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GENERALIZED WELL-POSEDNESS OF HYPERBOLIC VOLTERRA EQUATIONS OF NON-SCALAR TYPE*

Marko Kostić†

Abstract

In the present paper, we introduce the class of (A, k)-regularized C-pseudoresolvent families, analyze themes like generation, hyperbolic perturbations, regularity and local properties, and furnish several illustrative examples. The study of differentiability of (A, k)-regularized C-pseudoresolvent families seems to be new even in the case $k(t) \equiv 1$ and $C \equiv I$.

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1 Introduction and preliminaries

Our intention in this paper is to enquire into the basic structural properties of a fairly general class of (local) (A, k)-regularized C-pseudoresolvent families. This class of pseudoresolvent families is one of the main tools in the analysis of ill-posed hyperbolic Volterra equations of non-scalar type. It is worthwhile to mention here that there are by now only a few references concerning non-scalar evolutionary Volterra equations (cf. [10]-[11] and [23]).

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†marco.s@verat.net Faculty of Technical Sciences Trg Dositeja Obradovića 6 21125 Novi Sad Serbia; Partially supported by grant 174024 of Ministry of Science and Technological Development, Republic of Serbia.

We analyze Hille-Yosida type theorems, perturbations, differential and analytical properties of solutions of non-scalar operator equations, and remove density assumptions from the previously known concepts.

We shall henceforth assume that X and Y are Banach spaces and that Y is continuously embedded in X. Let $L(X) \ni C$ be injective and let $\tau \in (0,\infty]$. The norm in X, resp. Y, will be denoted by $||\cdot||_X$, resp. $||\cdot||_Y$; $[R(C)]$ denotes the Banach space $R(C)$ equipped with the norm $||x||_{R(C)} = ||C^{-1}x||_X$, $x \in R(C)$ and, for a given closed linear operator A in X, $[D(A)]$ denotes the Banach space $D(A)$ equipped with the graph norm $||x||_{D(A)} = ||x||_X + ||Ax||_X$, $x \in D(A)$. Suppose F is a subspace of X. Then the part of A in F, denoted by $A_{|F}$, is a linear operator defined by $D(A_{|F}) := \{x \in D(A) \cap F : Ax \in F\}$ and $A_{|F}x := Ax$, $x \in D(A_{|F})$. Let $A(t)$ be a locally integrable function from $[0,\tau)$ into $L(Y,X)$. Unless stated otherwise, we assume that $A(t)$ is not of scalar type, i.e., that there does not exist $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, and a closed linear operator A in X such that $Y = [D(A)]$ and that $A(t) = a(t)A$ for a.e. $t \in [0,\tau)$ (cf. also the short discussion preceding Proposition 1 for full details). We refer the reader to [14] and references cited there for further information concerning ill-posed abstract Volterra equations of scalar type.

In the sequel, the meaning of symbol A is clear from the context. We mainly use the following condition

(P1): $k(t)$ is Laplace transformable, i.e., it is locally integrable on $[0,\infty)$ and there exists $\beta \in \mathbb{R}$ so that $\tilde{k}(\lambda) := \mathcal{L}(k)(\lambda) := \lim_{b\to\infty} \int_0^b e^{-\lambda t} k(t)\,dt := \int_0^\infty e^{-\lambda t} k(t)\,dt$ exists for all $\lambda \in \mathbb{C}$ with $Re(\lambda) > \beta$. Put $\text{abs}(k) := \inf\{Re(\lambda) : \tilde{k}(\lambda) \text{ exists}\}$.

Let us recall that a function $k \in L^1_{loc}([0,\tau))$ is called a *kernel,* if for every $\phi \in C([0,\tau))$, the preassumption $\int_0^t k(t-s)\phi(s)\,ds = 0$, $t \in [0,\tau)$ implies $\phi(t) = 0$, $t \in [0,\tau)$. Thanks to the famous E. C. Titchmarsh's theorem, the condition $0 \in \text{supp}k$ implies that $k(t)$ is a kernel. Set $\Theta(t) := \int_0^t k(s)\,ds$, $t \in [0,\tau)$ and recall that the *C-resolvent set of A*, $\rho_C(A)$ in short, is defined by

$$\rho_C(A) := \big\{\lambda \in \mathbb{C} : \lambda - A \text{ is injective and } R(C) \subseteq R(\lambda - A)\big\};$$

the resolvent set of A is also denoted by $\rho(A)$. The principal branch is always used to take the powers and the abbreviation $*$ stands for the finite convolution product. Set $g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$ ($\alpha > 0$, $t > 0$), where $\Gamma(\cdot)$ denotes the Gamma function.

From now on, we basically follow the notation employed in the monograph of J. Prüss [23]. The notions of (a,k)-regularized C-resolvent families, (a,C)-regularized resolvent families as well as local (K-convoluted)

C-semigroups and cosine functions will be understood in the sense of [14] and [16].

2 (A, k)-regularized C-pseudoresolvent families

Definition 1 *Let $k \in C([0,\tau))$ and $k \neq 0$. Consider the linear Volterra equation:*

$$u(t) = f(t) + \int_0^t A(t-s)u(s)\,ds,\ t \in [0,\tau), \tag{1}$$

where $\tau \in (0,\infty]$, $f \in C([0,\tau) : X)$ and $A \in L^1_{loc}([0,\tau) : L(Y,X))$. Then a function $u \in C([0,\tau) : X)$ is said to be:

(i) a strong solution of (1) iff $u \in L^\infty_{loc}([0,\tau) : Y)$ and (1) holds on $[0,\tau)$,

(ii) a mild solution of (1) iff there exist a sequence (f_n) in $C([0,\tau) : X)$ and a sequence (u_n) in $C([0,\tau) : X)$ such that $u_n(t)$ is a strong solution of (1) with $f(t)$ replaced by $f_n(t)$ and that $\lim_{n\to\infty} f_n(t) = f(t)$ as well as $\lim_{n\to\infty} u_n(t) = u(t)$, uniformly on compact subsets of $[0,\tau)$.

The abstract Cauchy problem (1) is said to be (kC)-well posed (C-well posed, if $k(t) \equiv 1$) iff for every $y \in Y$, there exists a unique strong solution of

$$u(t;y) = k(t)Cy + \int_0^t A(t-s)u(s;y)\,ds,\ t \in [0,\tau) \tag{2}$$

and if $u(t;y_n) \to 0$ in X, uniformly on compact subsets of $[0,\tau)$, whenever (y_n) is a zero sequence in Y; (1) is said to be a-regularly (kC)-well posed (a-regularly C-well posed, if $k(t) \equiv 1$), where $a \in L^1_{loc}([0,\tau))$, iff (1) is (kC)-well posed and if the equation

$$u(t) = (a * k)(t)Cx + \int_0^t A(t-s)u(s)\,ds,\ t \in [0,\tau)$$

admits a unique strong solution for every $x \in X$.

It is clear that every strong solution of (1) is also a mild solution of (1).

Definition 2 *Let $\tau \in (0,\infty]$, $k \in C([0,\tau))$, $k \neq 0$ and $A \in L^1_{loc}([0,\tau) : L(Y,X))$. A family $(S(t))_{t\in[0,\tau)}$ in $L(X)$ is called an (A,k)-regularized C-pseudoresolvent family iff the following holds:*

(S1) The mapping $t \mapsto S(t)x$, $t \in [0,\tau)$ *is continuous in* X *for every fixed* $x \in X$, $S(0) = k(0)C$ *and* $S(t)C = CS(t)$, $t \in [0,\tau)$.

(S2) Put $U(t)x := \int_0^t S(s)x\,ds$, $x \in X$, $t \in [0,\tau)$. *Then (S2) means* $U(t)Y \subseteq Y$, $U(t)_{|Y} \in L(Y)$, $t \in [0,\tau)$ *and* $(U(t)_{|Y})_{t\in[0,\tau)}$ *is locally Lipschitz continuous in* $L(Y)$.

(S3) The resolvent equations

$$S(t)y = k(t)Cy + \int_0^t A(t-s)\,dU(s)y,\ \ t \in [0,\tau),\ \ y \in Y, \tag{3}$$

$$S(t)y = k(t)Cy + \int_0^t S(t-s)A(s)y\,ds,\ \ t \in [0,\tau),\ \ y \in Y, \tag{4}$$

hold; (3), resp. (4), is called the first resolvent equation, resp. the second resolvent equation.

An (A,k)*-regularized* C*-pseudoresolvent family* $(S(t))_{t\in[0,\tau)}$ *is said to be an* (A,k)*-regularized* C*-resolvent family if additionally:*

(S4) For every $y \in Y$, $S(\cdot)y \in L^{\infty}_{loc}([0,\tau) : Y)$.

An operator family $(S(t))_{t\in[0,\tau)}$ *in* $L(X)$ *is called a weak* (A,k)*-regularized* C*-pseudoresolvent family iff (S1) and (4) hold. A weak* (A,k)*-regularized* C*-pseudoresolvent family* $(S(t))_{t\geq 0}$ *is said to be exponentially bounded iff there exist* $M \geq 1$ *and* $\omega \geq 0$ *such that* $||S(t)||_{L(X)} \leq Me^{\omega t}$, $t \geq 0$. *Finally, a weak* (A,k)*-regularized* C*-pseudoresolvent family* $(S(t))_{t\in[0,\tau)}$ *is said to be* a*-regular* ($a \in L^1_{loc}([0,\tau))$) *iff* $a * S(\cdot)x \in C([0,\tau) : Y)$, $x \in \overline{Y}^X$.

In this paragraph, we will ascertain a few lexicographical agreements. A (weak) (A,k)-regularized C-(pseudo)resolvent family with $k(t) \equiv g_{\alpha+1}(t)$, where $\alpha \geq 0$, is also called a (weak) α-times integrated A-regularized C-(pseudo)resolvent family, whereas a (weak) 0-times integrated A-regularized C-(pseudo)resolvent family is also said to be a (weak) A-regularized C-(pseudo)resolvent family. A (weak) (A,k)-regularized C-(pseudo)resolvent family is also called a (weak) (A,k)-regularized (pseudo)resolvent family ((weak) A-regularized (pseudo)resolvent family) if $C = I$ (if $C = I$ and $k(t) \equiv 1$).

It is worth noting that the integral appearing in the first resolvent equation (3) is understood in the sense of discussion following [23, Definition 6.2, p. 152] and that M. Jung considered in [10] a slightly different notion of A-regularized (pseudo)resolvent families. Moreover, (S3) can be rewritten in the following equivalent form:

(S3)'

$$U(t)y = \Theta(t)Cy + \int_0^t A(t-s)U(s)y\,ds,\ t \in [0,\tau),\ y \in Y,$$

$$U(t)y = \Theta(t)Cy + \int_0^t U(t-s)A(s)y\,ds,\ t \in [0,\tau),\ y \in Y.$$

By the norm continuity we mean the continuity in $L(X)$ and, in many places, we do not distinguish $S(\cdot)$ ($U(\cdot)$) and its restriction to Y. The main reason why we assume that $A(t)$ is not of scalar type is the following: Let A be a subgenerator of a (local) (a,k)-regularized C-resolvent family $(S(t))_{t\in[0,\tau)}$ in the sense of [14, Definition 2.1], let $Y = [D(A)]$ and let $A(t) = a(t)A$ for a.e. $t \in [0,\tau)$. Then $(S(t))_{t\in[0,\tau)}$ is an (A,k)-regularized C-resolvent family in the sense of Definition 2, $S(t) \in L(Y)$, $t \in [0,\tau)$ and, for every $y \in Y$, $S(\cdot)y \in C([0,\tau) : Y)$ and the mapping $t \mapsto U(t)y$, $t \in [0,\tau)$ is continuously differentiable in Y with $\frac{d}{dt}U(t)y = S(t)y$, $t \in [0,\tau)$ (cf. also Remark 2 as well as the proofs of Theorem 1, Theorem 2 and Theorem 6). Assume conversely $A(t) = a(t)A$ for a.e. $t \in [0,\tau)$, $Y = [D(A)]$ and $(S(t))_{t\in[0,\tau)}$ is an (A,k)-regularized C-resolvent family in the sense of Definition 2. If $CA \subseteq AC$ and $a(t)$ is kernel, then $(S(t))_{t\in[0,\tau)}$ is an (a,k)-regularized C-resolvent family in the sense of [14, Definition 2.1]. In order to verify this, notice that the second equality in (S3)' implies after differentiation $S(t)x = k(t)Cx + \int_0^t S(t-s)a(s)Ax\,ds = k(t)Cx + \int_0^t a(t-s)S(s)Ax\,ds$, $t \in [0,\tau)$, $x \in D(A)$, so that it suffices to show that $S(t)A \subseteq AS(t)$, $t \in [0,\tau)$. Combined with the first equality in (S3)', we get that, for every $t \in [0,\tau)$ and $x \in D(A)$:

$$\frac{d}{dt}\int_0^t a(t-s)AU(s)x\,ds = S(t)x - k(t)Cx = \int_0^t a(t-s)S(s)Ax\,ds$$

and

$$\int_0^t a(t-s)AU(s)x\,ds = \int_0^t\int_0^s a(s-r)S(r)Ax\,dr\,ds = \int_0^t a(t-s)U(s)Ax\,ds.$$

Hence, $A\int_0^t S(s)x\,ds = \int_0^t S(s)Ax\,ds$, $t \in [0,\tau)$, $x \in D(A)$. Then the closedness of A yields $S(t)A \subseteq AS(t)$, $t \in [0,\tau)$, as required. In the formulations of Proposition 4, Theorem 3, Corollary 1(i) as well as in the analyses given in Example 1, Example 2 and the paragraph preceding it, we also allow that $A(t)$ $((A+B)(t))$ is of scalar type; if this is the case, then the notion of a corresponding (weak) (A,k)-regularized $((A+B,k)$-regularized) C-(pseudo)resolvent family will be always understood in the sense of Definition 2.

The subsequent propositions can be proved by means of the argumentation given in [23].

Proposition 1 *(i) Suppose that $(S_i(t))_{t\in[0,\tau)}$ is an (A,k_i)-regularized C-pseudoresolvent family, $i=1,2$. Then $(k_2*R_1)(t)x=(k_1*R_2)(t)x$, $t\in[0,\tau)$, $x\in\overline{Y}^X$.*

(ii) Let $(S_i(t))_{t\in[0,\tau)}$ be an (A,k)-regularized C-pseudoresolvent family, $i=1,2$ and let $k(t)$ be a kernel. Then $S_1(t)x=S_2(t)x$, $t\in[0,\tau)$, $x\in\overline{Y}^X$.

(iii) Let $(S(t))_{t\in[0,\tau)}$ be an (A,k)-regularized C-pseudoresolvent family. Assume any of the following conditions:

(a) Y has the Radon-Nikodym property.

(b) There exists a dense subset Z of Y such that $A(t)z\in Y$ for a.e. $t\in[0,\tau)$, $A(\cdot)z\in L^1_{loc}([0,\tau):Y)$, $z\in Z$ and $C(Y)\subseteq Y$.

(c) $(S(t))_{t\in[0,\tau)}$ is a-regular, $A(t)=(a*dB)(t)$ for a.e. $t\in[0,\tau)$, where $a\in L^1_{loc}([0,\tau))$, $C(Y)\subseteq Y$ and $B\in BV_{loc}([0,\tau):L(Y,X))$ is such that $B(\cdot)y$ has a locally bounded Radon-Nikodym derivative w.r.t. $b(t)=VarB|_0^t$, $t\in[0,\tau)$, $y\in Y$.

Then $(S(t))_{t\in[0,\tau)}$ is an (A,k)-regularized C-resolvent family. Furthermore, if Y is reflexive, then $S(t)(Y)\subseteq Y$, $t\in[0,\tau)$ and the mapping $t\mapsto S(t)y$, $t\in[0,\tau)$ is weakly continuous in Y for all $y\in Y$. In cases (b) *and* (c)*, the mapping $t\mapsto S(t)y$, $t\in[0,\tau)$ is even continuous in Y for all $y\in Y$.*

Proposition 2 *(i) Assume that $(S(t))_{t\in[0,\tau)}$ is a weak (A,k)-regularized C-pseudoresolvent family, $f\in C([0,\tau):X)$ and $u(t)$ is a mild solution of* (1)*. Then $(kC*u)(t)=(S*f)(t)$, $t\in[0,\tau)$. In particular, mild solutions of* (1) *are unique provided that $k(t)$ is a kernel.*

(ii) Assume $n\in\mathbb{N}$, $(S(t))_{t\in[0,\tau)}$ is an $(n-1)$-times integrated A-regularized C-pseudoresolvent family, $C^{-1}f\in C^{n-1}([0,\tau):X)$ and $f^{(i)}(0)=0$, $0\le i\le n-1$. Then the following assertions hold:

(a) *Let* $(C^{-1}f)^{(n-1)} \in AC_{loc}([0,\tau) : Y)$ *and* $(C^{-1}f)^{(n)} \in L^1_{loc}([0,\tau) : Y)$. *Then the function* $t \mapsto u(t)$, $t \in [0,\tau)$ *given by*

$$u(t) = \int_0^t S(t-s)(C^{-1}f)^{(n)}(s)\,ds = \int_0^t dU(s)(C^{-1}f)^{(n)}(t-s)$$

is a unique strong solution of (1). *Moreover,* $u \in C([0,\tau) : Y)$.

(b) *Let* $(C^{-1}f)^{(n)} \in L^1_{loc}([0,\tau) : X)$ *and* $\overline{Y}^X = X$. *Then the function* $u(t) = \int_0^t S(t-s)(C^{-1}f)^{(n)}(s)\,ds$, $t \in [0,\tau)$ *is a unique mild solution of* (1).

(c) *Let* $C^{-1}g \in W^{n,1}_{loc}([0,\tau) : \overline{Y}^X)$, $a \in L^1_{loc}([0,\tau))$, $f(t) = (g_n * a * g^{(n)})(t)$, $t \in [0,\tau)$ *and let* $(S(t))_{t\in[0,\tau)}$ *be* a*-regular. Then the function* $u(t) = \int_0^t S(t-s)(a * (C^{-1}g)^{(n)})(s)\,ds$, $t \in [0,\tau)$ *is a unique strong solution of* (1).

Proposition 3 *(i) Let* $(S(t))_{t\in[0,\tau)}$ *be an* (A,k)*-regularized* C*-resolvent family. Put* $u(t;y) := S(t)y$, $t \in [0,\tau)$, $y \in Y$. *Then* $u(t;y)$ *is a strong solution of* (2), *and* (2) *is* (kC)*-well posed if* $k(t)$ *is a kernel.*

(ii) Assume $\overline{Y}^X = X$, (2) *is* (kC)*-well posed, all suppositions quoted in the formulation of* Proposition 1(iii)(b) *hold and* $A(t)Cz = CA(t)z$ *for all* $z \in Z$ *and a.e.* $t \in [0,\tau)$. *Then* (1) *admits an* (A,k)*-regularized* C*-resolvent family.*

(iii) Assume $\overline{Y}^X = X$, $L^1_{loc}([0,\tau)) \ni a$ *is a kernel and* $A(t)Cy = CA(t)y$ *for all* $y \in Y$ *and a.e.* $t \in [0,\tau)$. *Then* (2) *is* a*-regularly* (kC)*-well posed iff* (1) *admits an* a*-regular* (A,k)*-regularized* C*-resolvent family.*

Before proceeding further, we would like to mention that Proposition 2(ii) enables one to simply reveal the formula [26, (2.5)] for a solution of the problem (ACP_n); for more details in this direction, we refer the reader to [26, Theorem 2.4, Theorem 3.1]. It would take too long to consider some other applications of (A,k)-regularized C-pseudoresolvent families to higher order abstract differential equations ([25]).

Proposition 4 *Assume* $A \in L^1_{loc}([0,\tau) : L([D(A)],X))$ *is of the form*

$$A(t) = a(t)A + \int_0^t a(t-s)\,dB(s) \text{ for a.e. } t \in [0,\tau), \tag{5}$$

where $a \in L^1_{loc}([0,\tau))$, $B \in BV_{loc}([0,\tau) : L([D(A)], X))$ *is left continuous,* $B(0) = B(0+) = 0$ *and* A *is a closed linear operator such that* $\rho(A) \neq \emptyset$. *Let* $(S(t))_{t\in[0,\tau)}$ *be an* (A,k)*-regularized* C*-pseudoresolvent family. Then* $(S(t))_{t\in[0,\tau)}$ *is* a*-regular.*

Proof. Let $\mu \in \rho(A)$ and $K(t) := -B(t)(\mu - A)^{-1}$, $t \in [0,\tau)$. Then it is clear that $K \in BV_{loc}([0,\tau) : L(X))$. We define recursively $K_0(t) := K(t)$, $t \in [0,\tau)$ and $K_{n+1}(t) := \int_0^t dK(\tau)K_n(t-\tau)$, $t \in [0,\tau)$, $n \in \mathbb{N}$. By the proof of [23, Theorem 0.5, p. 13], the series $L(t) := \sum_{n=0}^{\infty}(-1)^n K_n(t)$, $t \in [0,\tau)$ converges absolutely in the norm of $BV^0([0,\tau) : L(X))$, $L \in BV^0([0,\tau) : L(X))$ and $L = K - dK * L = K - L * dK$. Repeating literally the proof of [23, Proposition 6.4, p. 137], we obtain that, for every $y \in Y$:

$$A(a * S(\cdot)y) = S(\cdot)y - k(\cdot)Cy - dL * (S(\cdot)y - k(\cdot)Cy - \mu(a * S(\cdot))y).$$

Then the closedness of A immediately implies that, for every $x \in \overline{Y}^X$, one has $A(a * S(\cdot))x \in C([0,\tau) : X)$ and $a * S(\cdot)x \in C([0,\tau) : [D(A)])$.

The Hille-Yosida theorem for (A,k)-regularized C-pseudoresolvent families reads as follows.

Theorem 1 *Assume* $A \in L^1_{loc}([0,\tau) : L(Y,X))$, $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, $a(t)$ *and* $k(t)$ *satisfy* (P1), $\epsilon_0 \geq 0$ *and*

$$\int_0^\infty e^{-\epsilon t}\|A(t)\|_{L(Y,X)}\, dt < \infty,\ \epsilon > \epsilon_0. \tag{6}$$

(i) Let $(S(t))_{t\geq 0}$ *be an* (A,k)*-regularized* C*-pseudoresolvent family such that there exists* $\omega \geq 0$ *with*

$$\sup_{t>0} e^{-\omega t}\Big(\|S(t)\|_{L(X)} + \sup_{0<s<t}(t-s)^{-1}\|U(t)-U(s)\|_{L(Y)}\Big) < \infty. \tag{7}$$

Put $\omega_0 := \max(\omega, abs(k), \epsilon_0)$ *and* $H(\lambda)x := \int_0^\infty e^{-\lambda t}S(t)x\, dt$, $x \in X$, $Re(\lambda) > \omega_0$. *Then the following holds:*

(N1) $C(Y) \subseteq Y$, $(\tilde{A}(\lambda))_{Re(\lambda)>\epsilon_0}$ *is analytic in* $L(Y,X)$, $R(C_{|Y}) \subseteq R(I - \tilde{A}(\lambda))$, $Re(\lambda) > \omega_0$, $\tilde{k}(\lambda) \neq 0$, *and* $I - \tilde{A}(\lambda)$ *is injective,* $Re(\lambda) > \omega_0$, $\tilde{k}(\lambda) \neq 0$.

(N2) $H(\lambda)y = \lambda\tilde{U}(\lambda)y$, $y \in Y$, $Re(\lambda) > \omega_0$, $(I - \tilde{A}(\lambda))^{-1}C_{|Y} \in L(Y)$, $Re(\lambda) > \omega_0$, $\tilde{k}(\lambda) \neq 0$, $(H(\lambda))_{Re(\lambda)>\omega_0}$ *is analytic in both spaces,*

$L(X)$ and $L(Y)$, $H(\lambda)C = CH(\lambda)$, $Re(\lambda) > \omega_0$, and for every $y \in Y$ and $\lambda \in \mathbb{C}$ with $Re(\lambda) > \omega_0$ and $\tilde{k}(\lambda) \neq 0$:

$$H(\lambda)(I - \tilde{A}(\lambda))y = (I - \tilde{A}(\lambda))H(\lambda)y = \tilde{k}(\lambda)Cy. \tag{8}$$

(N3)

$$\sup_{n\in\mathbb{N}_0} \sup_{\lambda>\omega_0,\ \tilde{k}(\lambda)\neq 0} \frac{(\lambda-\omega)^{n+1}}{n!}\Big(\Big\|\frac{d^n}{d\lambda^n}H(\lambda)\Big\|_{L(X)} + \Big\|\frac{d^n}{d\lambda^n}H(\lambda)\Big\|_{L(Y)}\Big) < \infty.$$

(ii) *Assume that* (N1)-(N3) *hold. Then there exists an exponentially bounded (A,Θ)-regularized C-resolvent family $(S_1(t))_{t\geq 0}$.*

(iii) *Assume that* (N1)-(N3) *hold and $\overline{Y}^X = X$. Then there exists an exponentially bounded (A,k)-regularized C-pseudoresolvent family $(S(t))_{t\geq 0}$ such that* (7) *holds.*

(iv) *Assume $(S(t))_{t\geq 0}$ is an (A,k)-regularized C-pseudoresolvent family, there exists $\omega \geq 0$ such that* (7) *holds and $\omega' \geq \omega$. Then $(S(t))_{t\geq 0}$ is a-regular and $\sup_{t\geq 0} e^{-\omega' t}||a * S(t)||_{L(\overline{Y}^X, Y)} < \infty$ iff there exists a number $\omega_1 \geq \max(\omega, \omega', abs(a), abs(k), \epsilon_0)$ such that*

$$\sup_{n\in\mathbb{N}_0} \sup_{\lambda>\omega_1,\ \tilde{k}(\lambda)\neq 0} \frac{(\lambda-\omega')^{n+1}}{n!}\Big\|\frac{d^n}{d\lambda^n}\big(\tilde{a}(\lambda)H(\lambda)\big)\Big\|_{L(\overline{Y}^X, Y)} < \infty. \tag{9}$$

Proof. In order to prove (i), notice that $\tilde{U}(\lambda) = H(\lambda)/\lambda$, $Re(\lambda) > \omega_0$. Furthermore, $(\tilde{A}(\lambda))_{Re(\lambda)>\omega_0}$ is analytic in $L(Y,X)$ and (7) in combination with (S1) yields that $(H(\lambda))_{Re(\lambda)>\omega_0} \subseteq L(X) \cap L(Y)$ is analytic in both spaces, $L(X)$ and $L(Y)$, and that $H(\lambda)C = CH(\lambda)$, $Re(\lambda) > \omega_0$. Fix, for the time being, $\lambda \in \mathbb{C}$ with $Re(\lambda) > \omega_0$ and $\tilde{k}(\lambda) \neq 0$. Using (S3)', one gets (8), $C(Y) \subseteq Y$, $R(C_{|Y}) \subseteq R(I - \tilde{A}(\lambda))$, $(I - \tilde{A}(\lambda))^{-1}C_{|Y} = (\lambda\tilde{U}(\lambda)/\tilde{k}(\lambda)) \in L(Y)$ and the injectiveness of the operator $I - \tilde{A}(\lambda)$. Therefore, we have proved (N1)-(N2). The assertion (N3) is an immediate consequence of [25, Theorem 2.1, p. 7], which completes the proof of (i). Assume now (N1)-(N3). By [25, Theorem 2.1], we obtain that there exist $M \geq 1$ and continuous functions $S_1 : [0,\infty) \to L(X)$ and $S_1^Y : [0,\infty) \to L(Y)$ such that $S_1(0) = S_1^Y(0) = 0$,

$$\sup_{t>0} e^{-\omega t}\Big(\sup_{0<s<t}(t-s)^{-1}\big\|S_1(t) - S_1(s)\big\|_{L(X)} + \sup_{0<s<t}(t-s)^{-1}\big\|S_1^Y(t) - S_1^Y(s)\big\|_{L(Y)}\Big) < \infty, \tag{10}$$

$$H(\lambda)x = \lambda \int_0^\infty e^{-\lambda t} S_1(t)x\,dt,\ x \in X,\ Re(\lambda) > \omega_0 \tag{11}$$

and

$$H(\lambda)y = \lambda \int_0^\infty e^{-\lambda t} S_1^Y(t)y\,dt,\ y \in Y,\ Re(\lambda) > \omega_0. \tag{12}$$

Making use of the inverse Laplace transform, (N2) and (11)-(12), we infer that $(S_1(t))_{t\geq 0}$ commutes with C and $S_1(t)y = S_1^Y(t)y$, $t \geq 0$, $y \in Y$. It is evident that the mapping $t \mapsto S_1(t)y$, $t \geq 0$ is continuous in Y for every fixed $y \in Y$ and that $(U_1(t) \equiv \int_0^t S_1(s)\,ds)_{t\geq 0}$ is continuously differentiable in $L(Y)$ with $U_1'(t) = S_1^Y(t)$, $t \geq 0$. The above assures that (S1), (S2) and (S4) hold for $(S_1(t))_{t\geq 0}$. Combining the inverse Laplace transform and (8), one gets that $(S_1(t))_{t\geq 0}$ satisfies (S3)', which completes the proof of (ii). If $\overline{Y}^X = X$, then the proof of [25, Theorem 3.4, p. 14] implies that there exists a strongly continuous operator family $(S(t))_{t\geq 0}$ in $L(X)$ such that $S_1(t)x = \int_0^t S(s)x\,ds$, $t \geq 0$, $x \in X$. The estimate (7) is a consequence of (10) and the remaining part of the proof of (iii) essentially follows from the corresponding part of the proof of [23, Theorem 6.2, p. 164]. Assuming $M' \geq 1$, $\omega' \geq 0$, a-regularity of $(S(t))_{t\geq 0}$ and $||a * S(t)x||_Y \leq M'e^{\omega' t}||x||_X$, $t \geq 0$, $x \in \overline{Y}^X$, the estimate (9) follows from a straightforward computation. The converse implication in (iv) follows from [25, Theorem 2.1], the uniform boundedness principle and the final part of the proof of [23, Theorem 6.2, p. 165].

Remark 1 *Assume $A(t)$ is of the form (5) and $a(t)$ as well as $B(t)$, in addition to the assumptions prescribed in Proposition 4, are of exponential growth. Owing to the proof of [23, Corollary 6.4, p. 166], the condition (N3) can be replaced by a slightly weaker condition:*

(N3)'

$$\sup_{n\in\mathbb{N}_0} \sup_{\lambda>\omega_0,\ \tilde{k}(\lambda)\neq 0} \frac{(\lambda-\omega)^{n+1}}{n!} \Big\| \frac{d^n}{d\lambda^n} H(\lambda) \Big\|_{L(X)} < \infty.$$

Now we state the complex characterization theorem for (A, k)-regularized C-pseudoresolvent families.

Theorem 2 *(i) Assume $A(t)$ satisfies* (6) *with some $\epsilon_0 \geq 0$, $k(t)$ satisfies* (P1), *$\omega_1 \geq \max(abs(k), \epsilon_0)$ and there exists an analytic mapping f :*

$\{\lambda \in \mathbb{C} : Re(\lambda) > \omega_1\} \to L(X)$ *such that* $f(\lambda)C = Cf(\lambda)$, $Re(\lambda) > \omega_1$,

$$f(\lambda)(I - \tilde{A}(\lambda))y = \tilde{k}(\lambda)Cy,\ Re(\lambda) > \omega_1,\ \tilde{k}(\lambda) \neq 0,\ y \in Y$$

and

$$\|f(\lambda)\|_{L(X)} \leq M|\lambda|^r,\ Re(\lambda) > \omega_1 \text{ for some } M \geq 1 \text{ and } r > 1.$$

Then, for every $\alpha > 1$, *there exists a norm continuous, exponentially bounded weak* $(A, k * g_{r+\alpha})$*-regularized* C*-pseudoresolvent family* $(S_\alpha(t))_{t\geq 0}$.

(ii) *Let* $(S_\alpha(t))_{t\geq 0}$ *be as in* (i) *and let* $a(t)$ *satisfy* (P1). *Then* $(S_\alpha(t))_{t\geq 0}$ *is* a*-regular provided that there exist* $M_1 \geq 1$, $r_1 > 1$, *a set* $P \subseteq \mathbb{C}$, *which has a limit point in* $\{\lambda \in \mathbb{C} : Re(\lambda) > \max(\omega_1, abs(a))\}$, *and an analytic mapping* $h : \{\lambda \in \mathbb{C} : Re(\lambda) > \max(\omega_1, abs(a))\} \to L(\overline{Y}^X, Y)$ *such that*

$$h(\lambda)(I - \tilde{A}(\lambda))y = \tilde{a}(\lambda)\frac{\tilde{k}(\lambda)}{\lambda^{r+\alpha}}Cy,\ y \in Y,\ Re(\lambda) > \max(\omega_1, abs(a)),$$

$$\|h(\lambda)\|_{L(\overline{Y}^X, Y)} \leq M_1|\lambda|^{-r_1},\ Re(\lambda) > \max(\omega_1, abs(a)),$$

and that $(I - \tilde{A}(\lambda))^{-1} : \overline{Y}^X \to Y$ *exists for all* $\lambda \in P$.

(iii) *Let, in addition to the assumptions given in* (i), *the mapping* $\lambda \mapsto f(\lambda) \in L(Y)$, $Re(\lambda) > \omega_1$ *be analytic in* $L(Y)$. *Suppose*

$$(I - \tilde{A}(\lambda))f(\lambda)y = \tilde{k}(\lambda)Cy,\ Re(\lambda) > \omega_1,\ \tilde{k}(\lambda) \neq 0,\ y \in Y \tag{13}$$

and

$$\|f(\lambda)\|_{L(Y)} \leq M|\lambda|^r,\ Re(\lambda) > \omega_1. \tag{14}$$

Then, for every $\alpha > 1$, $(S_\alpha(t))_{t\geq 0}$ *is a norm continuous, exponentially bounded* $(A, k * g_{r+\alpha})$*-regularized* C*-resolvent family, and* $(U_\alpha(t) \equiv \int_0^t S_\alpha(s)\,ds)_{t\geq 0}$ *is continuously differentiable in* $L(Y)$.

Proof. To prove (i), fix an $\alpha > 1$ and notice that $(f(\lambda) - \tilde{A}(\lambda)f(\lambda))/\lambda^{r+\alpha} = \tilde{k}(\lambda)/\lambda^{r+\alpha}Cy$, $y \in Y$, $Re(\lambda) > \omega_1$, $\tilde{k}(\lambda) \neq 0$. By [1, Theorem 2.5.1], one gets that there exists an exponentially bounded, continuous function $S_\alpha : [0,\infty) \to L(X)$ such that $S_\alpha(0) = 0$ and $\widetilde{S_\alpha}(\lambda) = f(\lambda)/\lambda^{r+\alpha}$, $Re(\lambda) > \omega_1$. Using the inverse Laplace transform, one immediately obtains that $(S_\alpha(t))_{t\geq 0}$ commutes with C and that the second resolvent equation holds, which completes the proof of (i). To prove (ii), one can use again [1, Theorem 2.5.1] in order to see that there exists an exponentially bounded function $S_\alpha^a : [0,\infty) \to L(\overline{Y}^X, Y)$ such that $S_\alpha^a(0) = 0$ and $\widetilde{S_\alpha^a}(\lambda) = h(\lambda)$, $Re(\lambda) > \omega_1$. It is checked at once that

$$\big(\widetilde{S_\alpha^a}(\lambda) - \tilde{a}(\lambda)\widetilde{S_\alpha}(\lambda)\big)(I - \tilde{A}(\lambda))y = 0,\ y \in Y,\ Re(\lambda) > \omega_1. \tag{15}$$

Since the mapping $(I - \tilde{A}(\lambda))^{-1} : \overline{Y}^X \to Y$ exists for all $\lambda \in P$, (15) implies $(\widetilde{S_\alpha^a}(\lambda) - \tilde{a}(\lambda)\widetilde{S_\alpha}(\lambda))x = 0$, $x \in \overline{Y}^X$, $\lambda \in P$. Hence, $(\widetilde{S_\alpha^a}(\lambda) - \tilde{a}(\lambda)\widetilde{S_\alpha}(\lambda))x = 0$, $x \in \overline{Y}^X$, $Re(\lambda) > \omega_1$ and $S_\alpha^a(t)x = (a * S_\alpha)(t)x$, $t \geq 0$, $x \in \overline{Y}^X$, which shows that $(S_\alpha(t))_{t\geq 0}$ is a-regular. To prove (iii), it suffices to notice that (14) implies $S_\alpha \in C([0,\infty) : L(Y))$, $U_\alpha'(t) = S_\alpha(t)$, $t \geq 0$ in $L(Y)$ and that the first resolvent equation is a consequence of (13).

Remark 2 *Assume $a \in L^1_{loc}([0,\tau))$, $(S(t))_{t\in[0,\tau)}$ is a (weak, weak a-regular) (A,k)-regularized C-(pseudo)resolvent family and $L^1_{loc}([0,\tau)) \ni b$ satisfies $b * k \neq 0$. Set $S_b(t)x := (b * S)(t)x$, $t \in [0,\tau)$, $x \in X$. Then it readily follows that $(S_b(t))_{t\in[0,\tau)}$ is a (weak, weak a-regular) $(A, b * k)$-regularized C-(pseudo)resolvent family. Furthermore, $(U_b(t)_{|Y})_{t\in[0,\tau)}$ is continuously differentiable in $L(Y)$ (cf. the proofs of [1, Proposition 1.3.6, Proposition 1.3.7]), provided that (S2) holds for $(S(t))_{t\in[0,\tau)}$, and $a * S_b(\cdot)x \in AC_{loc}([0,\tau) : Y)$, $x \in \overline{Y}^X$, provided that $(S(t))_{t\in[0,\tau)}$ is a-regular.*

Now we will transfer the assertion of [21, Proposition 2.5] to non-scalar Volterra equations.

Proposition 5 *Let $k \in AC_{loc}([0,\tau))$, $k(0) \neq 0$ and let $(S(t))_{t\in[0,\tau)}$ be a (weak, weak a-regular) (A,k)-regularized C-(pseudo)resolvent family. Then there exists $b \in L^1_{loc}([0,\tau))$ such that $(R(t) \equiv k(0)^{-1}S(t) + (b * S)(t))_{t\in[0,\tau)}$ is a (weak, weak a-regular) A-regularized C-(pseudo)resolvent family.*

Proof. Let $b \in L^1_{loc}([0,\tau))$ be such that $(b*k')(t) = -k(0)^{-1}k'(t) - k(0)b(t)$, $t \in [0,\tau)$ and

$$(b * k)(t) + k(0)^{-1}k'(t) = 1,\ t \in [0,\tau). \tag{16}$$

If $k(t) = k(0)$, $t \in [0,\tau)$ then (16) implies $(b*k)(t) = 0$, $t \in [0,\tau)$ and $b(t) = 0$ for a.e. $t \in [0,\tau)$; in this case, the statement of proposition is trivial. Assume now $b * k \neq 0$. By Remark 2, it suffices to show that $(R(t))_{t\in[0,\tau)}$ satisfies (S3)' if $(S(t))_{t\in[0,\tau)}$ satisfies it. Towards this end, fix $t \in [0,\tau)$, $y \in Y$ and put $U_R(s)x := \int_0^s R(r)x\,dr$, $s \in [0,\tau)$, $x \in X$. Integrating (16) and using (S3)' for $(S(t))_{t\in[0,\tau)}$, we obtain:

$$\begin{aligned} U_R(t)y &= \frac{1}{k(0)}\big(\Theta(t)Cx + A * U\big)(t)y + b * \big(\Theta C + A * U\big)(t)y \\ &= tCy + \frac{1}{k(0)}\big(A * U\big)(t)y + \big(b * A * U\big)(t)y = tCy + \big(A * U_R\big)(t)y. \end{aligned}$$

Similarly one can prove that $U_R(t)y = tCy + (U_R * A)(t)y$.

Concerning hyperbolic perturbation results, we have the following.

Theorem 3 *Assume* $L^1_{loc}([0,\tau)) \ni a$ *is a kernel,* $C(Y) \subseteq Y$, $\overline{Y}^X = X$, $B \in L^1_{loc}([0,\tau) : L(Y,[R(C)]))$ *is of the form*

$$B(t)y = B_0(t)y + \big(a * B_1\big)(t)y, \ t \in [0,\tau), \ y \in Y,$$

where $(B_0(t))_{t\in[0,\tau)} \subseteq L(Y) \cap L(X,[R(C)])$, $(B_1(t))_{t\in[0,\tau)} \subseteq L(Y,[R(C)])$,

(i) $C^{-1}B_0(\cdot)y \in BV_{loc}([0,\tau) : Y)$ *for all* $y \in Y$, $C^{-1}B_0(\cdot)x \in BV_{loc}([0,\tau) : X)$ *for all* $x \in X$,

(ii) $C^{-1}B_1(\cdot)y \in BV_{loc}([0,\tau) : X)$ *for all* $y \in Y$, *and*

(iii) $CB(t)y = B(t)Cy$, $y \in Y$, $t \in [0,\tau)$.

Then the existence of an a-regular A-regularized C-(pseudo)resolvent family $(S(t))_{t\in[0,\tau)}$ *is equivalent with the existence of an a-regular* $(A+B)$*-regularized C-(pseudo)resolvent family* $(R(t))_{t\in[0,\tau)}$.

Proof. Theorem 3 can be shown following the lines of the proof of [23, Theorem 6.1, p. 159] with $K_0 = S * C^{-1}B_0$ and $K_1 = S * C^{-1}B_1$. Assuming $(S(t))_{t\in[0,\tau)}$ is an a-regular A-regularized C-(pseudo)resolvent family, we will only prove that the resulting a-regular $(A+B)$-regularized C-(pseudo)resolvent family $(R(t))_{t\in[0,\tau)}$ commutes with C. In order to do that, define a family $(W(t))_{t\in[0,\tau)}$ in $L(X,Y)$ as a unique solution of the equation

$$W(t)x = (a * S)(t)x + d\big[K_0 + a * K_1\big] * W(t)x, \ t \in [0,\tau), \ x \in X.$$

Using the condition (iii), we obtain that $(K_0 + a * K_1)(t)Cy = C(K_0 + a * K_1)(t)y$, $t \in [0,\tau)$, $y \in Y$. Keeping in mind [23, Corollary 0.3, p. 15; (0.36), (0.38), p. 14] and the proof of [23, Theorem 0.5, p. 13], it follows that

$W(t)Cx = CW(t)x$, $t \in [0,\tau)$, $x \in X$. On the other hand, $(R(t))_{t\in[0,\tau)}$ is defined by

$$R(t)x = S(t)x + dK_1 * W(t)x + dK_0 * R(t)x,\ t \in [0,\tau),\ x \in X,$$

and the following equality holds $W(t)x = (a * R)(t)x$, $t \in [0,\tau)$, $x \in X$ (cf. [23, p. 160, l. -2]). Since $a(t)$ is a kernel and $W(t)Cx = CW(t)x$, $t \in [0,\tau)$, $x \in X$, the above implies that $(R(t))_{t\in[0,\tau)}$ commutes with C.

It is worthwhile to mention here that it is not clear how one can prove an analogue of Theorem 3 in the case of a general a-regular (A,k)-regularized C-(pseudo)resolvent family $(S(t))_{t\in[0,\tau)}$. From a practical point of view, the following corollary is crucially important; notice only that one can remove density assumptions in the cases set out below since the mapping $t \mapsto (a * S)(t)x$, $t \in [0,\tau)$ is continuous in Y for every fixed $x \in X$ (cf. [23, p. 160, l. -9] and [14]):

Corollary 1 *(i) Assume $L^1_{loc}([0,\tau)) \ni a$ is a kernel, A is a subgenerator of an a-regularized C-resolvent family $(S(t))_{t\in[0,\tau)}$, $Y = [D(A)]$ and*

$$A(t) = a(t)A + \big(a * B_1\big)(t) + B_0(t),\ t \in [0,\tau),$$

where $B_0(\cdot)$ and $B_1(\cdot)$ satisfy the assumptions of Theorem 3. *Assume that the following condition holds:*

$$A\int_0^t a(t-s)S(s)x\,ds = S(t)x - Cx,\ t \in [0,\tau),\ x \in E.$$

Then there is an a-regular A-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$.

(ii) Let A be a subgenerator of a (local) C-regularized semigroup $(S(t))_{t\in[0,\tau)}$. If $B_0(\cdot)$ and $B_1(\cdot)$ satisfy the assumptions of Theorem 3 *with $Y = [D(A)]$, then for every $x \in D(A)$ there exists a unique solution of the problem*

$$\begin{cases} u \in C^1([0,\tau) : X)\ \cap\ C([0,\tau) : [D(A)]), \\ u'(t) = Au(t) + \big(dB_0 * u\big)(t)x + \big(B_1 * u\big)(t) + Cx,\ t \in [0,\tau), \\ u(0) = 0. \end{cases}$$

Furthermore, the mapping $t \mapsto u(t)$, $t \in [0,\tau)$ is locally Lipschitz continuous in $[D(A)]$.

(iii) *Let A be a subgenerator of a (local) C-regularized cosine function $(C(t))_{t\in[0,\tau)}$. If $B_0(\cdot)$ and $B_1(\cdot)$ satisfy the assumptions of* Theorem 3 *with $Y = [D(A)]$, then for every $x \in D(A)$ there exists a unique solution of the problem*

$$\begin{cases} u \in C^2([0,\tau) : X) \ \cap \ C([0,\tau) : [D(A)]), \\ u''(t) = Au(t) + \big(dB_0 * u'\big)(t)x + \big(B_1 * u\big)(t) + Cx, \ t \in [0,\tau), \\ u(0) = u'(0) = 0. \end{cases}$$

Furthermore, the mapping $t \mapsto u(t)$, $t \in [0,\tau)$ is continuously differentiable in $[D(A)]$ and the mapping $t \mapsto u'(t)$, $t \in [0,\tau)$ is locally Lipschitz continuous in $[D(A)]$.

It is clear that Corollary 1 can be applied to a wide class of integro-differential equations in Banach spaces and that all aspects of application cannot be easily perceived.

Example 1 *Assume $1 \leq p \leq \infty$, $0 < \tau \leq \infty$, $n \in \mathbb{N}$, $X = L^p(\mathbb{R}^n)$ or $X = C_b(\mathbb{R}^n)$, $P(\cdot)$ is an elliptic polynomial of degree $m \in \mathbb{N}$, $\omega = \sup_{x\in\mathbb{R}^n} Re(P(x)) < \infty$ and $A = P(D)$. (Possible applications can be also made to non-elliptic abstract differential operators; cf. [25] and [31].) Then, for every $\omega' > \omega$ and $r > n|1/2-1/p|$, A generates an exponentially bounded $(\omega' - A)^{-r}$-regularized semigroup in X (cf. for example [19, Theorem 3.7] and [16, Theorem 2.3.26]), where the complex power $(\omega' - A)^{-r}$ is defined in the sense of [16, Subsection 1.4.2]. Let a completely positive kernel $a(t)$ satisfy (P1) and let $B_0(\cdot)$ and $B_1(\cdot)$ satisfy the assumptions of Corollary 2.13(i). Then [7, Theorem 2.8(ii)] (cf. also [20, Lemma 4.2]) implies that A is the integral generator of an exponentially bounded $(a, (\omega' - A)^{-r})$-regularized resolvent family provided $X = L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$; clearly, the above assertion holds if $a(t) \equiv 1$ and $X = L^\infty(\mathbb{R}^n)$ $(C_b(\mathbb{R}^n))$. An application of Corollary 1 gives that, in any of these cases, there exists an a-regular A-regularized $(\omega' - A)^{-r}$-resolvent family $(R(t))_{t\in[0,\tau)}$, where $A(t) = a(t)P(D) + (a * B_1)(t) + B_0(t)$, $t \in [0,\tau)$. By means of [7, Theorem 2.8(iii)], the preceding example can be set, with some obvious modifications, in the framework of the theory of C-regularized cosine functions. We refer the reader to [5], [7], [9]-[12], [16] and [30] for various examples of differential operators generating C-regularized cosine functions.*

The application of (A, k)-regularized C-pseudoresolvent families to problems in linear (thermo-)viscoelasticity and electrodynamics with memory (cf. [23, Chapter 9]) is almost completely confined to the case in which the underlying space X is Hilbert. In this context, we would like to propose the following problem (cf. also [23, p. 240] for the analysis of viscoelastic Timoshenko beam in case of non-synchronous materials).

Problem. Suppose $\mu_0 > 0$, $\varepsilon_0 > 0$, $\Omega_1 \subseteq \mathbb{R}^3$ is an open set with smooth boundary Γ, $\Omega_2 = \mathbb{R}^3 \setminus \Omega_1$ and $n(x)$ denotes the outer normal at $x \in \Gamma$ of Ω_1. Let $X := L^p(\Omega_1 : \mathbb{R}^3) \times L^p(\Omega_2 : \mathbb{R}^3) \times L^p(\Omega_1 : \mathbb{R}^3) \times L^p(\Omega_2 : \mathbb{R}^3)$, $p \in [1, \infty] \setminus \{2\}$, and $||(u_1, u_2, u_3, u_4)|| := (\mu_0||u_1||^2 + \varepsilon_0||u_2||^2 + \mu_0||u_3||^2 + \varepsilon_0||u_4||^2)^{1/2}$, $u_1,\ u_3 \in L^p(\Omega_1 : \mathbb{R}^3)$, $u_2,\ u_4 \in L^p(\Omega_2 : \mathbb{R}^3)$. Define the operator A_0 in X by setting

$$D(A_0) := \big\{u \in X : u_1,\ u_2 \in H^{1,p}(\Omega_1 : \mathbb{R}^3),\ u_3,\ u_4 \in H^{1,p}(\Omega_2 : \mathbb{R}^3),$$
$$n \times (u_1 - u_3) = n \times (u_2 - u_4) = 0\big\}$$

and

$$A_0 u := \big(-\mu_0^{-1}\text{curl } u_2, \varepsilon_0^{-1}\text{curl } u_1, -\mu_0^{-1}\text{curl } u_4, \varepsilon_0^{-1}\text{curl } u_3\big),\ u \in D(A_0).$$

Then one can simply prove that A_0 is closable. Does there exist an injective operator $C \in L(X)$ such that $\overline{A_0}$ generates a (local, global exponentially bounded) C-regularized semigroup in X?

Assuming the answer to the previous problem is in the affirmative and the functions $\varepsilon_i(\cdot)$, $\mu_i(\cdot)$, $\sigma_i(\cdot)$, $\nu_i(\cdot)$ and $\eta_i(\cdot)$ satisfy certain conditions (cf. [23, Subsection 9.6, pp. 251–253]), one can apply Corollary 1(ii) in the study of C-wellposedness of transmission problem for media with memory.

3 Smoothing properties of (A, k)-regularized C-pseudoresolvent families

Let (L_p) be a sequence of positive real numbers such that $L_0 = 1$,

(M.1) $L_p^{2p} \leq L_{p+1}^{p+1} L_{p-1}^{p-1},\ p \in \mathbb{N}$,

(M.2) $L_n^n \leq AH^n \min_{p,q\in\mathbb{N},\ p+q=n} L_p^p L_q^q,\ n \in \mathbb{N}$ for some $A > 1$ and $H > 1$, and

(M.3)' $\sum_{p=1}^{\infty} \frac{L_{p-1}^{p-1}}{L_p^p} < \infty$.

The Gevrey sequences $(p!^{s/p})$, (p^s) and $(\Gamma(1+ps)^{1/p})$ satisfy the above conditions with $s > 1$. The associated function of (L_p) is defined by $M(\lambda) := \sup_{p\in\mathbb{N}_0} \ln(|\lambda|^p/L_p^p)$, $\lambda \in \mathbb{C} \setminus \{0\}$, $M(0) := 0$. Recall, the mapping $t \mapsto$

$M(t),\ t \geq 0$ is increasing, absolutely continuous, $\lim_{t\to\infty} M(t) = +\infty$ and $\lim_{t\to\infty}(M(t)/t) = 0$. Define $\omega_L(t) := \sum_{p=0}^{\infty}(t^p/L_p^p),\ t \geq 0,\ M_p := L_p^p$ and, for every $\alpha \in (0,\pi]$, $\Sigma_\alpha := \{\lambda \in \mathbb{C} : \lambda \neq 0,\ |\arg(\lambda)| < \alpha\}$.

Definition 3 *Let* $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $k \neq 0$, $A \in L^1_{loc}([0,\tau) : L(Y,X))$ *and* $\alpha \in (0,\pi]$.

(i) Assume $(S(t))_{t\geq 0}$ *is a (weak)* (A,k)*-regularized* C*-(pseudo)resolvent family. Then it is said that* $(S(t))_{t\geq 0}$ *is an analytic (weak)* (A,k)*-regularized* C*-(pseudo)resolvent family of angle* α, *if there exists an analytic function* $\mathbf{S} : \Sigma_\alpha \to L(X)$ *satisfying* $\mathbf{S}(t) = S(t),\ t > 0$ *and* $\lim_{z\to 0,\ z\in\Sigma_\gamma} \mathbf{S}(z)x = k(0)Cx$ *for all* $\gamma \in (0,\alpha)$ *and* $x \in X$. *It is said that* $(S(t))_{t\geq 0}$ *is an exponentially bounded, analytic (weak)* (A,k)*-regularized* C*-(pseudo)resolvent family, resp. bounded analytic (weak)* (A,k)*-regularized* C*-(pseudo)resolvent family of angle* α, *if for every* $\gamma \in (0,\alpha)$, *there exist* $M_\gamma > 0$ *and* $\omega_\gamma \geq 0$, *resp.* $\omega_\gamma = 0$, *such that* $||\mathbf{S}(z)||_{L(X)} \leq M_\gamma e^{\omega_\gamma |z|},\ z \in \Sigma_\gamma$.

Since no confusion seems likely, we shall also write $S(\cdot)$ *for* $\mathbf{S}(\cdot)$.

(ii) Assume $(S(t))_{t\in[0,\tau)}$ *is a (weak)* (A,k)*-regularized* C*-(pseudo)resolvent family and the mapping* $t \mapsto S(t),\ t \in (0,\tau)$ *is infinitely differentiable (in the strong topology of* $L(X)$*). Then it is said that* $(S(t))_{t\in[0,\tau)}$ *is of class* C^L, *resp. of class* C_L, *iff for every compact set* $K \subseteq (0,\tau)$ *there exists* $h_K > 0$, *resp. for every compact set* $K \subseteq (0,\tau)$ *and for every* $h > 0$:

$$\sup_{t\in K,\ p\in\mathbb{N}_0} \Big\| \frac{h_K^p \frac{d^p}{dt^p} S(t)}{L_p^p} \Big\|_{L(X)} < \infty, \ \textit{resp.} \ \sup_{t\in K,\ p\in\mathbb{N}_0} \Big\| \frac{h^p \frac{d^p}{dt^p} S(t)}{L_p^p} \Big\|_{L(X)} < \infty;$$

$(S(t))_{t\in[0,\tau)}$ *is said to be* ρ*-hypoanalytic,* $1 \leq \rho < \infty$, *if* $(S(t))_{t\in[0,\tau)}$ *is of class* C^L *with* $L_p = p!^{\rho/p}$.

The careful inspection of the proofs of structural characterizations of analytic K-convoluted C-semigroups (cf. [16, Section 2.4]) implies the validity of the following theorem.

Theorem 4 *(i) Assume* $\epsilon_0 \geq 0$, $k(t)$ *satisfies* (P1), $\omega \geq \max(abs(k), \epsilon_0)$, (6) *holds,* $(S(t))_{t\geq 0}$ *is a weak analytic* (A,k)*-regularized* C*-pseudoresolvent family of angle* $\alpha \in (0,\pi/2]$ *and*

$$\sup_{z\in\Sigma_\gamma} \big\| e^{-\omega z} S(z) \big\|_{L(X)} < \infty \ \textit{for all}\ \gamma \in (0,\alpha). \tag{17}$$

Then there exists an analytic mapping $H : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(X)$ *such that*

(a) $H(\lambda)(I - \tilde{A}(\lambda))y = \tilde{k}(\lambda)Cy,\ y \in Y,\ Re(\lambda) > \omega,\ \tilde{k}(\lambda) \neq 0;$ $H(\lambda)C = CH(\lambda),\ Re(\lambda) > \omega,$

(b) $\sup_{\lambda \in \omega + \Sigma_{\frac{\pi}{2}+\gamma}} \|(\lambda - \omega)H(\lambda)\|_{L(X)} < \infty,\ \gamma \in (0, \alpha)$ *and*

(c) $\lim_{\lambda \to +\infty,\ \tilde{k}(\lambda) \neq 0} \lambda H(\lambda)x = k(0)Cx,\ x \in X.$

(ii) *Assume* $\epsilon_0 \geq 0$, $k(t)$ *satisfies* (P1), (6) *holds,* $\omega \geq \max(abs(k), \epsilon_0)$, $\alpha \in (0, \pi/2]$, *there exists an analytic mapping* $H : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(X)$ *such that* (a) *and* (b) *of the item* (i) *hold and that, in the case* $\overline{Y}^X \neq X$, (c) *also holds. Then there exists a weak analytic* (A, k)*-regularized* C*-pseudoresolvent family* $(S(t))_{t \geq 0}$ *of angle* α *such that* (17) *holds.*

Theorem 5 (i) *Assume* $\epsilon_0 \geq 0$, $k(t)$ *satisfies* (P1), $\omega_0 \geq \max(abs(k), \epsilon_0)$, (6) *holds,* $\alpha \in (0, \pi/2]$, $(S(t))_{t \geq 0}$ *is an analytic* (A, k)*-regularized* C*-resolvent family of angle* α*, the mapping* $t \mapsto U(t) \in L(Y)$, $t > 0$ *can be analytically extended to the sector* Σ_α *(we shall denote the analytical extensions of* $U(\cdot)$ *and* $S(\cdot)$ *by the same symbols), and*

$$\sup_{z \in \Sigma_\gamma} \|e^{-\omega_0 z} S(z)\|_{L(X)} + \sup_{z \in \Sigma_\gamma} \|e^{-\omega_0 z} S(z)\|_{L(Y)} < \infty \text{ for all } \gamma \in (0, \alpha). \tag{18}$$

Denote $H(\lambda)x = \int_0^\infty e^{-\lambda t} S(t)x\, dt,\ x \in X,\ Re(\lambda) > \omega_0$. *Then* (N1)-(N2) *hold,*

(a) $\sup_{\lambda \in \omega_0 + \Sigma_{\frac{\pi}{2}+\gamma}} (\|(\lambda - \omega_0)H(\lambda)\|_{L(X)} + \|(\lambda - \omega_0)H(\lambda)\|_{L(Y)}) < \infty$ *for all* $\gamma \in (0, \alpha)$, $H(\lambda)C = CH(\lambda)$, $Re(\lambda) > \omega_0$, *and*

(b) $\lim_{\lambda \to +\infty,\ \tilde{k}(\lambda) \neq 0} \lambda H(\lambda)x = k(0)Cx,\ x \in X.$

(ii) *Assume* $\alpha \in (0, \pi/2]$, $\epsilon_0 \geq 0$, $k(t)$ *satisfies* (P1), (6) *and* (N1)-(N2) *hold. Let* $\omega_0 \geq \max(abs(k), \epsilon_0)$. *Assume that* (a) *of the item* (i) *of this theorem holds and that, in the case* $\overline{Y}^X \neq X$, (b) *also holds. Then there exists an analytic* (A, k)*-regularized* C*-resolvent family* $(S(t))_{t \geq 0}$ *of angle* α *such that* (18) *holds and that the mapping* $t \mapsto U(t) \in L(Y)$, $t > 0$ *can be analytically extended to the sector* Σ_α.

Proof. Let $\gamma \in (0, \alpha)$ and $x \in X$. The validity of conditions (N1)-(N2) follows from the argumentation given in the proof of Theorem 1(i). The estimate $\sup_{\lambda \in \omega_0 + \Sigma_{\frac{\pi}{2}+\gamma}} ||(\lambda - \omega_0)H(\lambda)||_{L(X)} < \infty$ and the equality stated in (b) are consequences of [1, Theorem 2.6.1, Theorem 2.6.4(a)]. Since the mapping $t \mapsto U(t) \in L(Y)$, $t > 0$ can be analytically extended to the sector Σ_α, we easily obtain $U'(z) = S(z)$, $z \in \Sigma_\alpha$ in $L(Y)$. By [1, Theorem 2.6.1], $\sup_{\lambda \in \omega_0 + \Sigma_{\frac{\pi}{2}+\gamma}} ||(\lambda - \omega_0)H(\lambda)||_{L(Y)} < \infty$. Clearly, $H(\lambda)C = CH(\lambda)$, $Re(\lambda) > \omega_0$ and this completes the proof of (i). Let us prove (ii). By (N2), $(H(\lambda))_{Re(\lambda)>\omega_0}$ is analytic in both spaces, $L(X)$ and $L(Y)$. Using the condition (a) and [1, Theorem 2.6.1], we obtain the existence of analytic functions $S : \Sigma_\alpha \to L(X)$ and $S^Y : \Sigma_\alpha \to L(Y)$ such that $H(\lambda) = \int_0^\infty e^{-\lambda t} S(t)\, dt$, $Re(\lambda) > \omega_0$, $H(\lambda) = \int_0^\infty e^{-\lambda t} S^Y(t)\, dt$, $Re(\lambda) > \omega_0$ and that, for every $\gamma \in (0, \alpha)$, $\sup_{z \in \Sigma_\gamma} e^{-\omega_0 Re(z)}(||S(z)||_{L(X)} + ||S^Y(z)||_{L(Y)}) < \infty$. Set $S(0) := k(0)C$. Then, by the uniqueness theorem for Laplace transform, $S(t)C = CS(t)$, $t \geq 0$ and $S(t)y = S^Y(t)y$, $t > 0$, $y \in Y$, which simply implies that the mapping $t \mapsto U(t) \in L(Y)$, $t > 0$ can be analytically extended to the sector Σ_α as well as that (S2) and (S4) hold for $(S(t))_{t\geq 0}$. The strong continuity of $(S(t))_{t\geq 0}$ on any closed subsector of $\Sigma_\alpha \cup \{0\}$ follows from the condition (b) and [1, Proposition 2.6.3, Theorem 2.6.4(a)]. In particular, $(S(t))_{t\geq 0}$ satisfies (S1). By (8) and the inverse Laplace transform, one gets that (S3)' holds for $(S(t))_{t\geq 0}$. Hence, $(S(t))_{t\geq 0}$ is an analytic (A, k)-regularized C-resolvent family $(S(t))_{t\geq 0}$ of angle α. Assume now $\overline{Y}^X = X$. By the previous consideration, $S(t)y - k(t)Cy = \int_0^t S(t - s)A(s)y\, ds$, $t \geq 0$, $y \in Y$, which clearly implies $\lim_{t\downarrow 0} S(t)y = k(0)Cy$, $y \in Y$. Using the exponential boundedness of $(S(t))_{t\geq 0}$ and the standard limit procedure, we obtain $\lim_{t\downarrow 0} S(t)x = k(0)Cx$, $x \in X$. The above equality implies (b) by [1, Theorem 2.6.4(a)].

The main objective in the subsequent theorems is to clarify the basic differential properties of (A, k)-regularized C-pseudoresolvent families.

Theorem 6 *Assume $k(t)$ satisfies* (P1), $r \geq -1$ *and* (6) *holds with some* $\epsilon_0 \geq 0$. *Assume that there exists* $\omega \geq \max(abs(k), \epsilon_0)$ *such that, for every* $\sigma > 0$, *there exist* $C_\sigma > 0$, $M_\sigma > 0$, *an open neighborhood* $\Omega_{\sigma,\omega}$ *of the region*

$$\Lambda_{\sigma,\omega} = \big\{\lambda \in \mathbb{C} : Re(\lambda) \leq \omega,\ Re(\lambda) \geq -\sigma \ln |Im(\lambda)| + C_\sigma\big\} \\ \bigcup \big\{\lambda \in \mathbb{C} : Re(\lambda) \geq \omega\big\},$$

and an analytic mapping $h_\sigma : \Omega_{\sigma,\omega} \to L(X)$ *such that* $h_\sigma(\lambda)C = Ch_\sigma(\lambda)$, $Re(\lambda) > \omega$,

$$h_\sigma(\lambda)(I - \tilde{A}(\lambda))y = \tilde{k}(\lambda)Cy,\ y \in Y,\ Re(\lambda) > \omega,\ \tilde{k}(\lambda) \neq 0, \qquad (19)$$

and $||h_\sigma(\lambda)||_{L(X)} \leq M_\sigma|\lambda|^r,\ \lambda \in \Lambda_{\sigma,\omega}$. *Then, for every* $\zeta > 1$, *there exists a norm continuous, exponentially bounded weak* $(A, k * g_{r+\zeta})$*-regularized* C*-pseudoresolvent family* $(S_\zeta(t))_{t\geq 0}$ *satisfying that the mapping* $t \mapsto S_\zeta(t),\ t > 0$ *is infinitely differentiable in* $L(X)$. *If, additionally,* $h_\sigma(\lambda) \in L(Y)$ *for all* $\sigma > 0$, *and if the mapping* $\lambda \mapsto h_\sigma(\lambda),\ \lambda \in \Omega_{\sigma,\omega}$ *is analytic in* $L(Y)$ *as well as*

$$(I - \tilde{A}(\lambda))h_\sigma(\lambda)y = \tilde{k}(\lambda)Cy,\ y \in Y,\ Re(\lambda) > \omega,\ \tilde{k}(\lambda) \neq 0, \qquad (20)$$

and $||h_\sigma(\lambda)||_{L(Y)} \leq M_\sigma|\lambda|^r,\ \lambda \in \Lambda_{\sigma,\omega}$, *then* $(S_\zeta(t))_{t\geq 0}$ *is a norm continuous, exponentially bounded* $(A, k * g_{r+\zeta})$*-regularized* C*-resolvent family satisfying that the mapping* $t \mapsto S_\zeta(t),\ t \geq 0$ *is continuous in* $L(Y)$ *and that the mapping* $t \mapsto S_\zeta(t),\ t > 0$ *is infinitely differentiable in* $L(Y)$.

Proof. Assume $\zeta > 1$, $\sigma > 0$, $\varsigma > 0$, $\omega_0 > \omega$ and set $\Gamma^1 := \{\lambda \in \mathbb{C} : Re(\lambda) = 2C_\sigma - \sigma \ln(-Im(\lambda)),\ -\infty < Im(\lambda) \leq -e^{\frac{2C_\sigma}{\sigma}}\}$, $\Gamma^2 := \{\lambda \in \mathbb{C} : Re(\lambda) = \omega_0,\ -e^{\frac{2C_\sigma}{\sigma}} \leq Im(\lambda) \leq e^{\frac{2C_\sigma}{\sigma}}\}$, $\Gamma^3 := \{\lambda \in \mathbb{C} : Re(\lambda) = 2C_\sigma - \sigma \ln(Im(\lambda)),\ e^{\frac{2C_\sigma}{\sigma}} \leq Im(\lambda) < +\infty\}$, $\Gamma := \Gamma^1 \cup \Gamma^2 \cup \Gamma^3$ and $\Gamma_k := \{\lambda \in \Gamma : |\lambda| \leq k\}$, $k \in \mathbb{N}$. Let $k_0 \in \mathbb{N}$ be sufficiently large. Then we assume that the curves Γ and Γ_k are oriented so that $Im(\lambda)$ increases along Γ and Γ_k, $k \in \mathbb{N}$, $k \geq k_0$. Set $S^k_\zeta(t) := \frac{1}{2\pi i}\int_{\Gamma_k} e^{\lambda t}\lambda^{-r-\zeta}h_\sigma(\lambda)\,d\lambda$, $t \geq 0$, $k \in \mathbb{N}$, $k \geq k_0$. Then it is straightforward to verify that $\frac{d^j}{dt^j}S^k_\zeta(t) = \frac{1}{2\pi i}\int_{\Gamma_k} e^{\lambda t}\lambda^{j-r-\zeta}h_\sigma(\lambda)\,d\lambda$, $t \geq 0$, $k,\ j \in \mathbb{N}$, $k \geq k_0$. Furthermore, the proof of [15, Theorem 2.5] implies that, for every $j \in \mathbb{N}_0$, the sequence $(\frac{d^j}{dt^j}S^k_\zeta(t))_{k\geq k_0}$ is convergent in $L(X)$ for $t > \max(0, \frac{j+1-\zeta}{\sigma}) =: a_{j,\sigma,\zeta}$ and that the convergence is uniform on every compact subset of $[a_{j,\sigma,\zeta} + \varsigma, \infty)$. Put $S_{j,\zeta}(t) := \lim_{k\to\infty}\frac{d^j}{dt^j}S^k_\zeta(t)$, $j \in \mathbb{N}_0$, $t > a_{j,\sigma,\zeta}$. Then the mapping $t \mapsto S_{0,\zeta}(t),\ t > a_{j+1,\sigma,\zeta} + \varsigma$ is j-times differentiable in $L(X)$, $\frac{d^j}{dt^j}S_{0,\zeta}(t) = S_{j,\zeta}(t),\ t > a_{j+1,\sigma,\zeta} + \varsigma$,

$$S_{0,\zeta}(t) = S_\zeta(t) := \frac{1}{2\pi i}\int\limits_{\omega_0 - i\infty}^{\omega_0 + i\infty} e^{\lambda t}\frac{h_\sigma(\lambda)}{\lambda^{r+\zeta}}\,d\lambda,\ t \geq \frac{1}{\sigma}, \qquad (21)$$

$S_\zeta(t)C = CS_\zeta(t)$, $t \geq 0$ and $S_\zeta(0) = 0$. The arbitrariness of $\sigma > 0$ combined with the proof of [1, Theorem 2.5.1] yields that the mapping $t \mapsto S_\zeta(t),\ t \geq 0$ is continuous in $L(X)$ and that the mapping $t \mapsto S_\zeta(t),\ t > 0$ is infinitely differentiable in $L(X)$. Using the inverse Laplace transform, we easily get from (19) that $(S_\zeta(t))_{t\geq 0}$ is a weak $(A, k * g_{r+\zeta})$-regularized C-pseudoresolvent family $(S_\zeta(t))_{t\geq 0}$, finishing the proof of the first part of theorem. Assume now $h_\sigma(\lambda) \in L(Y)$, $\sigma > 0$, the mapping $\lambda \mapsto h_\sigma(\lambda),\ \lambda \in \Omega_{\sigma,\omega}$ is analytic in $L(Y)$, (20) holds and $||h_\sigma(\lambda)||_{L(Y)} \leq M_\sigma|\lambda|^r,\ \lambda \in \Lambda_{\sigma,\omega}$. Then the improper integral appearing in (21) converges in $L(Y)$ and the above arguments imply that the mapping $t \mapsto S_\zeta(t),\ t \geq 0$ is continuous in $L(Y)$. Furthermore, the mapping $t \mapsto S_\zeta(t),\ t > 0$ is infinitely differentiable in $L(Y)$. Denote $U_\zeta(t)y = \int_0^t S_\zeta(s)y\,ds,\ t \geq 0,\ y \in Y$. Certainly, $U'_\zeta(t) = S_\zeta(t),\ t \geq 0$ in $L(Y)$. The conditions (S2) and (S4) for $(S_\zeta(t))_{t\geq 0}$ follows easily from the previous equality whereas the condition (S3)' follows from the equality (20) by performing the inverse Laplace transform.

Notice that it is not clear in which way one can transfer the assertions of [15, Theorem 2.8(iii)-(iv)] to non-scalar Volterra equations.

Theorem 7 *Suppose* $k(t)$ *is a kernel and satisfies* (P1), (6) *holds with some* $\epsilon_0 \geq 0$, (M.1)-(M.3)' *hold for* (L_p), $(S(t))_{t\in[0,\tau)}$ *is a (local) weak* (A,k)*-regularized* C*-pseudoresolvent family,* $\omega \geq \max(abs(k), \epsilon_0)$, $m \in \mathbb{N}$ *and* $\overline{Y}^X = X$. *Set, for every* $\varepsilon \in (0,1)$ *and a corresponding* $K_\varepsilon > 0$,

$$F_{\varepsilon,\omega} := \big\{\lambda \in \mathbb{C} : Re(\lambda) \geq -\ln \omega_L(K_\varepsilon |Im(\lambda)|) + \omega\big\}.$$

Assume that, for every $\varepsilon \in (0,1)$, *there exist* $C_\varepsilon > 0$, $M_\varepsilon > 0$, *an open neighborhood* $O_{\varepsilon,\omega}$ *of the region* $G_{\varepsilon,\omega} = \{\lambda \in \mathbb{C} : Re(\lambda) \geq \omega,\ \tilde{k}(\lambda) \neq 0\} \cup \{\lambda \in F_{\varepsilon,\omega} : Re(\lambda) \leq \omega\}$, *and analytic mappings* $f_\varepsilon : O_{\varepsilon,\omega} \to \mathbb{C}$, $g_\varepsilon : O_{\varepsilon,\omega} \to L(Y,X)$ *and* $h_\varepsilon : O_{\varepsilon,\omega} \to L(X)$ *such that:*

(i) $f_\varepsilon(\lambda) = \tilde{k}(\lambda),\ Re(\lambda) > \omega,\ g_\varepsilon(\lambda) = \tilde{A}(\lambda),\ Re(\lambda) > \omega,\ h_\varepsilon(\lambda)C = Ch_\varepsilon(\lambda),\ Re(\lambda) > \omega$,

(ii) $h_\varepsilon(\lambda)(I - g_\varepsilon(\lambda))y = f_\varepsilon(\lambda)Cy,\ y \in Y,\ \lambda \in F_{\varepsilon,\omega}$,

(iii) $||h_\varepsilon(\lambda)||_{L(X)} \leq M_\varepsilon(1+|\lambda|)^m e^{\varepsilon|Re(\lambda)|},\ \lambda \in F_{\varepsilon,\omega},\ Re(\lambda) \leq \omega$ *and* $||h_\varepsilon(\lambda)||_{L(X)} \leq M_\varepsilon(1+|\lambda|)^m,\ Re(\lambda) \geq \omega$.

Then $(S(t))_{t\in[0,\tau)}$ *is of class* C^L. *Assume* $(S(t))_{t\in[0,\tau)}$ *is an* (A,k)*-regularized* C*-resolvent family and, in addition to the above assumptions,* $h_\varepsilon(\lambda) \in L(Y)$ *for all* $\varepsilon \in (0,1)$ *and* $\lambda \in O_{\varepsilon,\omega}$. *Let the mapping* $\lambda \mapsto h_\varepsilon(\lambda),\ \lambda \in O_{\varepsilon,\omega}$ *be analytic in* $L(Y)$ *and let:*

(ii)' $(I - g_\varepsilon(\lambda))h_\varepsilon(\lambda)y = f_\varepsilon(\lambda)Cy,\ y \in Y,\ \lambda \in F_{\varepsilon,\omega}$,

(iii)' $||h_\varepsilon(\lambda)||_{L(Y)} \leq M_\varepsilon(1+|\lambda|)^m e^{\varepsilon|Re(\lambda)|},\ \lambda \in F_{\varepsilon,\omega},\ Re(\lambda) \leq \omega$ *and* $||h_\varepsilon(\lambda)||_{L(Y)} \leq M_\varepsilon(1+|\lambda|)^m,\ Re(\lambda) \geq \omega$ *for all* $\varepsilon \in (0,1)$.

Then, for every compact set $K \subseteq (0,\tau)$, *there exists* $h_K > 0$ *such that*

$$\sup_{t\in K,\ p\in\mathbb{N}_0} \Big\| \frac{h_K^p \frac{d^p}{dt^p} S(t)}{L_p^p} \Big\|_{L(Y)} < \infty.$$

Proof. Combining Theorem 2(i), Cauchy formula, the proof of [1, Theorem 2.5.1] and (iii), it follows that there exists an exponentially bounded, weak $(A, k * g_{m+2})$-regularized C-pseudoresolvent family $(S_{m+2}(t))_{t\geq 0}$ such that, for every $\varepsilon \in (0,1)$, $x \in X$ and $t \in [0,\tau)$, one has $S_{m+2}(t)x = \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} e^{\lambda t}\lambda^{-m-2}h_\varepsilon(\lambda)x\,d\lambda$. Making use of Proposition 1(ii), we get that $S_{m+2}(t)x = \int_0^t g_{m+2}(t-s)S(s)x\,ds$, $x \in \overline{Y}^X = X$. On the other hand, (M.3)' holds for (L_n), which implies by [13, (4.5), (4.7), p. 56] that $\lim_{\lambda\to+\infty}(M(\lambda)/\lambda) = 0$ and $\lim_{n\to\infty}(n/m_n) = 0$. Hence, there exists $c > 0$ such that $M(\lambda) \leq c\lambda$, $\lambda \geq 0$ and

$$\frac{\omega_L'(t)}{\omega_L(t)} = \frac{\sum\limits_{n=1}^{\infty}\frac{nt^{n-1}}{M_n}}{\sum\limits_{n=0}^{\infty}\frac{t^n}{M_n}} \leq c\frac{\sum\limits_{n=1}^{\infty}\frac{t^{n-1}}{M_{n-1}}}{\sum\limits_{n=0}^{\infty}\frac{t^n}{M_n}} = c,\ t \geq 0. \tag{22}$$

It is evident that, for every $\varepsilon \in (0,1)$, there exists a unique number $a_\varepsilon > 0$ such that $\omega_L(K_\varepsilon a_\varepsilon) = 1$. Define now $\Gamma_\varepsilon := \Gamma_{1,\varepsilon} \cup \Gamma_{2,\varepsilon} \cup \Gamma_{3,\varepsilon}$, where $\Gamma_{1,\varepsilon} := \{-\ln(K_\varepsilon s) + \omega + is : s \in (-\infty, -a_\varepsilon]\}$, $\Gamma_{2,\varepsilon} := \{\omega + is : s \in [-a_\varepsilon, a_\varepsilon]\}$ and $\Gamma_{3,\varepsilon} := \{-\ln(K_\varepsilon s) + \omega + is : s \in [a_\varepsilon, \infty)\}$. Set, for every $\varepsilon \in (0,1)$ and $x \in X$,

$$S_{m+2,\varepsilon}(t)x := \frac{1}{2\pi i}\int\limits_{\Gamma_\varepsilon} e^{\lambda t}\frac{h_\varepsilon(\lambda)x}{\lambda^{m+2}}\,d\lambda,\ t > \varepsilon. \tag{23}$$

By the proof of [13, Proposition 4.5, p. 58], we have $\omega_L(s) \leq 2e^{M(2s)}$, $t \geq 0$ and $\ln\omega_L(K_\varepsilon s) \leq \ln 2 + M(2K_\varepsilon s)$, $s \geq 0$. Using (22) and (iii), we obtain that there exists $c_\varepsilon > 0$ such that, for every $x \in X$ and $t > \varepsilon$:

$$\begin{aligned}\left\|S_{m+2,\varepsilon}(t)\right\|_{L(X)} \leq \frac{1}{2\pi}\Big(&c_\varepsilon + 2e^{(\omega+\varepsilon)t}\\ &\times\int_{a_\varepsilon}^{\infty}\omega_L(K_\varepsilon s)^{\varepsilon-t}(1+\omega+s+\ln 2+2K_\varepsilon cs)^{-2}ds\Big),\end{aligned}$$

which implies that the improper integral appearing in (23) is convergent and $S_{m+2,\varepsilon}(t) \in L(X)$, $t > \varepsilon$. An elementary contour argument shows that $S_{m+2}(t) = S_{m+2,\varepsilon}(t)$, $t > \varepsilon$. Making use of the dominated convergence theorem, we obtain similarly that the mapping $t \mapsto S_{m+2}(t)$, $t > 0$ is infinitely differentiable in $L(X)$ with

$$\frac{d^n}{d\lambda^n}S_{m+2}(t)x = \frac{1}{2\pi i}\int\limits_{\Gamma_\varepsilon} e^{\lambda t}\lambda^{n-m-2}h_\varepsilon(\lambda)x\,d\lambda,\ t > \varepsilon,\ x \in X,\ n \in \mathbb{N}_0. \tag{24}$$

Suppose $K \subseteq (0,\tau)$ is compact. Let $k \in \mathbb{N}$, $\varepsilon \in (0,1)$ and let $\inf K - \varepsilon > k^{-1}$. Then there exists $c'_\varepsilon > 1$ such that $|-\ln \omega_L(K_\varepsilon s) + \omega + is| \leq c'_\varepsilon s,\ s \geq a_\varepsilon$. Let $h_K \in (0, K_\varepsilon / c'_\varepsilon)$. By (M.2), it follows inductively that

$$M_{kn} \leq A^{k-1} H^{k(k+1)/2} M_n^k,\ n \in \mathbb{N}_0. \tag{25}$$

Now one can apply (24)-(25) in order to see that there exists $c_K > 0$ such that, for every $n \in \mathbb{N}_0$ and $t \in K$:

$$\begin{aligned}
&\Big\| \frac{h_K^n \frac{d^n}{d\lambda^n} S_{m+2}(t)}{M_n} \Big\|_{L(X)} \\
&\leq \frac{c_K}{2\pi} \Big(\omega_L\big(h_K(\omega + a_\varepsilon)\big) + 2e^{(\omega+\varepsilon)t} \int\limits_{a_\varepsilon}^{\infty} \omega_L\big(K_\varepsilon s\big)^{-1/k} \frac{\big(c'_\varepsilon h_K s\big)^n}{M_n} s^{-2}\, ds \Big) \\
&\leq \frac{c_K}{2\pi} \Big(\omega_L\big(h_K(\omega + a_\varepsilon)\big) + 2e^{(\omega+\varepsilon)t} \int\limits_{a_\varepsilon}^{\infty} \frac{M_{kn}^{1/k}}{M_n} \frac{\big(c'_\varepsilon h_K s\big)^n s^{-2}}{\big(K_\varepsilon s\big)^n}\, ds \Big) \\
&\leq \frac{c_K}{2\pi} \Big(\omega_L\big(h_K(\omega + a_\varepsilon)\big) + \frac{2}{a_\varepsilon} e^{(\omega+\varepsilon)t} A^{(k-1)/k} H^{(k+1)/2} \Big(\frac{c'_\varepsilon h_K}{K_\varepsilon}\Big)^n \Big) \\
&\leq \frac{c_K}{2\pi} \Big(\omega_L\big(h_K(\omega + a_\varepsilon)\big) + \frac{2}{a_\varepsilon} e^{(\omega+\varepsilon)t} A^{(k-1)/k} H^{(k+1)/2} \Big).
\end{aligned}$$

This implies that the set $\{(h_K^n \frac{d^n}{dt^n} S_{m+2}(t)/M_n) : t \in K,\ n \in \mathbb{N}_0\}$ is bounded in $L(X)$. As a consequence of the condition (M.2), the set $\{(h_K^n \frac{d^n}{dt^n} S(t)/M_n) : t \in K,\ n \in \mathbb{N}_0\}$ is also bounded in $L(X)$, which shows that $(S(t))_{t\in[0,\tau)}$ is of class C^L. The remaining part of proof follows exactly in the same way as in the proof of Theorem 6.

Note that (M.3)' does not hold if $L_p = p!^{1/p}$ and that the preceding theorem remains true in this case; then, in fact, we obtain the sufficient conditions for the generation of real analytic C-(pseudo)resolvents. Furthemore, [7, Theorem 2.24] can be reformulated in non-scalar case and the set $F_{\varepsilon,\omega}$ appearing in the formulation of Theorem 7 can be interchanged by the set $F_{\varepsilon,\omega,\rho} = \{\lambda \in \mathbb{C} : Re(\lambda) \geq -K_\varepsilon |Im(\lambda)|^{1/\rho} + \omega\}$, provided $L_p = p!^{\rho/p}$ and $1 \leq \rho < \infty$.

Several examples of (differentiable) (a, C)-regularized resolvent families of class C^L (C_L) can be found in [3], [15], [24] and [28]. Combining with Corollary 1(i) and the following observation, one can simply construct examples of (differentiable, in general, non-analytic) A-regularized C-resolvent families of class C^L (C_L). Let $(S(t))_{t\in[0,\tau)}$ be an (a, C)-regularized resolvent family of class C^L (C_L) and let the assumptions of Theorem 3 hold with

$Y = [D(A)]$ and $B_1 = 0$. Assume, in addition, $C^{-1}B_0 \in C^{\infty}([0,\tau) : L(X))$ is of class C^L (C_L), with the notion understood in the sense of Definition 3(ii), and $(C^{-1}B_0)^{(i)}(0) = 0$, $i \in \mathbb{N}_0$. Denote by L the solution of the equation $L = K_0 + dK_0 * L$ in $BV_{loc}([0,\tau) : L(X))$, where $K_0(t) = (S * C^{-1}B_0)(t)$, $t \in [0,\tau)$. Let $A(t) = a(t)A + B_0(t)$, $t \in [0,\tau)$ and let $(R(t))_{t\in[0,\tau)}$ be an A-regularized C-resolvent family given by Corollary 2.13(i). Then one can straightforwardly check that $L \in C^{\infty}([0,\tau) : L(X))$ is of class C^L (C_L) and that $L^{(i)}(0) = 0$, $i \in \mathbb{N}_0$. Taking into account the proof of [23, Theorem 6.1] (cf. also [23, (6.20), p. 160] and [23, Corollary 0.3, p. 15]), it follows that $R^{(n)}(t) = S^{(n)}(t) + \int_0^t L^{(n+1)}(t-s)S(s)\,ds$, $t \in [0,\tau)$, $n \in \mathbb{N}_0$. This implies that $(R(t))_{t\in[0,\tau)}$ is of class C^L (C_L). Using the same method, we are in a position to construct examples of analytic A-regularized C-resolvent families (in general, the angle of analyticity of such resolvent families may be strictly greater than $\pi/2$, cf. [2, Theorem 3.3] and [16, Theorem 2.4.19]):

Example 2 *The isothermal motion of a one-dimensional body with small viscosity and capillarity ([4], [8], [29]) is described, in the simplest situation, by the system:*

$$\begin{cases} u_t = 2au_{xx} + bv_x - cv_{xxx}, \\ v_t = u_x, \\ u(0) = u_0,\ v(0) = v_0, \end{cases}$$

where a, b *and* c *are positive constants. The associated matrix of polynomials (cf. [17] and [28]-[29] for more details)* $P(x) \equiv \begin{bmatrix} -2ax^2 & ibx + icx^3 \\ ix & 0 \end{bmatrix}$ *is Shilov* 2*-parabolic. Let* $X = L^p(\mathbb{R}) \times L^p(\mathbb{R})$ $(1 \leq p < \infty)$ *be equipped with the norm* $||(f,g)|| := ||f||_{L^p(\mathbb{R})} + ||g||_{L^p(\mathbb{R})}$, $f,\ g \in L^p(\mathbb{R})$. *Then it is well known that the operator* $P(D)$, *considered with its maximal distributional domain, is closed and densely defined in* X.

- *(i) ([17]) Let* $a^2 - c < 0$ *and* $r' \geq 1/2$. *Then* $P(D)$ *is the integral generator of an exponentially bounded, analytic* $(1-\Delta)^{-r'}$*-regularized semigroup* $(S_{r'}(t))_{t\geq 0}$ *of angle* $\arctan(a/\sqrt{c-a^2})$.
- *(ii) ([29]) Let* $a^2 - c = 0$ *and* $r' > 3/4$. *Then* $P(D)$ *is the integral generator of a bounded analytic* $(1-\Delta)^{-r'}$*-regularized semigroup* $(S_{r'}(t))_{t\geq 0}$ *of angle* $\pi/2$.
- *(iii) ([17]) Let* $a^2 - c > 0$ *and* $r' \geq 1/2$. *Then* $P(D)$ *is the integral generator of an exponentially bounded, analytic* $(1-\Delta)^{-r'}$*-regularized semigroup* $(S_{r'}(t))_{t\geq 0}$ *of angle* $\pi/2$.

Assume, in any of these cases, $\psi_1,\ \psi_2 \in \mathrm{S}^{2r',1}(\mathbb{R})$*, where the fractional Sobolev space* $\mathrm{S}^{2r',1}(\mathbb{R})$ *is defined in the sense of [22, Definition 12.3.1, p. 297],* $B_1 = 0$, $B_0(z)\binom{f}{g} = z\binom{\psi_1 * f}{\psi_2 * g}$ *and* $K(z)\binom{f}{g} = (S_{r'} * (1-\Delta)^{r'} B_0)(z)\binom{f}{g}$, $z \in \Sigma_\alpha$, $f,\ g \in L^p(\mathbb{R})$*, where* $\alpha = \arctan(a/\sqrt{c-a^2})$*, provided that (i) holds, resp.* $\alpha = \pi/2$*, provided that (ii) or (iii) holds. Let* $K \subseteq \Sigma_\alpha$ *be compact and let* $\gamma \in (0, \alpha)$ *satisfy* $K \subseteq \Sigma_\gamma$*. Then there exist*

$$\delta \in \Bigl(0, \frac{1}{(1+\sup K)\bigl(1 + \bigl\|(1-\Delta)^{r'}\psi_1\bigr\|_{L^1(\mathbb{R})} + \bigl\|(1-\Delta)^{r'}\psi_2\bigr\|_{L^1(\mathbb{R})}\bigr)}\Bigr),$$

$M_\gamma \geq 1$, $\omega_\gamma \geq 0$ *and* $\omega'_\gamma > \omega_\gamma$ *such that*

$$\Bigl\|S^{(-1)}(z) \equiv \int_0^z S(s)\,ds\Bigr\|_{L(X)} \leq M_\gamma |z| e^{\omega_\gamma Re(z)} \leq \delta e^{\omega'_\gamma Re(z)},\ z \in \Sigma_\gamma.$$

Hence, $\|\int_0^z S^{(-1)}(z-s)S^{(-1)}(s)\,ds\|_{L(X)} \leq \delta^2 |z| e^{\omega'_\gamma Re(z)}$, $z \in \Sigma_\gamma$*. Define* $(K_n(z))$ *by* $K_0(z) := K(z)$, $z \in \Sigma_\alpha$ *and* $K_{n+1}(z) := \int_0^z dK(s) K_n(z-s)$, $z \in \Sigma_\alpha$, $n \in \mathbb{N}_0$*. Then, for every* $z \in \Sigma_\alpha$ *and* $n \in \mathbb{N}$, $K_n(z) = (\underbrace{K' * \cdots * K'}_{n} * K)(z)$*. By Young's inequality,*

$$\bigl\|K'_1(z)\bigr\|_{L(X)} \leq \delta^2 |z| \Bigl(\bigl\|(1-\Delta)^{r'}\psi_1\bigr\|_{L^1(\mathbb{R})} + \bigl\|(1-\Delta)^{r'}\psi_2\bigr\|_{L^1(\mathbb{R})}\Bigr)^2 e^{\omega'_\gamma Re(z)},$$

for any $z \in \Sigma_\gamma$*. Going on inductively, we obtain*

$$\begin{aligned} &\bigl\|K'_{n+1}(z)\bigr\|_{L(X)} \\ &\quad \leq \delta^{n+1} |z|^n \Bigl(\bigl\|(1-\Delta)^{r'}\psi_1\bigr\|_{L^1(\mathbb{R})} + \bigl\|(1-\Delta)^{r'}\psi_2\bigr\|_{L^1(\mathbb{R})}\Bigr)^{n+1} e^{\omega'_\gamma Re(z)}, \end{aligned}$$

for any $z \in \Sigma_\gamma$ *and* $n \in \mathbb{N}_0$*. Taken together, the preceding estimate and the Weierstrass theorem imply that the function* $z \mapsto \int_0^z \sum_{n=0}^\infty K'_n(z-s)S(s)\,ds$, $z \in \Sigma_\alpha$ *is analytic and that there exist* $M'_\gamma \geq 1$ *and* $\omega''_\gamma > \omega'_\gamma$ *such that* $\|\int_0^z \sum_{n=0}^\infty K'_n(z-s)S(s)\,ds\|_{L(X)} \leq M'_\gamma e^{\omega''_\gamma Re(z)}$, $z \in \Sigma_\gamma$*. Let* $(R_{r'}(t))_{t\geq 0}$ *be an* A*-regularized* C*-resolvent family given by Corollary 1(i). Since* $R(t) = S(t) + \int_0^t \sum_{n=0}^\infty K'_n(t-s)S(s)\,ds$, $t \geq 0$*, we have that* $(R_{r'}(t))_{t\geq 0}$ *is an exponentially bounded, analytic* 1*-regular* A*-regularized* C*-resolvent family of angle* α*. On the other hand,* $P(D)$ *does not generate a strongly continuous semigroup in* $L^1(\mathbb{R}) \times L^1(\mathbb{R})$ *([8]) and* $\rho(P(D)) \neq \emptyset$ *([17]). Combining this with Theorem 3 and Proposition 4, we get that there does not exist a local* A*-regularized pseudoresolvent family provided* $p = 1$*.*

Example 3 *Let* $X = L^p(\mathbb{R}),\ 1 \leq p \leq \infty$. *Consider the next multiplication operators with maximal domain in* X :

$$Af(x) =: 2xf(x),\ Bf(x) := \big(-x^4 + x^2 - 1\big)f(x),\ x \in \mathbb{R}.$$

Notice that $D(B) \subseteq D(A)$. *Let* $Y := [D(B)]$ *and let* $A \in L^1_{loc}([0,\infty) : L(Y,X))$ *be given by* $A(t)f := Af + tBf$, $t \geq 0$, $f \in D(B)$. *Assume, further,* $s \in (1,2)$, $\delta = 1/s$, $L_p = p!^{s/p}$ *and* $K_\delta(t) = \mathcal{L}^{-1}(\exp(-\lambda^\delta))(t),\ t \geq 0$, *where* $\mathcal{L}^{-1}$ *denotes the inverse Laplace transform. Then there exists a global (not exponentially bounded)* (A, K_δ)*-regularized resolvent family. Towards this end, it suffices to show that, for every* $\tau > 0$, *there exists a local* (A, K_δ)*-regularized resolvent family on* $[0,\tau)$. *Denote by* $M(t)$ *the associated function of the sequence* (L_p) *and denote* $\Lambda_{\alpha,\beta,\gamma} = \{\lambda \in \mathbb{C} : Re(\lambda) \geq \gamma^{-1}M(\alpha\lambda) + \beta\}$, $\alpha,\ \beta,\ \gamma > 0$. *It is obvious that there exists* $C_s > 0$ *such that* $M(\lambda) \leq C_s|\lambda|^{1/s},\ \lambda \in \mathbb{C}$. *Given* $\tau > 0$ *and* $d > 0$ *in advance, one can find* $\alpha > 0$ *and* $\beta > 0$ *such that* $\tau \leq \cos(\delta\pi/2)/(C_s\alpha^{1/s})$ *and that* $|\lambda^2 - 2x\lambda + (x^4 - x^2 + 1)| \geq d,\ \lambda \in \Lambda_{\alpha,\beta,1},\ x \in \mathbb{R}$. *Denote by* Γ *the upwards oriented frontier of the ultra-logarithmic region* $\Lambda_{\alpha,\beta,1}$, *and define, for every* $f \in X$, $x \in \mathbb{R}$ *and* $t \in [0, \cos(\delta\pi/2)/(C_s\alpha^{1/s}))$,

$$\big(S_\delta(t)f\big)(x) := \frac{1}{2\pi i}\int\limits_\Gamma \frac{\lambda^2 e^{\lambda t - \lambda^\delta} f(x)}{\lambda^2 - 2x\lambda + (x^4 - x^2 + 1)}\, d\lambda.$$

Then one can simply prove that $(S_\delta(t))_{t\in[0,\tau)}$ *is a local* (A, K_δ)*-regularized resolvent family and that the mapping* $t \mapsto S_\delta(t)$, $t \geq 0$ *is infinitely differentiable in the strong topologies of* $L(X)$ *and* $L(Y)$. *Moreover, in both spaces,* $L(X)$ *and* $L(Y)$,

$$\Big(\frac{d^p}{dt^p}S_\delta(t)f\Big)(x) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{\lambda^{p+2} e^{\lambda t - \lambda^\delta} f(x)}{\lambda^2 - 2x\lambda + (x^4 - x^2 + 1)}\, d\lambda,$$

for any $p \in \mathbb{N}_0$, $x \in \mathbb{R}$ *and* $f \in X$. *This implies that, for every compact set* $K \subseteq [0,\infty)$, *there exists* $h_K > 0$ *such that*

$$\sup_{t\in K,\ p\in\mathbb{N}_0}\Big(\Big\|\frac{h_K^p \frac{d^p}{dt^p}S_\delta(t)}{L_p^p}\Big\|_{L(X)} + \Big\|\frac{h_K^p \frac{d^p}{dt^p}S_\delta(t)}{L_p^p}\Big\|_{L(Y)}\Big) < \infty.$$

In particular, $(S_\delta(t))_{t\geq 0}$ *is s-hypoanalytic. Define now the function* $K_{1/2}(t)$ *by* $K_{1/2}(t) := \mathcal{L}^{-1}(\exp(-\lambda^{1/2}))(t),\ t \geq 0$. *Then we obtain similarly that* *there exists* $\tau_0 > 0$ *such that there exists a local* 2*-hypoanalytic* $(A, K_{1/2})$*-regularized resolvent family on* $[0, \tau_0)$*. Note also that the use of Fourier multipliers enables one to reveal that the preceding conclusions remain true in the case of the corresponding differential operators* $\pm A(t)$*, where*

$$A(t)f = -tf'''' - tf'' - 2if' - tf,\ t \geq 0,\ 1 < p < \infty,\ f \in Y = \mathrm{S}^{4,p}(\mathbb{R}).$$

Finally, the non-scalar equations on the line

$$u(t) = \int_0^\infty A(s)u(t-s)\,ds + \int_{-\infty}^t k(t-s)g'(s)\,ds,$$

where $g : \mathbb{R} \to X$, $A \in L^1_{loc}([0,\infty) : L(Y,X))$, $A \neq 0$, $k \in C([0,\infty))$, $k \neq 0$, and

$$u(t) = f(t) + \int_0^t A(t-s)u(s)\,ds,\ t \in (-\tau, \tau),$$

where $\tau \in (0, \infty]$, $f \in C((-\tau,\tau) : X)$ and $A \in L^1_{loc}((-\tau,\tau) : L(Y,X))$, $A \neq 0$ can be treated without any substantial changes ([18]).

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