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ON SOME UNSTEADY MOTIONS OF SECOND GRADE FLUIDS IN A RECTANGULAR EDGE*

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Abstract

A mixed boundary value problem is studied for the unsteady motion of a second grade fluid in a rectangular edge. A part of the boundary applies a shear stress ft^a to the fluid and the other one is moving in its plane with the velocity gt^b. Dimensionless velocity and shear stresses are obtained using integral transforms. They satisfy all imposed initial and boundary conditions and can easily be reduced to constantly accelerating boundary conditions. Finally, some characteristics of the fluid motion are graphically underlined.

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1 Introduction

The behavior of many materials such as clay coating, drilling muds, suspensions, certain oils and greases, polymer melts, elastomers and different

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emulsions cannot be described by Navier-Stokes equations. For this reason, many non-Newtonian models have been proposed. One of the most popular among them is the model of second grade fluids. This is particularly so due to the fact that the calculations will generally be simpler. Usually, the equation of motion for incompressible second grade fluids is of higher order than the corresponding Navier-Stokes equation. A marked difference between the Navier-Stokes theory and that of second grade fluids is that, ignoring the non-linearity in the Navier-Stokes equation does not lower the order of the equation. However, ignoring the higher order non-linearities in the case of second grade fluids reduce the order of the equation. The no-slip boundary condition is sufficient for a Newtonian fluid but for a second grade fluid, it may not be sufficient. A critical review on the boundary conditions, the existence and uniqueness of solution has been given by Rajagopal [1] and a listing of some problems that have been solved for such fluids may be found in [2] and [3]. The first exact solutions for unsteady unidirectional flows of second grade fluids seem to be those obtained by Ting [4].

The Rayleigh-Stokes problem for an edge, as well as the first problem of Stokes for the flat plate, has received much attention due to its practical importance and fundamental value for theory. One of the most interesting solutions for this problem was given by Zierep [5] for Newtonian fluids. Its extension to the motion induced by a constantly accelerating edge has been realized in [6] and [7] for Newtonian and Maxwell, second grade and Oldroyd-B fluids. However, there is no result in the literature in which the shear stress is given on the edge or on one of its sides. The first exact solutions for motions of second grade fluids in which the shear stress is given on a part of the boundary seem to be those of Bandelli and Rajagopal [8]. These solutions have been recently extended to second grade fluids with fractional derivatives in [9-11].

The purpose of this paper is to study a similar problem whose solution leads to a mixed boundary value problem. More exactly, we intend to study the problem in which a side of the edge applies a shear ft to the fluid while the other part is moving in its plane with a velocity gt. For completness, the more general boundary conditions ft^a and gt^b are considered and the solutions are obtained using integral transforms. These solutions, presented in integral form, satisfy all imposed initial and boundary conditions and can easily be reduced to give the similar solutions corresponding to different values of a and b greater than zero. Finally, some characteristics of the fluid motion are brought to light by graphical illustrations.

2 Governing Equations

The Cauchy stress tensor $\mathbf{T}$ for an incompressible second grade fluid is related to the fluid motion in the following manner [4,8]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S},\ \mathbf{S} = \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1}$$

where $-p\mathbf{I}$ is the indeterminate part of the stress due to the constraint of incompressibility, $\mathbf{S}$ is the extra-stress tensor, μ the dynamic viscosity, α_1 and α_2 are normal stress moduli and $\mathbf{A_1}$ and $\mathbf{A_2}$ are the first two Rivlin-Ericksen tensors. The Clausius-Duhem inequality and the assumption that the specific Helmholtz free energy is minimum at equilibrium provide the following restrictions for material parameters [12]

$$\mu \geq 0,\ \alpha_1 \geq 0 \ \text{ and } \ \alpha_1 + \alpha_2 = 0.$$

The sign of the material moduli α_1 and α_2 has been the subject to much controversy. A comprehensive discussion on the restrictions for μ, α_1 and α_2 can be found in the work by Dunn and Rajagopal [13]. If the second inequality is reversed, so that $\alpha_1 < 0$, then the corresponding fluid model leads to an unacceptable instability. In the following we are looking for a velocity field of the form [6,7]

$$\mathbf{V} = \mathbf{V}(y, z, t) = u(y, z, t)\mathbf{i}, \tag{2}$$

where $\mathbf{i}$ is the unit vector along the x-direction of the Cartesian coordinate system x, y and z. For such flows the constraint of incompressibility is automatically satisfied. In the absence of a pressure gradient in the flow direction, the governing equation is [14]

$$\frac{\partial u(y,z,t)}{\partial t} = (\nu + \alpha\frac{\partial}{\partial t})\left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] u(y,z,t), \tag{3}$$

where $\nu = \mu/\rho$ is the kinematic viscosity, $\alpha = \alpha_1/\rho$ and ρ is the constant density of the fluid. The non-trivial shear stresses $\tau_1(y,z,t) = S_{xy}(y,z,t)$ and $\tau_2(y,z,t) = S_{xz}(y,z,t)$ are given by

$$\tau_1(y,z,t) = (\mu + \alpha_1\frac{\partial}{\partial t})\frac{\partial u(y,z,t)}{\partial y}, \qquad \tau_2(y,z,t) = (\mu + \alpha_1\frac{\partial}{\partial t})\frac{\partial u(y,z,t)}{\partial z}. \tag{4}$$

The governing equation (3) with appropriate initial and boundary conditions can be solved by different methods. We shall use the Laplace transform to eliminate the time variable and the Fourier sine transform for the spatial variable z.

3 Flow within an infinite edge

Suppose that an incompressible second grade fluid occupies the space of the first dial of a rectangular edge [5-7] $(-\infty < x < \infty\,, y \geq 0\,, z \geq 0)$. At time $t = 0^+$ a side of the boundary is pulled in its plane with a time-dependent shear stress ft^a and the other one is subject to a translation motion in its plane of velocity gt^b. Due to the shear the fluid is gradually moved. Its velocity is of the form (2), the governing equations are given by Eqs. (3) and (4) while the initial and boundary conditions are given by

$$u(y,z,0) = 0, \qquad y,z \geq 0, \tag{5}$$

$$\tau_1(0,z,t) = (\mu + \alpha_1 \frac{\partial}{\partial t})\frac{\partial u(y,z,t)}{\partial y}\Big|_{y=0} = ft^a, \qquad z,t \geq 0,\ a > 0, \tag{6}$$

$$u(y,0,t) = gt^b, \qquad y,t \geq 0,\ b > 0, \tag{7}$$

where a, b, f and g are constants. Furthermore, the natural condition

$$u(y,z,t)\ \to 0 \quad as \quad y,\ z \to \infty, \tag{8}$$

has to be also satisfied.

Introducing the dimensionless variables

$$t^* = \frac{t}{(\frac{\alpha}{\nu})}\,,\ y^* = \frac{y}{\frac{\mu g}{f}(\frac{\alpha}{\nu})^{b-a}}\,,\ z^* = \frac{z}{\frac{\mu g}{f}(\frac{\alpha}{\nu})^{b-a}}\,,\ u^* = \frac{u}{g(\frac{\alpha}{\nu})^b}\,,\ \tau_1^* = \frac{\tau_1}{f(\frac{\alpha}{\nu})^a},$$
$$\tau_2^* = \frac{\tau_2}{f(\frac{\alpha}{\nu})^a}, \tag{9}$$

the governing equation (3) takes the form (for simplicity the $*$ notation was neglected)

$$\frac{\partial u(y,z,t)}{\partial t} = \frac{1}{R_e}(1 + \frac{\partial}{\partial t})\left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]u(y,z,t), \tag{10}$$

where $R_e = \frac{1}{\alpha}\left[\frac{\mu g(\frac{\alpha}{\nu})^{b-a}}{f}\right]^2$ is the Reynolds number. The dimensionless non-trivial shear stresses $\tau_1(y,z,t)$ and $\tau_2(y,z,t)$ are given by

$$\tau_1(y,z,t) = (1 + \frac{\partial}{\partial t})\frac{\partial u(y,z,t)}{\partial y}, \qquad \tau_2(y,z,t) = (1 + \frac{\partial}{\partial t})\frac{\partial u(y,z,t)}{\partial z}, \tag{11}$$

while the initial and boundary conditions become

$$u(y,z,0) = 0,\ \ \tau_1(0,z,t) = (1 + \frac{\partial}{\partial t})\frac{\partial u(y,z,t)}{\partial y}\Big|_{y=0} = t^a,\ \ u(y,0,t) = t^b,$$
$$u(y,z,t) \to 0 \quad as \quad y,\ z \to \infty. \tag{12}$$

3.1 Calculation of the velocity field

Applying the Laplace transform to Eq. (10) and using the initial condition we find that [15]

$$q\bar{u}(y,z,q) = \frac{1}{R_e}(1+q)\left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\bar{u}(y,z,q). \tag{13}$$

The Laplace transform $\bar{u}(y,z,q)$ of $u(y,z,t)$ has to satisfy the conditions

$$\left.\frac{\partial \bar{u}(y,z,q)}{\partial y}\right|_{y=0} = \frac{\Gamma(a+1)}{q^{a+1}(1+q)}; \quad \bar{u}(y,0,q) = \frac{\Gamma(b+1)}{q^{b+1}}, \tag{14}$$

$$\bar{u}(y,z,q) \to 0 \quad as \quad y^2 + z^2 \to \infty, \tag{15}$$

where $\Gamma(\cdot)$ is the Gamma function.
Now multiplying Eq. (13) by $\sqrt{2/\pi}\sin(\eta z)$ and integrating the result with respect to z from 0 to infinity, we get

$$\frac{\partial^2 \bar{u}_s(y,\eta,q)}{\partial y^2} - \left[\frac{qR_e + (1+q)\eta^2}{(1+q)}\right]\bar{u}_s(y,\eta,q) = -\sqrt{\frac{2}{\pi}}\frac{\Gamma(b+1)}{q^{b+1}}\eta, \tag{16}$$

where the Fourier sine transform

$$\bar{u}_s(y,\eta,q) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{u}(y,z,q)\sin(\eta z)dz,$$

of $\bar{u}(y,z,q)$ has to satisfy the conditions

$$\bar{u_s}(y,\eta,q) \to 0 \quad as\ y \to \infty\ and\ \eta \to 0\ , \quad \left.\frac{\partial \bar{u}_s(y,\eta,q)}{\partial y}\right|_{y=0} = \sqrt{\frac{2}{\pi}}\frac{\Gamma(a+1)}{\eta q^{a+1}(1+q)}. \tag{17}$$

Solution of the ordinary differential equation (16) with the boundary conditions (17) is

$$\bar{u}_s(y,\eta,q) = \sqrt{\tfrac{2}{\pi}}\tfrac{\Gamma(b+1)}{q^{b+1}}\eta\tfrac{(1+q)}{qR_e+(1+q)\eta^2} - \sqrt{\tfrac{2}{\pi}}\tfrac{\Gamma(a+1)}{q^{a+1}(1+q)}\times \tfrac{1}{\eta\sqrt{W(\eta,q)}}e^{-y\sqrt{W(\eta,q)}}, \tag{18}$$

where

$$W(\eta,q) = \frac{qR_e + (1+q)\eta^2}{(1+q)}. \tag{19}$$

Applying the inverse Laplace transform [16] to the first term

$$\bar{u}_{s1}(\eta,q) = \sqrt{\frac{2}{\pi}}\eta\frac{\Gamma(b+1)}{q^{b+1}}\left[\frac{1}{R_e+\eta^2} + \frac{R_e}{R_e+\eta^2}\cdot\frac{1}{(R_e+\eta^2)q+\eta^2}\right], \tag{20}$$

of Eq. (18) and using the convolution theorem, we find that

$$u_{s1}(\eta,t)=\sqrt{\frac{2}{\pi}}\frac{\eta}{\eta^2+R_e}t^b+\sqrt{\frac{2}{\pi}}\frac{\eta R_e}{(\eta^2+R_e)^2}\int_0^t (t-s)^b\, exp\left(-\frac{\eta^2 s}{\eta^2+R_e^2}\right)ds. \tag{21}$$

The last term of Eq. (18) can be written as a product of two functions

$$\begin{aligned}&\bar{u}_{s2}(\eta,q)=-\sqrt{\tfrac{2}{\pi}}\tfrac{\Gamma(a+1)}{q^{a+1}(1+q)}\tfrac{1}{\eta}\sqrt{W(\eta,q)}\quad \text{and}\\ &\bar{u}_{s3}(y,\eta,q)=\tfrac{1}{W(\eta,q)}e^{-y\sqrt{W(\eta,q)}}.\end{aligned} \tag{22}$$

The inverse Laplace transforms of $\bar{u}_{s2}(\eta,q)$ and $\bar{u}_{s3}(y,\eta,q)$

$$\begin{aligned}&u_{s2}(\eta,t)=-\sqrt{\tfrac{2}{\pi}}\tfrac{\sqrt{(\eta^2+R_e)}}{\eta}\int_0^t (t-s)^a I_0\left(\tfrac{sR_e}{2(\eta^2+R_e)}\right)\times\\ &exp\left(-\tfrac{(2\eta^2+R_e)s}{2(\eta^2+R_e)}\right)ds+\sqrt{\tfrac{2}{\pi}}\tfrac{R_e}{\eta\sqrt{(\eta^2+R_e)}}\int_0^t\int_0^\sigma(\sigma-s)^a e^{-s} I_0\left(\tfrac{(t-\sigma)R_e}{2(\eta^2+R_e)}\right)\times\\ &exp\left(-\tfrac{(2\eta^2+R_e)(t-\sigma)}{2(\eta^2+R_e)}\right)d\sigma ds.\end{aligned} \tag{23}$$

$$\begin{aligned}u_{s3}(y,\eta,t)=&\int_0^\infty\sqrt{\tfrac{uR_e}{t}}e^{-t}erfc\left(\tfrac{y}{2\sqrt{u}}\right)I_1\left(2\sqrt{uR_e t}\right)e^{-u(\eta^2+R_e)}du+\\ &\tfrac{1}{\eta^2+R_e}e^{-y\sqrt{\eta^2+R_e}}\delta(t),\end{aligned} \tag{24}$$

where $\delta(\cdot)$ is Dirac delta function, are obtained using Eqs. (A.1)-(A.4) from Appendix and the convolution theorem. Finally writing

$$u_s(y,\eta,t)=u_{s1}(\eta,t)+(u_{s2}*u_{s3})(y,\eta,t), \tag{25}$$

where the $*$ denotes the convolution product, we obtain

$$\begin{aligned}u_s(y,\eta,t)=&\sqrt{\frac{2}{\pi}}\frac{\eta}{\eta^2+R_e}t^b+\sqrt{\frac{2}{\pi}}\frac{\eta R_e}{(\eta^2+R_e)^2}\int_0^t(t-s)^b exp\left(-\frac{\eta^2 s}{\eta^2+R_e}\right)ds\\ &-\sqrt{\frac{2}{\pi}}\frac{e^{-y\sqrt{\eta^2+R_e}}}{\eta\sqrt{\eta^2+R_e}}\int_0^t(t-s)^a I_0\left(\frac{sR_e}{2(\eta^2+R_e)}\right)\exp\left(-\frac{(2\eta^2+R_e)}{2(\eta^2+R_e)}s\right)ds\\ &+\sqrt{\frac{2}{\pi}}\frac{R_e e^{-y\sqrt{\eta^2+R_e}}}{\eta(\sqrt{\eta^2+R_e})^3}\times\int_0^t\int_0^\sigma(\sigma-s)^a I_0\left(\frac{(t-\sigma)R_e}{2(\eta^2+R_e)}\right)\\ &\exp\left(-\frac{(2\eta^2+R_e)}{2(\eta^2+R_e)}(t-\sigma)-s\right)dsd\sigma-\sqrt{\frac{2}{\pi}}\frac{\sqrt{\eta^2+R_e}}{\eta}\times\\ &\int_0^\infty\int_0^t\int_0^\sigma(\sigma-s)^a\sqrt{\frac{uR_e}{t-\sigma}}\,erfc\left(\frac{y}{2\sqrt{u}}\right)I_0\left(\frac{sR_e}{2(\eta^2+R_e)}\right)\times\end{aligned}$$

$$I_1\left(2\sqrt{uR_e(t-\sigma)}\right)\times\exp\left(-\frac{(2\eta^2+R_e)}{2(\eta^2+R_e)}s-(t-\sigma)-u(\eta^2+R_e)\right)\,ds\,d\sigma du$$
$$+\sqrt{\frac{2}{\pi}}\frac{R_e}{\eta\sqrt{\eta^2+R_e}}\times\int_0^\infty\int_0^t\int_0^\tau\int_0^\sigma(\sigma-s)^a\sqrt{\frac{uR_e}{t-\tau}}\,erfc\left(\frac{y}{2\sqrt{u}}\right)\times$$
$$I_0\left(\frac{(\tau-\sigma)R_e}{2(\eta^2+R_e)}\right)I_1\left(2\sqrt{uR_e(t-\tau)}\right)\times$$
$$\exp\left(-\frac{(2\eta^2+R_e)}{2(\eta^2+R_e)}(\tau-\sigma)-s-(t-\tau)-u(\eta^2+R_e)\right)ds\,d\sigma\,d\tau\,du. \quad (26)$$

Now, applying the inverse Fourier sine transform to Eq. (26) we get the velocity field

$$u(y,z,t)=t^b\,e^{-z\sqrt{R_e}}+\frac{2R_e}{\pi}\int_0^\infty\int_0^t\frac{\eta\sin(\eta z)}{(\eta^2+R_e)^2}(t-s)^b\,\exp\left(-\frac{\eta^2 s}{\eta^2+R_e}\right)dsd\eta$$
$$-\frac{2}{\pi}\int_0^\infty\int_0^t\frac{e^{-y\sqrt{\eta^2+R_e}}}{\eta\sqrt{\eta^2+R_e}}\sin(\eta z)(t-s)^a I_0\left(\frac{sR_e}{2(\eta^2+R_e)}\right)\exp\left(\frac{-(2\eta^2+R_e)}{2(\eta^2+R_e)}s\right)dsd\eta$$
$$+\frac{2R_e}{\pi}\int_0^\infty\int_0^t\int_0^\sigma\frac{\sin(\eta z)e^{-y\sqrt{\eta^2+R_e}}}{\eta(\sqrt{\eta^2+R_e})^3}(\sigma-s)^a I_0\left(\frac{(t-\sigma)R_e}{2(\eta^2+R_e)}\right)\times$$
$$\exp\left(-\frac{2\eta^2+R_e}{2(\eta^2+R_e)}(t-\sigma)-s\right)dsd\sigma d\eta-\frac{2}{\pi}\int_0^\infty\int_0^\infty\int_0^t\int_0^\sigma\frac{\sqrt{\eta^2+R_e}}{\eta}\sin(\eta z)\times$$
$$(\sigma-s)^a\sqrt{\frac{uR_e}{t-\sigma}}\,erfc\left(\frac{y}{2\sqrt{u}}\right)I_0\left(\frac{sR_e}{2(\eta^2+R_e)}\right)I_1\left(2\sqrt{uR_e(t-\sigma)}\right)$$
$$\exp\left(-\frac{(2\eta^2+R_e)}{2(\eta^2+R_e)}s-(t-\sigma)-u(\eta^2+R_e)\right)ds\,d\sigma\,du\,d\eta+$$
$$\frac{2R_e}{\pi}\int_0^\infty\int_0^\infty\int_0^t\int_0^\tau\int_0^\sigma\frac{\sin(\eta z)}{\eta\sqrt{(\eta^2+R_e)}}(\sigma-s)^a\sqrt{\frac{uR_e}{t-\tau}}\,erfc\left(\frac{y}{2\sqrt{u}}\right)\times$$
$$I_0\left(\frac{(t-\tau)R_e}{2(\eta^2+R_e)}\right)I_1\left(2\sqrt{uR_e(t-\tau)}\right)\times$$
$$\exp\left(-\frac{(2\eta^2+R_e)}{2(\eta^2+R_e)}(\tau-\sigma)-s-(t-\tau)-u(\eta^2+R_e)\right)ds\,d\sigma\,d\tau\,du\,d\eta. \quad (27)$$

3.2 Calculation of shear stresses

Applying the Laplace transform to Eqs. (11) and then the Fourier sine transform to the first relation, we find that

$$\bar{\tau}_{s1}(y,\eta,q) = (1+q)\frac{\partial \bar{u}_s(y,\eta,q)}{\partial y}, \quad \bar{\tau}_2(y,z,q) = (1+q)\frac{\partial \bar{u}(y,z,q)}{\partial z}. \tag{28}$$

Introducing Eq. (18) in Eq. $(28)_1$, it results that

$$\bar{\tau}_{s1}(y,\eta,q) = \sqrt{\frac{2}{\pi}}\frac{\Gamma(a+1)}{q^{a+1}}\frac{1}{\eta} - \sqrt{\frac{2}{\pi}}\frac{\Gamma(a+1)}{q^{a+1}}\frac{1}{\eta}\left[\frac{qR_e+(1+q)\eta^2}{(1+q)}\right] \times \frac{1-e^{-y\sqrt{W(\eta,q)}}}{W(\eta,q)}. \tag{29}$$

Let us denote by

$$\bar{T}_{s1}(\eta,q) = \sqrt{\frac{2}{\pi}}\frac{\Gamma(a+1)}{q^{a+1}}\frac{1}{\eta}, \bar{T}_{s2}(\eta,q) = -\sqrt{\frac{2}{\pi}}\frac{\Gamma(a+1)}{q^{a+1}}\frac{1}{\eta}\left[\frac{qR_e+(1+q)\eta^2}{(1+q)}\right], \tag{30}$$

and

$$\bar{T}_{s3}(y,\eta,q) = \frac{1-e^{-y\sqrt{W(\eta,q)}}}{W(\eta,q)}. \tag{31}$$

Applying the inverse Laplace transform to Eq. (30) we find that

$$T_{s1}(\eta,t) = \sqrt{\frac{2}{\pi}}\frac{t^a}{\eta}, \quad T_{s2}(\eta,t) = -\sqrt{\frac{2}{\pi}}\frac{\eta^2+R_e}{\eta}t^a + \sqrt{\frac{2}{\pi}}\frac{R_e}{\eta}\int_0^t (t-s)^a e^{-s} ds. \tag{32}$$

As regards the last term $\bar{T}_{s3}(y,\eta,t)$, in view of the identities $(A.2)_2$ and (A.3), it results that

$$T_{s3}(\eta,t) = \int_0^\infty \sqrt{\frac{uR_e}{t}} e^{-t} erf\left(\frac{y}{2\sqrt{u}}\right) I_1\left(2\sqrt{uR_e t}\right) e^{-u(\eta^2+R_e)} du + \frac{1-e^{-y(\eta^2+R_e)}}{\eta^2+R_e}\delta(t). \tag{33}$$

Combining the above results, it is easy to show that

$$\tau_{s1}(\eta,t)=\sqrt{\frac{2}{\pi}}\frac{t^a}{\eta}-\sqrt{\frac{2}{\pi}}t^a\frac{1-e^{-y(\eta^2+R_e)}}{\eta}+\sqrt{\frac{2}{\pi}}\frac{R_e}{\eta}\frac{1-e^{-y(\eta^2+R_e)}}{\eta^2+R_e}\times$$
$$\int_0^t (t-s)^a e^{-s}ds-\sqrt{\frac{2}{\pi}}\frac{\eta^2+R_e}{\eta}\int_0^\infty\int_0^t s^a\sqrt{\frac{uR_e}{t-s}}erf\left(\frac{y}{2\sqrt{u}}\right)\times$$
$$I_1\left(2\sqrt{uR_e(t-s)}\right)\exp\left(-u(\eta^2+R_e)-(t-s)\right)ds\,du$$
$$+\sqrt{\frac{2}{\pi}}\frac{R_e}{\eta}\int_0^\infty\int_0^t\int_0^\sigma(\sigma-s)^a\sqrt{\frac{uR_e}{t-\sigma}}erf\left(\frac{y}{2\sqrt{u}}\right)$$
$$I_1\left(2\sqrt{uR_e(t-\sigma)}\right)\exp\left(-u(\eta^2+R_e)-(t-\sigma)-s\right)ds\,d\sigma\,du. \quad (34)$$

Apply the inverse Fourier sine transform to Eq. (34) we get

$$\tau_1(y,z,t)=\frac{2}{\pi}t^a\int_0^\infty\frac{e^{-y\sqrt{\eta^2+R_e}}}{\eta}sin(\eta z)d\eta+\frac{2}{\pi}R_e\int_0^\infty\int_0^t\frac{sin(\eta z)}{\eta}\times$$
$$\frac{1-e^{-y\sqrt{\eta^2+R_e}}}{\eta^2+R_e}(t-s)^a e^{-s}\,ds\,d\eta-\frac{2}{\pi}\int_0^\infty\int_0^\infty\int_0^t\frac{\eta^2+R_e}{\eta}\sin(\eta z)\times$$
$$(t-s)^a\sqrt{\frac{uR_e}{s}}erf\left(\frac{y}{2\sqrt{u}}\right)I_1\left(2\sqrt{R_e us}\right)\exp\left(-u(\eta^2+R_e)-s\right)ds\,du\,d\eta$$
$$+\frac{2}{\pi}R_e\int_0^\infty\int_0^\infty\int_0^t\int_0^\sigma\frac{sin(\eta z)}{\eta}(\sigma-s)^a\sqrt{\frac{uR_e}{t-\sigma}}erf\left(\frac{y}{2\sqrt{u}}\right)\times$$
$$I_1\left(2\sqrt{uR_e(t-\sigma)}\right)\exp\left(-u(\eta^2+R_e)-(t-\sigma)-s\right)ds\,d\sigma\,du\,d\eta. \quad (35)$$

In order to determine the second shear stress $\tau_2(y,z,t)$, we apply the inverse Fourier sine transform to Eq. (18) and introduce the result in Eq. $(28)_2$. It results that

$$\bar{\tau}_2(y,z,q)=\frac{2}{\pi}\int_0^\infty\eta^2cos(\eta z)\frac{\Gamma(b+1)}{q^{b+1}}\left[\frac{(q+1)^2}{qR_e+(1+q)\eta^2}\right]d\eta \quad (36)$$
$$-\frac{2}{\pi}\int_0^\infty cos(\eta z)\frac{\Gamma(a+1)}{q^{a+1}}\frac{e^{-y\sqrt{W(\eta,q)}}}{\sqrt{W(\eta,q)}}d\eta.$$

The inverse Laplace transforms of the two terms

$$\bar{T}_{21}(z,q)=\frac{2}{\pi}\int_0^\infty\eta^2cos(\eta z)\frac{\Gamma(b+1)}{q^{b+1}}\left[\frac{(q+1)^2}{qR_e+(1+q)\eta^2}\right]d\eta, \quad (37)$$

$$\bar{T}_{22}(y,z,q) = -\frac{2}{\pi}\int_0^\infty cos(\eta z)\frac{\Gamma(a+1)}{q^{a+1}}\frac{e^{-y\sqrt{W(\eta,q)}}}{\sqrt{W(\eta,q)}}d\eta, \qquad (38)$$

of Eq. (36) are (see also $(A.4)_2$ for the first of them)

$$T_{21}(z,t) = -b\sqrt{R_e}t^{b-1}e^{-z\sqrt{R_e}} - \frac{2R_e^2}{\pi}t^b\int_0^\infty \frac{\cos(\eta z)}{(\eta^2+R_e)^2}d\eta + \frac{2R_e^2}{\pi}\int_0^\infty\int_0^t \frac{\eta^2\cos(\eta z)}{(\eta^2+R_e)^3}(t-s)^b\exp\left[-\frac{\eta^2 s}{\eta^2+R_e}\right]ds\,d\eta, \qquad (39)$$

$$T_{22}(y,z,t) = -\frac{2}{\pi}\int_0^\infty\int_0^t \frac{e^{-y\sqrt{\eta^2+R_e}}}{\eta^2+R_e}\frac{\cos(\eta z)}{\sqrt{\eta^2+R_e}}\left[a(\eta^2+R_e)+\eta^2 s\right]s^{a-1}\times I_0\left[\frac{R_e(t-s)}{2(\eta^2+R_e)}\right]\exp\left[-\frac{2\eta^2+R_e}{2(\eta^2+R_e)}(t-s)-u(\eta^2+R_e)\right]ds\,d\eta$$
$$-\frac{2}{\pi}\int_0^\infty\int_0^\infty\int_0^t\int_0^\sigma \frac{\cos(\eta z)}{\sqrt{\eta^2+R_e}}\left[a(\eta^2+R_e)+\eta^2 s\right]s^{a-1}\sqrt{\frac{uR_e}{t-\sigma}}erfc\left(\frac{y}{2\sqrt{u}}\right)\times I_0\Big[\tfrac{R_e(\sigma-s)}{2(\eta^2+R_e)}\Big]\,I_1\left[2\sqrt{uR_e(t-\sigma)}\right]\times$$
$$\exp\left[-\frac{2\eta^2+R_e}{2(\eta^2+R_e)}(\sigma-s)-(t-\sigma)-u(\eta^2+R_e)\right]ds\,d\sigma\,du\,d\eta. \qquad (40)$$

Combining the above results and using again $(A.4)_2$, we obtain for $\tau_2(y,z,t)$ the expression

$$\tau_2(y,z,t) = -b\sqrt{R_e}t^{b-1}e^{-z\sqrt{R_e}} - \frac{\sqrt{R_e}}{2}t^b(z\sqrt{R_e}+1)e^{-z\sqrt{R_e}}$$
$$+\frac{2R_e^2}{\pi}\int_0^\infty\int_0^t \frac{\eta^2\cos(\eta z)}{(\eta^2+R_e)^3}(t-s)^b\exp\left[-\frac{\eta^2 s}{\eta^2+R_e}\right]ds\,d\eta$$
$$-\frac{2}{\pi}\int_0^\infty\int_0^t \frac{e^{-y\sqrt{\eta^2+R_e}}}{\eta^2+R_e}\cos(\eta z)\frac{\left[a(\eta^2+R_e)+\eta^2 s\right]}{\sqrt{\eta^2+R_e}}s^{a-1}\times$$
$$I_0\left[\frac{R_e(t-s)}{2(\eta^2+R_e)}\right]\exp\left[-\frac{2\eta^2+R_e}{2(\eta^2+R_e)}(t-s)\right]ds\,d\eta$$

$$-\frac{2}{\pi}\int_0^\infty\int_0^\infty\int_0^t\int_0^\sigma cos(\eta z)\sqrt{\frac{uR_e}{t-\sigma}}erfc\left(\frac{y}{2\sqrt{u}}\right)\times$$
$$I_1\left[2\sqrt{uR_e(t-\sigma)}\right]I_0\left[\frac{R_e(\sigma-s)}{2(\eta^2+R_e)}\right]\cdot\frac{\left[a(\eta^2+R_e)+\eta^2 s\right]s^{a-1}}{\sqrt{\eta^2+R_e}}\times$$
$$\exp\left[-\frac{2\eta^2+R_e}{2(\eta^2+R_e)}(\sigma-s)-(t-\sigma)-u(\eta^2+R_e)\right]ds\,d\sigma\,du\,d\eta. \quad (41)$$

4 Numerical results and conclusions

In this note a mixed initial and boundary-value problem has been solved by means of integral transforms. More accurately, solutions are established for the dimensionless velocity $u(y,z,t)$ and non-trivial shear stresses $\tau_1(y,z,t)$ and $\tau_2(y,z,t)$ corresponding to the motion of a second grade fluid in an edge. The motion of the fluid is due to the two sides of the edge. One of them (in the plane $y=0$) applies a time-dependent shear stress to the fluid and the other one (in the plane $z=0$) is moving in its plane parallel to the corner line with a prescribed velocity. Direct computations show that the solutions that have been obtained, in form of simple and multiple integrals, satisfy all imposed initial and boundary conditions.

In order to reveal some relevant physical aspects of the obtained results the diagrams of the velocity $u(y,z,t)$ and the shear stresses $\tau_1(y,z,t)$ and $\tau_2(y,z,t)$ have been drawn against z for different values of y, t and Reynolds number Re. A series of calculations were performed for different situations with typical values using the program Mathcad 14.0. From Figs. 1 it clearly results that the velocity of the fluid $u(y,z,t)$, as expected, decreases with respect to z and increases with regards to y. This is due to the skin friction $\tau_1(0,z,t)$ applied on the side $y=0$. Of course, the velocity of the fluid on the side $z=0$ is the same for each y. The influence of the Reynolds number Re on the fluid motion is shown by Figs. 2. The velocity of the fluid decreases for increasing Re. Last two figures give similar representations for the adequate shear stresses $\tau_1(y,z,t)$ and $\tau_2(y,z,t)$. The results of Figs. 3 are in accordance with those resulting from Figs. 1. The skin friction $\tau_1(y,z,t)$ in parallel planes to the bottom wall $y=0$ decreases with respect to z but is an increasing function of y. The second shear stress $\tau_2(y,z,t)$, as it results from Figs. 4, is a decreasing function with respect to both variables y and z. This result also seem to be a realistic one. The units of material constants are SI units in all figures.

Figure 1: Profiles of the velocity $u(y, 0, t)$ for $Re = 5$, $a = b = 1$ and for different values of y and t.

Figure 2: Profiles of the velocity $u(y, 0, t)$ for $y = 0.5$, $a = b = 0.5$ and for different values of Re and time.

Figure 3: Profiles of the shear stress $\tau_1(y,z,t)$ for $Re = 5$, $a = b = 1$ and for different values of y and t.

Figure 4: Profiles of the shear stress $\tau_2(y, z, t)$ for $y = 0.5$, $a = b = 0.5$ and for different values of Re and time.

Finally, it is worth pointing out that besides the velocity field we also provide exact solutions for the shear stresses that are induced due to the flow. Such solutions, in additions to serving as approximations to some specific initial-boundary value problems also serve a very important purpose, namely they can be used as tests to verify numerical schemes that are developed to study more complex unsteady flow problems. Of special interest is the case $a = b = 1$ corresponding to constantly accelerating velocity and shear stress on the boundary. However, in all cases the motion of the fluid is unsteady and remains unsteady.

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Appendix

$$L^{-1}\left[\frac{1}{\sqrt{(q+a)^2-b^2}}\right] = e^{-at}I_0(bt); \quad L^{-1}\left[e^{\frac{u}{q}}-1\right] = \sqrt{\frac{u}{t}}I_1\left(2\sqrt{ut}\right). \tag{A.1}$$

where I_0 and I_1 are the modified Bessel functions of first kind.

$$L^{-1}\left[\frac{e^{-y\sqrt{q}}}{q}\right] = erfc\left(\frac{y}{2\sqrt{t}}\right);$$

$$\int_0^\infty erfc\left(\frac{y}{2\sqrt{u}}\right)e^{-u(\eta^2+R_e)}du = \frac{1}{\eta^2+R_e}e^{-y\sqrt{\eta^2+R_e}}. \tag{A.2}$$

$$L^{-1}\left[\frac{e^{-y\sqrt{W(\eta,q)}}}{W(\eta,q)}\right] = \int_0^\infty erfc\left(\frac{y}{2\sqrt{u}}\right)g(u,t)du, g(u,t) = L^{-1}\left[e^{-uW(\eta,q)}\right]. \tag{A.3}$$

$$\int_0^\infty \frac{\eta\sin(\eta z)}{\eta^2+a^2}d\eta = \frac{\pi}{2}e^{-az}, Re(a)\geq 0;$$

$$\int_0^\infty \frac{\cos(bx)}{(x^2+a^2)^2}dx = \frac{\pi}{4a^3}(ab+1)e^{-ab};\ a,b>0. \tag{A.4}$$

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