the following formula

$$\prod_{j=1}^{n} \frac{4j^2}{4j^2-1} \sim \frac{\pi}{2}\left(1-\frac{1}{4n}+\frac{5}{32n^2}-\frac{11}{128n^3}+\frac{31}{768n^4}-\cdots\right),$$

which is an extension of the following formula presented by Hirschhorn in [12]:

$$\prod_{j=1}^{n} \frac{4j^2}{4j^2-1} \sim \frac{\pi}{2}-\frac{\pi}{8n}+O\left(\frac{1}{n^2}\right) \quad \text{as} \quad n\to\infty.$$

Undoubtedly the most used formula for estimating big factorials is the following

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

now known as Stirling's formula. Classical methods for constructing the corresponding asymptotic series use some equations involving numeric series and improper integrals, Euler-Maclaurin summation formula, Legendre duplication formula, or the analytic definition of Bernoulli numbers. The method proposed in [16] is quite elementary. For the asymptotic series

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\left(\sum_{k=1}^{\infty}\frac{a_k}{n^k}\right), \tag{5}$$

with $f(n)=n!$ and $g(n)=\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, we have in (3)

$$x_k = (-1)^k \frac{k-1}{2k(k+1)}.$$

The solution of the triangular system (4) is $a_1 = 1/12$, $a_2 = 0$, $a_3 = -1/360$, $a_4 = 0$, $a_5 = 1/1260$, $a_6 = 0$, $a_7 = -1/1680$, which are coefficients in (5).

It is presented in [17] the following asymptotic expansion in terms of Bernoulli numbers for every $p \in [0,1]$:

$$\Gamma(x+1) \sim \sqrt{2\pi e}\cdot e^{-p}\left(\frac{x+p}{e}\right)^{x+\frac{1}{2}} \cdot \exp\left\{\sum_{k=1}^{\infty}\frac{a_p(x)}{x^k}\right\}, \quad n\to\infty, \tag{6}$$

where

$$a_p(x) = \frac{1}{k(k+1)}\left[B_{k+1}-(-1)^k p^k\left(\left(p-\frac{1}{2}\right)k-\frac{1}{2}\right)\right].$$

The class of approximations (6) was also recently studied by Nemes [22]. Particular case $p = 1/2$ is Burnside series [6]:

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x+\frac{1}{2}}{e}\right)^{x+\frac{1}{2}} \exp\left(\sum_{k=1}^{\infty}\left(B_{k+1}+\frac{(-1)^k}{2^{k+1}}\right)\frac{1}{k(k+1)x^k}\right),$$

while $p = 1$ case provides the following formula:

$$\Gamma(x+1) \sim \sqrt{\frac{2\pi}{e}}\left(\frac{x+1}{e}\right)^{x+\frac{1}{2}} \exp\left(\sum_{k=1}^{\infty}\left(B_{k+1}-(-1)^k\frac{k-1}{2}\right)\frac{1}{k(k+1)x^k}\right).$$

As usually truncations of these series provide upper- and lower- estimates. The following double inequalities were presented in [17]:

Theorem 2. F*or every* $x \geq 1$, *we have*

$$\sqrt{2\pi}\left(\frac{x+\frac{1}{2}}{e}\right)^{x+\frac{1}{2}} \exp a(x) < \Gamma(x+1) < \sqrt{2\pi}\left(\frac{x+\frac{1}{2}}{e}\right)^{x+\frac{1}{2}} \exp b(x),$$

where

$$a(x) = -\frac{1}{24x}+\frac{1}{48x^2}-\frac{23}{2880x^3}+\frac{1}{640x^4}+\frac{11}{40\,320x^5}+\frac{1}{5376x^6}-\frac{143}{215\,040x^7}$$

and

$$b(x) = a(x) + \frac{143}{215\,040x^7}.$$

Theorem 3. *For every* $x \geq 1$, *we have*

$$\sqrt{\frac{2\pi}{e}}\left(\frac{x+1}{e}\right)^{x+\frac{1}{2}} \exp c(x) < \Gamma(x+1) < \sqrt{\frac{2\pi}{e}}\left(\frac{x+1}{e}\right)^{x+\frac{1}{2}} \exp d(x),$$

where

$$c(x) = \frac{1}{12x}-\frac{1}{12x^2}+\frac{29}{360x^3}-\frac{3}{40x^4}+\frac{17}{252x^5}-\frac{5}{84x^6}$$

and

$$d(x) = c(x) + \frac{89}{1680x^7}.$$

Liu [15] established the following integral version of Stirling's formula

$$\Gamma(n+1) = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \cdot \exp\left(\int_n^{\infty}\frac{\frac{1}{2}-\{t\}}{t}dt\right).$$

An extension to Nemes' family was presented in [18]. The following formula is valid for every $p \in [0,1]$:

$$\Gamma(n+1) = \sqrt{2\pi e} \cdot e^{-p} \left(\frac{n+p}{e} \right)^{n+\frac{1}{2}}$$

$$\cdot \exp \left(\int_n^\infty \left(\frac{\frac{3}{2} - p - \{t\}}{t+p} + \frac{p}{p\{t\} + [t]} - \frac{1}{t} \right) dt \right).$$

According to our discussion in general case, the Stirling series in terms of Bernoulli numbers

$$\Gamma(n+1) \sim \sqrt{2\pi n} \left(\frac{n}{e} \right)^n \exp \left\{ \sum_{k=1}^\infty \frac{B_{2k}}{2k(2k-1)n^{2k-1}} \right\} \tag{7}$$

is of best performance from the approximation point of view when it is truncated at every term. However better results can be obtained if we consider the truncations in (7) as rational functions of the form

$$\sum_{k=1}^m \frac{B_{2k}}{2k(2k-1)n^{2k-1}} = \frac{R_m(n^2)}{12nT_m(n^2)},$$

where R_m, T_m are polynomials of $(m-1)\,th$ degree, with the leading coefficients equal to unity. It is indicated in [19] how can be constructed polynomials P_m, Q_m of $(m-1)\,th$ degree such that the approximation

$$\Gamma(n+1) \sim \sqrt{2\pi n} \left(\frac{n}{e} \right)^n \exp \frac{P_m(n^2)}{12nQ_m(n^2)} \tag{8}$$

is the best possible among all approximations of the form

$$\Gamma(n+1) \sim \sqrt{2\pi n} \left(\frac{n}{e} \right)^n \exp \frac{P'_m(n^2)}{12nQ'_m(n^2)},$$

where P'_m and Q'_m are every polynomials of $(m-1)\,th$ degree with leading coefficient equal to unity. New obtained approximations (8) are more accurate than the mth approximation of the classical Stirling series (7). Initial approximations

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e} \right)^n \exp \frac{n^2 + \frac{53}{210}}{12n\left(n^2 + \frac{2}{7}\right)} =: \rho_1 \tag{9}$$

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp \frac{n^4+\frac{2559}{1430}n^2+\frac{22999}{90090}}{12n\left(n^4+\frac{782}{429}n^2+\frac{263}{858}\right)} =: \rho_2 \tag{10}$$

are more accurate than the classical approximations arising from Stirling series truncated at the second, respective at the third term, namely

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp \frac{30n^2-1}{360n^3} =: \sigma_1, \tag{11}$$

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp \frac{210n^4-7n^2+2}{2520n^5} =: \sigma_2. \tag{12}$$

In order to offer an initial image, we consider a comparison table to prove the superiority of (9)-(10) over (11)-(12).

n	$\ln(n!/\sigma_1)$	$\ln(\rho_1/n!)$
10	7.8×10^{-9}	3.6×10^{-11}
100	7.9×10^{-14}	3.6×10^{-18}
250	8.1×10^{-16}	6.0×10^{-21}

n	$\ln(\sigma_2/n!)$	$\ln(n!/\rho_2)$
10	5.8×10^{-11}	5.2×10^{-15}
100	5.9×10^{-18}	5.7×10^{-26}
250	9.7×10^{-21}	3.1×10^{-27}

Rigorous proofs of these facts are presented in [19]. Remark that the first approximations (8) are the approximations obtained by truncation the classical Stieltjes continued fraction to gamma function, but the proof of this result is left as an open problem in [19].

In order to show our method, let us search the best constants a_1, a_2 in $m=2$ case:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp \frac{n^2+a_1}{12\left(n^3+a_2 n\right)}.$$

For the relative error sequence z_n defined by

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp \frac{n^2+a_1}{12\left(n^3+a_2 n\right)} \exp z_n\ , \quad (n\geq 1)\,,$$

we used Maple software for symbolic computation to deduce

$$\begin{aligned} z_n - z_{n+1} &= \left(-\frac{1}{4}a_1+\frac{1}{4}a_2-\frac{1}{120}\right)\frac{1}{n^4} - 2\left(-\frac{1}{4}a_1+\frac{1}{4}a_2-\frac{1}{120}\right)\frac{1}{n^5} \\ &+ \left(\frac{5}{6}a_2-\frac{5}{6}a_1+\frac{5}{12}a_1a_2-\frac{5}{12}a_2^2-\frac{1}{42}\right)\frac{1}{n^6} \\ &+ \left(\frac{5}{4}a_1-\frac{5}{4}a_2-\frac{5}{4}a_1a_2+\frac{5}{4}a_2^2+\frac{5}{168}\right)\frac{1}{n^7} + O\left(\frac{1}{n^8}\right). \end{aligned}$$

Now the fastest sequence z_n is obtained when the first two coefficients in this power series vanish, that is $a_1=\frac{53}{210}$, $a_2=\frac{2}{7}$.

In case $m = 3$, we define the sequence t_n by

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\frac{n^4 + b_1 n^2 + b_2}{12\left(n^5 + b_3 n^3 + b_4 n\right)} \exp t_n \ , \quad (n \geq 1)\,.$$

As

$$t_n - t_{n+1} = \left(-\frac{1}{4}b_1 + \frac{1}{4}b_3 - \frac{1}{120}\right)\frac{1}{n^4} - 2\left(-\frac{1}{4}b_1 + \frac{1}{4}b_3 - \frac{1}{120}\right)\frac{1}{n^5}$$
$$+\left(\frac{5}{6}b_3 - \frac{5}{12}b_2 - \frac{5}{6}b_1 + \frac{5}{12}b_4 + \frac{5}{12}b_1 b_3 - \frac{5}{12}b_3^2 - \frac{1}{42}\right)\frac{1}{n^6} + O\left(\frac{1}{n^7}\right),$$

we get $b_1 = \frac{2559}{1430}$, $b_2 = \frac{22\,999}{90\,090}$, $b_3 = \frac{782}{429}$, $b_4 = \frac{263}{858}$. In this case,

$$t_n - t_{n+1} = -\frac{80\,713}{12\,972\,960 n^{12}} + O\left(\frac{1}{n^{13}}\right).$$

The following estimates were stated in [19]:

Theorem 4. *For every positive integer n, we have*

$$\exp\left(\frac{P_2\left(n^2\right)}{12nQ_2\left(n^2\right)} - \frac{13}{35280n^7}\right) < \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} < \exp\frac{P_2\left(n^2\right)}{12nQ_2\left(n^2\right)}.$$

We illustrate our method by providing

The proof of Theorem 4. We have to prove that $a_n > 0$ and $b_n < 0$, where

$$a_n = \frac{P_2\left(n^2\right)}{12nQ_2\left(n^2\right)} - \ln\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n},$$

$$b_n = \frac{P_2\left(n^2\right)}{12nQ_2\left(n^2\right)} - \frac{13}{35280n^7} - \ln\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}.$$

As a_n, b_n converge to zero, it suffices to show that a_n is strictly decreasing, while b_n is strictly increasing. In this sense, $a_{n+1} - a_n = f(n)$, $b_{n+1} - b_n = g(n)$, where

$$f(x) = \left(x + \frac{1}{2}\right)\ln\left(1 + \frac{1}{x}\right) - 1 + \frac{P_2\left((x+1)^2\right)}{12(x+1)\,Q_2\left((x+1)^2\right)} - \frac{P_2\left(x^2\right)}{12xQ_2\left(x^2\right)}$$

and

$$g(x) = f(x) - \left(\frac{13}{35280\,(x+1)^7} - \frac{13}{35280x^7}\right).$$

The function f is strictly concave, while g is strictly convex with $f(\infty) = g(\infty) = 0$, so $f(x) < 0$ and $g(x) > 0$, for every $x \in [1, \infty)$ and the theorem is proved.□

In the same manner, the following result is stated in [19]:

Theorem 5. *For every positive integer n, we have*

$$\exp\left(\frac{P_3(n^2)}{12nQ_3(n^2)} - \frac{80713}{142702560n^{11}}\right) < \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} < \exp\frac{P_3(n^2)}{12nQ_3(n^2)}.$$

2 Landau constants

E. Landau studied the asymptotic behaviour of the constants

$$G_n = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 + ... + \left(\frac{1\cdot 3\cdot ...\cdot (2n-1)}{2\cdot 4\cdot ...\cdot (2n)}\right)^2,$$

(now known as Landau constants) proving the asymptotic formula $G_n \sim (1/\pi)\ln n$, see e.g. [14]. Then Watson [24] proposed

$$G_n = c_0 + \frac{1}{\pi}\ln(n+1) - \frac{1}{4\pi(n+1)} + O\left(\frac{1}{n^2}\right),$$

where $c_0 = \frac{1}{\pi}(\gamma + 4\ln 2) = 1.06627...$ and $\gamma = 0.577...$ is Euler-Mascheroni constant. Further improvements were presented by Brutman [5]

$$1 + \frac{1}{\pi}\ln(n+1) < G_n < 1.0663 + \frac{1}{\pi}\ln(n+1)$$

and Falaleev [9]

$$1.0662 + \frac{1}{\pi}\ln\left(n + \frac{3}{4}\right) < G_n < 1.0916 + \frac{1}{\pi}\ln\left(n + \frac{3}{4}\right). \tag{13}$$

It is showed in [20] that 3/4 is the best possible constant that can be used in (13). The proofs are based on inequalities

$$s(x) < \ln\Gamma(x+1) < t(x) \tag{14}$$

where

$$s(x) = \ln\sqrt{2\pi} + \left(x + \frac{1}{2}\right)\ln x - x + \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7}$$

and

$$t(x) = \ln\sqrt{2\pi} + \left(x+\frac{1}{2}\right)\ln x - x + \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \frac{1}{1188x^9}.$$

They are a consequence of a result of Alzer [2, Theorem 8].

By (14), we get

$$e^{u(x)} < \frac{1}{16^x}\left(\frac{\Gamma(2x+1)}{(\Gamma(x+1))^2}\right)^2 < e^{v(x)}, \tag{15}$$

where

$$u(x) = 2s(2x) - 4t(x) - x\ln 16\ , \quad v(x) = 2t(2x) - 4s(x) - x\ln 16.$$

Mortici [20] used (15) to establish the following

Theorem 6. *For every integer $n \geq 1$, we have*

$$c_0 + \frac{1}{\pi}\ln\left(n+\frac{3}{4}\right) < G_n < c_0 + \frac{1}{\pi}\ln\left(n+\frac{3}{4}+\frac{11}{192n}\right). \tag{16}$$

Proof. As $n = 1, 2$ cases can be easily proven, we assume $n \geq 3$. The sequence

$$a_n = G_n - c_0 - \frac{1}{\pi}\ln\left(n+\frac{3}{4}\right)$$

converges to zero and it suffices to show that $(a_n)_{n\geq 3}$ is strictly decreasing. As

$$a_n - a_{n-1} = \frac{1}{16^n}\left(\frac{\Gamma(2n+1)}{(\Gamma(n+1))^2}\right)^2 - \frac{1}{\pi}\ln\left(1+\frac{1}{n-\frac{1}{4}}\right) < e^{v(n)} - \frac{1}{\pi}\sum_{k=1}^{4}\frac{(-1)^{k-1}}{k\left(n-\frac{1}{4}\right)^k},$$

we have to prove that $f(x) < 0$, where

$$f(x) = v(x) - \ln\left(\frac{1}{\pi}\sum_{k=1}^{4}\frac{(-1)^{k-1}}{k\left(x-\frac{1}{4}\right)^k}\right).$$

This function has its derivative $f' > 0$ on $[3,\infty)$. Now f is strictly increasing on $[3,\infty)$, with $f(\infty) = 0$, so $f(x) < 0$, for every $x \in [3,\infty)$.

For the right-hand side inequality (16), define the sequence

$$b_n = G_n - c_0 - \frac{1}{\pi}\ln\left(n+\frac{3}{4}+\frac{11}{192n}\right)$$

and proceed as above. We have

$$b_n - b_{n-1} = \frac{1}{16^n}\left(\frac{\Gamma(2n+1)}{(\Gamma(n+1))^2}\right)^2 - \frac{1}{\pi}\ln\left(1+\frac{1+\frac{11}{192n}-\frac{11}{192(n-1)}}{n-\frac{1}{4}+\frac{11}{192(n-1)}}\right)$$

$$> e^{u(n)} - \frac{1}{\pi}\sum_{k=1}^{5}\frac{(-1)^{k-1}}{k\left(\frac{n-\frac{1}{4}+\frac{11}{192(n-1)}}{1+\frac{11}{192n}-\frac{11}{192(n-1)}}\right)^k}.$$

The function

$$g(x) = u(x) - \ln\left(\frac{1}{\pi}\sum_{k=1}^{5}\frac{(-1)^{k-1}}{k\left(\frac{x-\frac{1}{4}+\frac{11}{192(x-1)}}{1+\frac{11}{192x}-\frac{11}{192(x-1)}}\right)^k}\right),$$

is strictly decreasing on $[3,\infty)$, with $g(\infty)=0$, so $g(x)>0$, for every $x\in[3,\infty)$.□

Zhao [25] extended the asymptotic expansion of G_n to

$$G_n = c_0 + \frac{1}{\pi}\ln(n+1) - \frac{1}{4\pi(n+1)} + \frac{5}{192\pi(n+1)^2} + O\left(\frac{1}{(n+1)^3}\right),$$

then Mortici [20] proved the following improvement

Theorem 7. *For every integer $n \geq 1$, we have*

$$c_0 + \frac{1}{\pi}\ln(n+1) - \frac{1}{4\pi(n+1)} + \frac{5}{192\pi(n+1)^2} \tag{17}$$

$$+\frac{3}{128\pi(n+1)^3} - \frac{341}{122880\pi(n+1)^4} - \frac{75}{8192\pi(n+1)^5} < G_n$$

$$< c_0 + \frac{1}{\pi}\ln(n+1) - \frac{1}{4\pi(n+1)} + \frac{5}{192\pi(n+1)^2} + \frac{3}{128\pi(n+1)^3} -$$

$$-\frac{341}{122880\pi(n+1)^4}$$

and the following asymptotic formula holds as $n\to\infty$:

$$G_n = c_0 + \frac{1}{\pi}\ln(n+1) - \frac{1}{4\pi(n+1)} + \frac{5}{192\pi(n+1)^2}$$

$$+\frac{3}{128\pi(n+1)^3}-\frac{341}{122880\pi(n+1)^4}+O\left(\frac{1}{(n+1)^5}\right).$$

Cvijović and Klinowski [8] presented some estimates in terms of the digamma function

$$c_0+\frac{1}{\pi}\psi\left(n+\frac{5}{4}\right)<G_n<1.0725+\frac{1}{\pi}\psi\left(n+\frac{5}{4}\right)$$

and

$$0.9883+\frac{1}{\pi}\psi\left(n+\frac{3}{2}\right)<G_n<c_0+\frac{1}{\pi}\psi\left(n+\frac{3}{2}\right)\quad(n\geq 0),$$

as Alzer [1] proved the following double sharp inequality

$$c_0+\frac{1}{\pi}\psi(n+\alpha)<G_n<c_0+\frac{1}{\pi}\psi(n+\beta),\quad(n\geq 1),$$

where $\alpha=5/4$ and $\beta=\psi^{-1}(\pi(1-c_0))=1.26621...$.

Mortici [20] improved the above results of Cvijović, Klinowski and Alzer as follows:

Theorem 8. *For every positive integer n, we have*

$$c_0+\frac{1}{\pi}\psi\left(n+\frac{5}{4}\right)+\frac{1}{64\pi n^2}-\frac{3}{128\pi n^3}<G_n$$

$$<c_0+\frac{1}{\pi}\psi\left(n+\frac{5}{4}\right)+\frac{1}{64\pi n^2}-\frac{3}{128\pi n^3}+\frac{173}{8192\pi n^4}. \tag{18}$$

Cases $n=1,2$ are true, so we assume $n\geq 3$. The sequence

$$t_n=G_n-c_0-\frac{1}{\pi}\psi\left(n+\frac{5}{4}\right)-\frac{1}{64\pi n^2}+\frac{3}{128\pi n^3}$$

is strictly decreasing. As

$$t_n-t_{n-1}=\frac{1}{16^n}\frac{(\Gamma(2n+1))^2}{(\Gamma(n+1))^4}-\frac{1}{\pi\left(n+\frac{1}{4}\right)}$$

$$-\frac{1}{64\pi n^2}+\frac{3}{128\pi n^3}+\frac{1}{64\pi(n-1)^2}-\frac{3}{128\pi(n-1)^3},$$

we have to prove that $m<0$, where

$$m(x)=v(x)-$$

$$-\ln\left(\frac{1}{\pi\left(x+\frac{1}{4}\right)}+\frac{1}{64\pi x^2}-\frac{3}{128\pi x^3}-\frac{1}{64\pi\left(x-1\right)^2}+\frac{3}{128\pi\left(x-1\right)^3}\right).$$

But m is strictly increasing with $m\left(\infty\right)=0$, so $m<0$ on $[3,\infty)$.

For the right-hand side inequality (18), the sequence

$$z_n=G_n-c_0-\frac{1}{\pi}\psi\left(n+\frac{5}{4}\right)-\frac{1}{64\pi n^2}+\frac{3}{128\pi n^3}-\frac{173}{8192\pi n^4}$$

is strictly increasing and the argument is similar.□

Recent studies on Landau and Lebesgue constants were performed by Chen and Choi [7], Granath [10], or Nemes [23].

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