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TOEPLITZ OPERATORS WITH BOUNDED HARMONIC SYMBOLS*

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Abstract

In this paper we show that if A and B are two bounded linear operators on the Bergman space $L^2_a(\mathbb{D})$ and $AT_\phi B = T_\phi$ for all $\phi \in h^\infty(\mathbb{D})$ then $A = \alpha I$ and $B = \beta I$ for some $\alpha, \beta \in \mathbb{C}$ and $\alpha\beta = 1$. Here $h^\infty(\mathbb{D})$ is the space of all bounded harmonic functions on the open unit disk $\mathbb{D}$.

MSC: 47B35

Keywords: Toeplitz operators, Bergman space, bounded harmonic function, Bergman shift, Hardy space

1 Introduction

Let $n \in \mathbb{N}$ and $L^{2,n}_a(\mathbb{D})$ be the Hilbert space of all analytic functions f on $\mathbb{D}$ with finite norm

$$\|f\|^2_{L^{2,n}_a(\mathbb{D})} = \lim_{r\to 1} \int_{\overline{\mathbb{D}}} |f(rz)|^2 d\mu_n(z).$$

The measure $d\mu_1$ is the normalized Lebesgue arc length measure on the unit circle $\mathbb{T}$ and for $n \geq 2$ the measure $d\mu_n$ is the weighted Lebesgue area measure given by $d\mu_n(z) = (n-1)(1-|z|^2)^{n-2}dA(z), z \in \mathbb{D}$, where $dA(z) = \frac{dxdy}{\pi}, z = x + iy$, is the planar Lebesgue area measure normalized so that the unit disk $\mathbb{D}$ has area 1. The space $L^{2,1}_a(\mathbb{D}) = H^2(\mathbb{D})$, the standard Hardy space, the space $L^{2,2}_a(\mathbb{D}) = L^2_a(\mathbb{D})$, is the unweighted Bergman space and

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in general the space $L_a^{2,n}(\mathbb{D})$ is the standard weighted Bergman space. The norm of $L_a^{2,n}(\mathbb{D})$ is given by

$$\|f\|^2_{L_a^{2,n}(\mathbb{D})} = \sum_{k\geq 0} |a_k|^2 \mu_{n,k},$$

where $\mu_{n,k} = \frac{1}{\binom{k+n-1}{k}}$ for $k \geq 0$, using the power series expansion $f(z) = \sum_{k\geq 0} a_k z^k, z \in \mathbb{D}$, of the function $f \in L_a^{2,n}(\mathbb{D})$.

For any $n \geq 0, n \in \mathbb{Z}$, let $e_n(z) = \sqrt{n+1}z^n$. The sequence $\{e_n\}_{n=0}^{\infty}$ forms (7) an orthonormal basis for $L_a^2(\mathbb{D})$. Let

$$K(z,w) = \overline{K_z(w)} = \frac{1}{(1-z\overline{w})^2} = \sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)}.$$

The function $K(z,w)$ is called the Bergman kernel of $\mathbb{D}$ or the reproducing kernel of $L_a^2(\mathbb{D})$ because the formula:

$$f(z) = \int_{\mathbb{D}} f(w)K(z,w)dA(w)$$

reproduces each f in $L_a^2(\mathbb{D})$. Let $k_a(z) = \frac{K(z,a)}{\sqrt{K(a,a)}} = \frac{1-|a|^2}{(1-\bar{a}z)^2}$. These functions k_a are called the normalized reproducing kernels of $L_a^2(\mathbb{D})$; it is clear that they are unit vectors in $L_a^2(\mathbb{D})$.

Let $Aut(\mathbb{D})$ be the Lie group of all automorphisms (biholomorphic mappings) of $\mathbb{D}$. We can define for each $a \in \mathbb{D}$, an automorphism ϕ_a in $Aut(\mathbb{D})$ such that
(i) $(\phi_a \ o \ \phi_a)(z) \equiv z$;
(ii) $\phi_a(0) = a, \phi_a(a) = 0$;
(iii) ϕ_a has a unique fixed point in $\mathbb{D}$.
In fact, $\phi_a(z) = \frac{a-z}{1-\bar{a}z}$ for all a and z in $\mathbb{D}$. An easy calculation shows that the derivative of ϕ_a at z is equal to $-k_a(z)$. It follows that the real Jacobian determinant of ϕ_a at z is $J_{\phi_a}(z) = |k_a(z)|^2 = \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}$.

Let $\mathcal{L}(H)$ be the space of bounded linear operators from the Hilbert space H into itself. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $L^p(\mathbb{T}), 1 \leq p < \infty$ be the Lebesgue space of $\mathbb{T}$ induced by $\frac{d\theta}{2\pi}$ where $d\theta$ is the arc-length measure on $\mathbb{T}$. Since $d\theta$ is finite, $L^p(\mathbb{T}) \subset L^1(\mathbb{T})$ for all $p \geq 1$. Given $f \in L^1(\mathbb{T})$, the Fourier coefficients of f are:

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta, n \in \mathbb{Z}$$

where $\mathbb{Z}$ is the set of all integers. The Hardy space of $\mathbb{T}$, denoted by $H^2(\mathbb{T})$, is the subspace of $L^2(\mathbb{T})$ consisting of functions f with $\widehat{f}(n) = 0$ for all negative integers n. It is not very important (7) to distinguish $H^2(\mathbb{D})$ from $H^2(\mathbb{T})$. Let $L^\infty(\mathbb{T})$ be the space of all complex-valued, essentially bounded Lebesgue measurable functions on $\mathbb{T}$ with $\|f\|_\infty = \text{ess sup}_{z\in\mathbb{T}}|f(z)| < \infty$. Let

$$H^\infty(\mathbb{T}) = \{\phi \in L^\infty(\mathbb{T}) : \widehat{\phi}(n) = 0 \text{ for } n < 0, n \in \mathbb{Z}\}.$$

Since $H^2(\mathbb{T})$ is a closed subspace of the Hilbert space $L^2(\mathbb{T})$, there exists an orthogonal projection P_+ from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. For $\phi \in L^\infty(\mathbb{T})$, we define the Toeplitz operator L_ϕ on the Hardy space $H^2(\mathbb{T})$ as $L_\phi f = P_+(\phi f), f \in H^2(\mathbb{T})$. Let $L^\infty(\mathbb{D})$ be the space of all essentially bounded, Lebesgue measurable functions on $\mathbb{D}$ with the essential supremum norm and $h^\infty(\mathbb{D})$ be the space of all bounded harmonic functions on $\mathbb{D}$. For $\phi \in L^\infty(\mathbb{D})$, we define the Toeplitz operator on the Bergman space $L^2_a(\mathbb{D})$ as $T_\phi f = P(\phi f)$ where P denotes the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $L^2_a(\mathbb{D})$.

Let S denote the unilateral shift on $H^2(\mathbb{T})$. It is not hard to see that $S^* L_\phi S = L_\phi$ for all $\phi \in L^\infty(\mathbb{T})$. Brown and Halmos (1) showed that the converse also holds: if an operator $T \in \mathcal{L}(H^2(\mathbb{T}))$ satisfies $S^*TS = T$, then $T = L_\phi$ for some $\phi \in L^\infty(\mathbb{T})$. In (3), Englis showed that no such characterization is possible for Toeplitz operators on the Bergman space $L^2_a(\mathbb{D})$. In fact, he proved that if $A, B \in \mathcal{L}(L^2_a(\mathbb{D}))$ and $AT_\phi B = T_\phi$ for all $\phi \in L^\infty(\mathbb{D})$ then A and B are scalar multiples of the identity. Frankfurt (4) and Cao (2) proved that no bounded operator T on $L^2_a(\mathbb{D})$ satisfies the operator equation $T_z^* T T_z = T$, where T_z is the Bergman shift on $L^2_a(\mathbb{D})$. A function $q \in H^\infty(\mathbb{T})$ is said to be an inner function if $|q| = 1$ almost everywhere. Guo and Wang (5) established that if $T \in \mathcal{L}(H^2(\mathbb{T}))$ then T is a Toeplitz operator if and only if $L_q^* T L_q = T$ for each inner function $q \in H^\infty(\mathbb{T})$. Louhichi and Olofsson (6) also obtained a characterization of Toeplitz operators with bounded harmonic symbols.

2 Toeplitz operators on the Bergman space

In this section we shall show that if $A, B \in \mathcal{L}(L^2_a(\mathbb{D}))$ and $AT_\phi B = T_\phi$ for all $\phi \in h^\infty(\mathbb{D})$ then $A = \alpha I$ and $B = \beta I$ for some $\alpha, \beta \in \mathbb{C}$ and $\alpha\beta = 1$.

The set of vectors $\{z^n\}_{n=0}^\infty$ is the standard orthonormal basis for $H^2(\mathbb{D})$. Define the operator W as $Wz^n = \sqrt{n+1}z^n, n = 0, 1, 2, \ldots$. The operator W is an unitary operator from $H^2(\mathbb{D})$ onto $L^2_a(\mathbb{D})$ and it maps the standard orthonormal basis $\{z^n\}_{n=0}^\infty$ of $H^2(\mathbb{D})$ onto the basis $\{\sqrt{n+1}z^n\}_{n=0}^\infty$ of $L^2_a(\mathbb{D})$ and $W(\sum_{n=0}^\infty a_n z^n) = \sum_{n=0}^\infty a_n \sqrt{n+1} z^n$.

Let $T \in \mathcal{L}(L_a^2(\mathbb{D}))$. Now $[T, T_z] = TT_z - T_zT$ is compact if and only if

$$W^*TT_zW - W^*T_zTW$$

is compact. This is true if and only if

$$(W^*TW)(W^*T_zW) - (W^*T_zW)(W^*TW)$$

is compact. That is, if and only if

$$(W^*TW)\mathcal{S} - \mathcal{S}(W^*TW) = [W^*TW, \mathcal{S}]$$

is compact in $\mathcal{L}(H^2(\mathbb{D}))$ where $\mathcal{S}$ is the operator of multiplication by z on $H^2(\mathbb{D})$. This is so, since

$$(W^*T_zW) - \mathcal{S} = \mathcal{S} \cdot \operatorname{diag}\left(\sqrt{\frac{n+1}{n+2}} - 1\right)$$

is a compact operator.

Theorem 2.1. Suppose $A, B \in \mathcal{L}(L_a^2(\mathbb{D}))$ and $AT_\phi B = T_\phi$ for all $\phi \in h^\infty(\mathbb{D})$. Then $A = \alpha I$ and $B = \beta I$ for some $\alpha, \beta \in \mathbb{C}$ and $\alpha\beta = 1$.

Proof. Suppose $A, B \in \mathcal{L}(L_a^2(\mathbb{D}))$ and $AT_\phi B = T_\phi$ for all $\phi \in h^\infty(\mathbb{D})$. Then

$$(W^*AW)(W^*T_\phi W)(W^*BW) = W^*T_\phi W$$

for all $\phi \in h^\infty(\mathbb{D})$. Therefore

$$\begin{aligned}
(W^*AW)(W^*T_\phi W)(W^*BW)(W^*T_zW) &= (W^*T_\phi W)(W^*T_zW)\\
&= W^*T_\phi T_zW\\
&= W^*T_{\phi z}W\\
&= W^*AT_{\phi z}BW\\
&= W^*AT_\phi T_zBW\\
&= (W^*AW)(W^*T_\phi W)(W^*T_zW)(W^*BW).
\end{aligned}$$

Thus

$$(W^*AW)(W^*T_\phi W)\left[(W^*BW)(W^*T_zW) - (W^*T_zW)(W^*BW)\right] = 0.$$

Let $0 \neq f \in \operatorname{Ran}[(W^*BW)(W^*T_zW) - (W^*T_zW)(W^*BW)] \subset H^2(\mathbb{D})$. Then

$$(W^*AW)(W^*T_\phi W)f = 0$$

for all $\phi \in h^{\infty}(\mathbb{D})$. Hence the kernel of W^*AW contains the set

$$\mathcal{M} = \{(W^*T_\phi W)f : \phi \in h^{\infty}(\mathbb{D})\}.$$

Consider some $g \in H^2(\mathbb{D})$ orthogonal to $\mathcal{M}$. Then

$$\begin{aligned} 0 &= \langle g, (W^*T_\phi W)f\rangle \\ &= \langle Wg, P(\phi Wf)\rangle \\ &= \langle Wg, \phi Wf\rangle \\ &= \int_{\mathbb{D}} (Wg)(z)\overline{\phi(z)Wf(z)}dA(z) \end{aligned}$$

for all $\phi \in h^{\infty}(\mathbb{D})$.

Since $\overline{Wf}Wg \in L^1(\mathbb{D}, dA)$, we obtain $\overline{Wf}Wg = 0$ and this is possible if at least one of the analytic function Wf, Wg is identically zero. But $f \neq 0$, by assumption. Hence $Wf \neq 0$. Thus $Wg \equiv 0$ and therefore $g \equiv 0$.

It thus follows that $\overline{\mathcal{M}} = H^2(\mathbb{D})$. Since $\mathcal{M} \subset \ker(W^*AW)$, we obtain $W^*AW = 0$ and hence $A \equiv 0$. This implies $T_\phi = AT_\phi B = 0$ for all $\phi \in h^{\infty}(\mathbb{D})$. This is a contradiction. Hence $(W^*BW)(W^*T_zW) - (W^*T_zW)(W^* BW) = 0$ and therefore $BT_z - T_zB = 0$. Let $B1 = h \in L^2_a(\mathbb{D})$. Then $Bz^n = BT_z^n 1 = T_z^n B1 = z^n h$ for all $n \geq 0$ and, consequently, $Bp = hp$ for all polynomials $p(z)$.

For $f_1 \in L^2_a(\mathbb{D})$, take a sequence $\{p_n\}$ of polynomials converging to f_1 in the $L^2_a(\mathbb{D})$ norm. Then $Bp_n \to Bf_1$ in norm. Because point evaluations are continuous functionals, we have $p_n(z) \to f_1(z)$ and $(Bp_n)(z) \to (Bf_1)(z)$ for any $z \in \mathbb{D}$. On the other hand,

$$(Bp_n)(z) = (p_n h)(z) = p_n(z)h(z) \to f_1(z)h(z) \text{ for all } z \in \mathbb{D}.$$

Consequently, $Bf_1 = hf_1$ for all $f_1 \in L^2_a(\mathbb{D})$, i.e., B is the operator of multiplication by $h \in L^2_a(\mathbb{D})$. That is, $B = T_h$. Now $AT_\phi B = T_\phi$ for all $\phi \in h^{\infty}(\mathbb{D})$ implies $B^*T_\phi A^* = T_\phi$ for all $\phi \in h^{\infty}(\mathbb{D})$, thus, we can deduce in the same way that A^* is the operator of multiplication by some $k \in L^2_a(\mathbb{D})$. Hence $A^* = T_k$ and $A = T_{\overline{k}}$. Since $AB = I$ we obtain $T_{\overline{k}}T_h = T_{\overline{k}h} = I$ as $h \in L^2_a(\mathbb{D})$ and $\overline{k} \in \overline{L^2_a(\mathbb{D})}$. For $m, n \in \mathbb{N}\cup\{0\}$, $\langle \overline{k}hz^m, z^n\rangle = \langle z^m, z^n\rangle$. That is, $\int_{\mathbb{D}} z^m\bar{z}^n\overline{k(z)}h(z)dA(z) = \int_{\mathbb{D}} z^m\bar{z}^n dA(z)$. This implies that the finite measure $(\overline{k(z)}h(z)-1)dA(z)$ on $\mathbb{D}$ is annihilated by all monomials $z^m\bar{z}^n, m, n \geq 0$. By linearity and the Stone-Weierstrass theorem, it is annihilated by all functions continuous on $\overline{\mathbb{D}}$, and so is the zero measure and $\overline{k}h = 1$ on $\mathbb{D}$. But this means that $\overline{k} = \frac{1}{h}$ is both analytic and co-analytic and so must be constant. $\square$

Given $z \in \mathbb{D}$ and f any measurable function on $\mathbb{D}$, we define a function $U_z f(w) = k_z(w)f(\phi_z(w))$. Since $|k_z|^2$ is the real Jacobian determinant of the mapping ϕ_z (see (7)), U_z is easily seen to be a unitary operator on $L^2(\mathbb{D}, dA)$ and $L_a^2(\mathbb{D})$. It is also easy to check that $U_z^* = U_z$, thus U_z is a self-adjoint unitary operator. If $\phi \in L^\infty(\mathbb{D}, dA)$ and $z \in \mathbb{D}$ then $U_z T_\phi = T_{\phi\circ\phi_z} U_z$. This is because $PU_z = U_z P$ and for $f \in L_a^2$, $T_{\phi\circ\phi_z} U_z f = T_{\phi\circ\phi_z}((f \circ \phi_z)k_z) = P((\phi \circ \phi_z)(f \circ \phi_z)k_z) = P(U_z(\phi f)) = U_z P(\phi f) = U_z T_\phi f$.

Corollary 2.1. *Let $a \in \mathbb{D}$ and $A_a, B_a \in \mathcal{L}(L_a^2(\mathbb{D}))$. If $A_a T_\phi B_a = T_{\phi\circ\phi_a}$ for all $\phi \in h^\infty(\mathbb{D})$ then $A_a = \alpha U_a, B_a = \beta U_a$ and $\alpha\beta = 1$.*

Proof. Notice that $A_a T_\phi B_a = T_{\phi\circ\phi_a}$ for all $\phi \in h^\infty(\mathbb{D})$ implies

$$U_a A_a T_\phi B_a U_a = U_a T_{\phi\circ\phi_a} U_a = T_\phi$$

for all $\phi \in h^\infty(\mathbb{D})$. From Theorem 2.1, it follows that $U_a A_a = \alpha I$ and $B_a U_a = \beta I$ for some $\alpha, \beta \in \mathbb{C}$ and $\alpha\beta = 1$. Hence $A_a = \alpha U_a$, $B_a = \beta U_a$ and $\alpha\beta = 1$. □

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