Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 6, No. 1/2014

ON THE STABILITY OF THE ROTATING BÉNARD PROBLEM*

Lidia Palese†

Abstract

In this paper we study the nonlinear Lyapunov stability of the conduction-diffusion solution of the rotating Bénard problem.

We provide a method for a derivation of the optimum nonlinear stability bound. It allows us to derive a linearization principle in a larger sense, i.e. to prove that, if the principle of exchange of stabilities holds, the linear and nonlinear stability bounds are equal.
After reformulating the perturbation evolution equations in a suitable equivalent form, we derive the appropriate Lyapunov function and *for the first time* we find that the nonlinear stability bound is nothing else but the critical Rayleigh number obtained solving the linear instability problem of the conduction-diffusion solution.

MSC: 76E15, 76E30

Keywords: Stability - Bénard problem - Energy Method.

1 Introduction

The convective instability and the nonlinear stability of a homogeneous fluid in a gravitational field heated from below, the classical Bénard problem, is a well known interesting problem in several fields of fluid mechanics [1], [2], [3], [4], [5].

*Accepted for publication in revised form on April 28, 2013

†`palese@dm.uniba.it`, Dipartimento di Matematica, Università di Bari, Via E. Orabona 4, 70125 Bari, Italy;

The influence, on the stability problem of the mechanical equilibria, of effects such as a rotation field, a magnetic field, or some chemical reactions of reactive fluids, is a problem of a big importance in astrophisics, geophysics, oceanography, meteorology. This is why it has been largely studied, mostly in the Oberbeck- Boussinesq approximation [3]- [12].
In [6] is developed a nonlinear stability theory based on the choice of some modified energy function for the rotating Bénard problem, showing the dependence of the stability region on Prandtl number P_r, even if $P_r \geq 1$.
In previous articles [7] is found such a dependence only when $P_r < 1$.
In [7], in some range of values of the Taylor number T^2, is obtained the coincidence of the obtained critical Rayleigh number with that of the linear theory of Chandrasekhar [1].

In the case of rigid boundaries of the layer, from the linear stability theory is deduced the stabilizing effect of the rotation, while, in the nonlinear case, the rotation around a vertical axis has only a non-destabilizing effect [8]. In [11] [12] is studied the problem of the coincidence of the critical and nonlinear stability bounds, and in [11] the coincidence of linear and nonlinear stability parameters is deduced under some restriction on initial data.
In the magnetohydrodynamic case, in [13], for a fully ionized fluid is deduced the coincidence of linear and nonlinear stability parameters, , if the conduction diffusion solution is linearly stable, it is conditionally nonlinearly asymptotically stable.

The point of loss of linear stability is usually also a bifurcation point at which convective motions set in [4], [14]. In particular, subcritical instabilities may occur explaining unusual phenomena. Whence a special interest for the study of the relationsheep between linear and nonlinear stability bounds and, thus, of the linearization principle [15], [16], [17], [18], [19], [20], [21].

A linearization principle, in a larger sense of the coincidence of linear and nonlinear stability bounds, in convection problem was settled in [2], [22], where an energy, defined in terms of linear combinations of the concentration and temperature fields, was used.

In [23] is considered a Newtonian fluid mixture in a horizontal layer heated from below. The thermoanisotropic effects on the hidrodynamic stability of the mechanical equilibrium are evaluated.

Introducing some linear combinations of temperature and concentration, a system equivalent to the perturbation evolution equations is derived, generalizing the Joseph's method of parametric differentiation [23], [24], [25], [26], changing the given problem in an equivalent one with better symmetry properties, in order to obtain an optimum stability bound.

With symmetrization arguments for the involved linear operators the nonlinear stability bound is investigated and its detection is reduced to the solution of an algebraic system.

For reactive fluids of technological interest, chemical reactions [5] can give temperature and concentration gradients which influence the transport process and can alter hydrodynamic stabilities.
In [27] [28] is performed a nonlinear stability analysis of the conduction-diffusion solution of the Bénard problem, assuming the upper surface stress free and the lower one experiencing a catalyzed chemical reaction. In the case of coincidence of Prandtl and Schmidt numbers, the equality between linear and nonlinear stability bounds was proved, at least in the class of normal modes.

In this work, we consider a rotating Bénard problem for a homogeneous fluid in a horizontal layer with free boundaries, and we study the nonlinear stability of the thermodiffusive equilibrium.

Our idea was reformulating the mathematical problem of the nonlinear Lyapunov stability in one equivalent one, reducing the number of unknown fields, and, at the same time, obtaining a system of perturbation evolution equations in some *suitable* orthogonal subspaces, preserving the contribution of the skewsymmetric rotation term.

We consider the initial-boundary value problem for the perturbation fields (Section 2), formulate (Section 3) the mathematical problem in an equivalent form, in terms of suitable variables which represent solenoidal fields in a plane layer, that is poloidal and toroidal fields, and introduce (Section 4) some functions depending on some parameters [21], [23] *on a suitable linear space of admissible vector functions* and, *we derive a quadratic function* $E_{\mathcal{L}}$.

The inequality $\frac{dE_{\mathcal{L}}}{dt} \leq 0$, when $E_{\mathcal{L}}$ is positive definite, i.e. *a Lyapunov function*, represents a sufficient condition for global nonlinear Lyapunov stability.
Solving, with respect to normal modes, the Euler equations associated with the maximum problem arising from the energy inequality, and applying successively the Joseph's idea of differentiation of parameters [2], [23], we determine a sufficient condition of global nonlinear Lyapunov stability. *If the principle of exchange of stabilities holds, we recover the coincidence of the nonlinear stability parameter with the critical Rayleigh number of the linear instability obtained applying the classical normal modes technique to the eigenvalue problem governing the linear instability.*

2 The initial/boundary value problem for perturbation

In the framework of physics of continua, let us consider a homogeneous Newtonian fluid, subject to the gravity field $\vec{g}$, in a horizontal layer S bounded by the surfaces $z = 0$ and $z = d$ in a frame of reference $\{O, \vec{i}, \vec{j}, \vec{k}\}$, with $\vec{k}$ unit vector in the vertical upwards direction.

We assume the fluid, subject to a constant vertical adverse temperature gradient β, in rotation around the fixed vertical axis z with a constant angular velocity $\vec{\Omega} = \Omega\vec{k}$.

The motion which occurs in S, for an observer rotating around the same axis z with the same angular velocity $\vec{\Omega}$, in the Oberbeck-Boussinesq approximation, is described by the following equations [1]

$$\frac{\partial}{\partial t}\vec{v} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{\nabla p}{\rho_0} + [1 - \alpha(T - T_0)]\vec{g} + 2\vec{v}\times\vec{\Omega} + \nu\Delta\vec{v}, \qquad (1)$$

$$\frac{\partial}{\partial t}T + \vec{v}\cdot\nabla T = K_T\Delta\theta, \qquad (2)$$

$$\nabla\cdot\vec{v} = 0, \qquad (3)$$

where $\vec{v}$, T, p are the velocity, temperature and pressure fields. ρ_0, α, ν and K_T are positive constant which represent the density of the fluid at a reference temperature, the coefficient of the volume expansion, the kinematic viscosity and the thermal conductivity, respectively. T_0 is a constant temperature.

∇ and Δ stand for gradient and Laplacian operators.

We assume the boundaries of the layer S stress free and thermally conducting. In this case the boundary conditions read [1]

$$\begin{cases} \vec{v}\cdot\vec{n} = \vec{n}\times\vec{D}\cdot\vec{n} = \vec{0}, & z = 0, d \\ T = T^0 & z = 0, \qquad t \geq 0 \\ T = T^1 & z = d, \end{cases} \qquad (4)$$

where $\mathbf{D}$ is the stress tensor, $\mathbf{n}$ is the external normal to the layer boundary and T^0 and T^1 are prescribed temperatures on the walls of the layer.

Let us now perturb the zero solution corresponding to a motionless state,

$$\left\{\vec{v} = \vec{0}, \overline{T} = -\beta z + T^0, \overline{P} = \overline{P}(z)\right\}, \qquad (5)$$

$\beta = \frac{T^0 - T^1}{d}$, up to a cellular motion characterized by a velocity $\vec{u} = \vec{0} + \vec{u}$, a pressure $p = \overline{P} + p'$ and a temperature $T = \overline{T} + \theta$.

The perturbation fields $\vec{u}, p', \theta$ satisfy the following nondimensional equations

$$\frac{\partial}{\partial t}\vec{u} + (\vec{u}\cdot\nabla)\vec{u} = -\nabla p' + \mathcal{R}\theta\vec{k} + 2\vec{u}\times\vec{\Omega} + \Delta\vec{u}, \tag{6}$$

$$P_r(\frac{\partial}{\partial t}\theta + \vec{u}\cdot\nabla\theta) = \Delta\theta + \mathcal{R}w, \qquad (t,\vec{x}) \in (0,\infty)\times V \tag{7}$$

$$\nabla\cdot\vec{u} = 0, \tag{8}$$

in the following subset of the Sobolev space $W^{2,2}(V)$,

$$\mathcal{N} = \{(\vec{u}, p, \theta,) \in W^{2,2}(V) \mid \frac{\partial}{\partial z}u = \frac{\partial}{\partial z}v = w = \theta = 0 \,\text{on}\, \partial \mathrm{V}\}, \tag{9}$$

where $\vec{u} = (u, v, w)$, and $V = [0, \dfrac{2\pi}{k_1}] \times [0, \dfrac{2\pi}{k_2}] \times [0, 1]$, is the periodicity cell and its boundary is denoted by ∂V, after assuming the perturbation fields, depending on the time t and space $\vec{x} = (x, y, z)$, doubly periodic functions in x and y, of period $2\pi/k_1$ and $2\pi/k_2$.

$\mathcal{R}^2, P_r$ and are the Rayleigh and Prandtl numbers, respectively.

Let us suppose that each term of (6), as function of the space variable $\vec{x}$, belongs to the Sobolev space $W^{2,2}(V)$.

3 The evolution equations for the perturbation fields

In order to obtain some suitable perturbation evolution equations we consider the representation theorem of solenoidal vectors [3] in a plane layer, into toroidal and poloidal fields.

This allows us to reduce the number of scalar fields and, first of all, *to derive a system of perturbation evolution equations equivalent to (6), (7), (8), applying no more differential operators* in opposition to what happens in the linear instability theory.

In such a way we can *integrate* [3] the solenoidality equation (8) obtaining a system of equations in some suitable orthogonal subspaces of $L_2(V)$. If the mean value of $\vec{u}$ vanish over V, [29] that is if the condition

$$\int_V u dx dy = \int_V v dx dy = \int_V w dx dy = 0,$$

holds, the velocity perturbation $\vec{u}$ has the unique decomposition [3]

$$\vec{u} = \vec{u}_1 + \vec{u}_2, \tag{10}$$

with

$$\nabla \cdot \vec{u}_1 = \nabla \cdot \vec{u}_2 = \vec{k} \cdot \nabla \times \vec{u}_1 = \vec{k} \cdot \vec{u}_2 = 0, \tag{11}$$

$$\vec{u}_1 = \nabla \frac{\partial}{\partial z}\chi - \vec{k}\Delta\chi \equiv \nabla \times \nabla \times (\chi\vec{k}), \tag{12}$$

$$\vec{u}_2 = \vec{k} \times \nabla\psi = -\nabla \times (\vec{k}\psi), \tag{13}$$

where the poloidal and toroidal potentials χ and ψ are doubly periodic and satisfy the equations [3]

$$\Delta_1\chi \equiv \frac{\partial^2}{\partial x^2}\chi + \frac{\partial^2}{\partial y^2}\chi = -\vec{k}\vec{u} \tag{14}$$

$$\Delta_1\psi = \vec{k} \cdot \nabla \times \vec{u}. \tag{15}$$

From now going on, we denote $\frac{\partial}{\partial x} f \equiv f_x$, where f is an arbitrary function. The boundary conditions for χ and ψ, for free planar surfaces, are [3]:

$$\chi = \chi_{zz} = \psi_z = 0 \quad z = 0, 1. \tag{16}$$

From (11)-(13) it follows that

$$\vec{u} \cdot \vec{k} = \vec{u}_1 \cdot \vec{k} = -\Delta_1\chi, \tag{17}$$

while the projection of $\vec{u}$ orthogonal to $\vec{k}$ is given by

$$(\vec{I} - \vec{k} \otimes \vec{k})\vec{u} = \vec{u} - \vec{k}w \equiv \vec{u}_1^{\perp} + \vec{u}_2, \tag{18}$$

where I and $\otimes$ stand for the identity operator and the tensor product, respectively.

Explicitely, in terms of the poloidal and toroidal fields the projection of $\vec{u}$ orthogonal to $\vec{k}$ is

$$(\vec{I} - \vec{k} \otimes \vec{k})\vec{u} = (\chi_{xz} - \psi_y)\vec{i} + (\chi_{yz} + \psi_x)\vec{j}. \tag{19}$$

In order to derive the perturbation evolution equations in terms of the potential and toroidal field we apply to equation (6) the tensor operator $(\vec{I} - \vec{k} \otimes \vec{k})$, and we obtain:

$$\frac{\partial}{\partial t}(\vec{u}_1^{\perp} + \vec{u}_2) + (\vec{u} \cdot \nabla)(\vec{u}_1^{\perp} + \vec{u}_2) = -\nabla_1 p' + 2(\vec{u}_1^{\perp} + \vec{u}_2) \times \vec{\Omega} + \Delta(\vec{u}_1^{\perp} + \vec{u}_2), \tag{20}$$

where ∇_1 stands for the horizontal gradient operator.

The velocity field $\vec{u}_1^{\perp} + \vec{u}_2$ and the rotation term $2(\vec{u}_1^{\perp} + \vec{u}_2) \times \vec{\Omega}$ become

$$\vec{u}_1^{\perp} + \vec{u}_2 = \nabla_1\chi_z - \nabla \times (\psi\vec{k}), \tag{21}$$

$$(\vec{u}_1^{\perp} + \vec{u}_2) \times \vec{\Omega} = \Omega \nabla_1 \psi + \Omega \nabla \times \left(\chi_z \vec{k}\right). \tag{22}$$

Let us recall the Weyl decomposition theorem [4]

$$L^2(V) = G(V) \oplus N(V), \tag{23}$$

with $G(V)$ and $N(V)$ spaces of generalized solenoidal and potential vectors respectively.

So, the advective term in (20) can be uniquely obtained as

$$(\vec{u} \cdot \nabla)(\vec{u}_1^{\perp} + \vec{u}_2) = \nabla U + \nabla \times \vec{A}, \tag{24}$$

where U is a scalar function and $\vec{A}$ a vector field we specify as follows.

If we define the scalar and vector fields

$$\Phi = \nabla \cdot (\vec{u} \cdot \nabla(\vec{u}_1^{\perp} + \vec{u}_2)), \quad \vec{W} = \nabla \times (\vec{u} \cdot \nabla(\vec{u}_1^{\perp} + \vec{u}_2)), \tag{25}$$

the imbedding Sobolev theorems of $W^{2,2}(V)$ in the space of continuous functions $C(\overline{V})$ [30] allows us to prove the following identity

$$\nabla \times (\vec{u} \cdot \nabla(\vec{u}_1^{\perp} + \vec{u}_2)) \equiv \nabla \times (\vec{u} \cdot \nabla(\vec{u}_1^{\perp} + \vec{u}_2) - \nabla U). \tag{26}$$

Let us define

$$\vec{B} = \vec{u} \cdot \nabla(\vec{u}_1^{\perp} + \vec{u}_2) - \nabla U, \tag{27}$$

by choosing $\nabla \cdot \vec{B} = 0$, the scalar function U is (up to a constant) the solution of the interior Neumann problem [31] in the periodicity cell V

$$\Delta U = \Phi \tag{28}$$

$$\frac{\partial}{\partial \vec{n}} U = \Gamma, \tag{29}$$

where $\frac{\partial}{\partial \vec{n}} U$ is the normal derivative of U on the boundary ∂V of the periodicity cell V, and $\Gamma = -\vec{B} \cdot \vec{n}$.

The necessary condition

$$\int_V \Phi dv + \int_{\partial V} \Gamma dv = \int_{\partial V} (\vec{u} \cdot \nabla(\vec{u}_1^{\perp} + \vec{u}_2)) \cdot \vec{n} d\sigma - \int_V \nabla \cdot \vec{B} dv = 0, \tag{30}$$

is fulfilled, in order a solution of (28), (29) to exist.

Taking into account the solenoidality of $\vec{B}$, it follows that exists a vector field $\vec{A}$ such that $\vec{B} = \nabla \times \vec{A}$, i.e. (24).

The perturbation equation (20), taking into account (21), (22) and (24), becomes

$$\frac{\partial}{\partial t}\big(\nabla_1\chi_z - \nabla\times(\psi\vec{k})\big) + \nabla U + \nabla\times\vec{A} = -\nabla_1 p' + 2\Omega\nabla_1\psi + 2\Omega\nabla\times\big(\chi_z\vec{k}\big) \tag{31}$$

$$+\Delta(\nabla_1\chi_z - \nabla\times(\psi\vec{k})).$$

If we project this equation on the orthogonal subspaces of solenoidal and gradient vectors, taking into account that the only vector belonging to both previous subspaces is zero [4], from the imbedding Sobolev theorems of $W^{2,2}(V)$ in the space of continuous functions $C(\overline{V})$ [30], it follows that

$$\frac{\partial}{\partial t}\nabla_1\chi_z + \nabla U = -\nabla_1 p' + 2\Omega\nabla_1\psi + \Delta\nabla_1\chi_z \tag{32}$$

$$-\frac{\partial}{\partial t}\nabla\times(\psi\vec{k}) + \nabla\times\vec{A} = 2\Omega\nabla\times\chi_z\vec{k} - \Delta\nabla\times(\psi\vec{k}). \tag{33}$$

Then we can obtain the null contribution of the pressure term and of the nonlinear terms in the left hand side of (31).

4 Lyapunov stability

If we consider the inner product $(\cdot,\cdot)$ in $L^2(V)$ of (32) by the poloidal field $\vec{u}_1$, which is solenoidal, it follows that

$$\Big(\frac{\partial}{\partial t}\nabla_1\chi_z, \vec{u}_1\Big) = \Big(-\nabla U - \nabla_1 p', \vec{u}_1\Big) + \Big(2\Omega\nabla_1\psi + \Delta\nabla_1\chi_z, \vec{u}_1\Big) \tag{34}$$

where

$$\Big(\frac{\partial}{\partial t}\nabla_1\chi_z, \vec{u}_1\Big) = \frac{d}{dt}\int_V (\chi_{xz}^2 + \chi_{yz}^2)dV, \tag{35}$$

$$(\nabla U, \vec{u}_1) \equiv 0, \tag{36}$$

$$(-\nabla_1 p', \vec{u}_1) - (p'_z, w) = 0, \tag{37}$$

and

$$\Big(2\Omega\nabla_1\psi + \Delta\nabla_1\chi_z, \vec{u}_1\Big) = 2\Omega\Big((\chi_{xz}, \psi_x) + (\chi_{yz}, \psi_y)\Big) \tag{38}$$

$$+(\chi_{xz}, \Delta\chi_{xz}) + (\chi_{yz}, \Delta\chi_{yz}).$$

Finally, performing the sum $c(34)+cw\vec{k}(6)+\vec{u}(6)+b\theta(7)$, where b, c are positive parameter, integrating the obtained equation over V and taking into account the boundary conditions (16), we obtain

$$\frac{d}{dt}\frac{1}{2}\int_V \{(\vec{u}^2 + c\Big(\chi_{xz}^2 + \chi_{yz}^2 + (\Delta_1\chi)^2\Big) + b\theta^2\}dV = \tag{39}$$

$$\mathcal{R}(1 + c + \frac{b}{P_r})(\theta, w) + (\vec{u}, \Delta\vec{u}) + 2c\Omega\big\{(\chi_{xz}, \psi_x) + (\chi_{yz}, \psi_y)\big\}$$

$$+c\big\{(\chi_{xz}, \Delta\chi_{xz}) + (\chi_{yz}, \Delta\chi_{yz}) + (\Delta_1\chi, \Delta\Delta_1\chi)\big\} + \frac{b}{P_r}(\theta, \Delta\theta).$$

Taking into account the relations

$$(-\psi_y, \chi_{xz}) + (\psi_x, \chi_{yz}) = (\vec{u}_1^{\perp}, \vec{u}_2) = (\nabla_1\chi_z, -\nabla\times(\psi\vec{k})) \equiv 0, \tag{40}$$

$$(-\nabla\psi_y, \nabla\chi_{xz}) + (\nabla\psi_x, \nabla\chi_{yz}) = \tag{41}$$

$$\int_{\partial V}[-\big(\nabla\chi_z\cdot\nabla\psi_y\big)\vec{i}\cdot\vec{n} + \big(\nabla\chi_z\cdot\nabla\psi_x\big)\vec{j}\cdot\vec{n}]d\sigma \equiv 0,$$

the equations (14) and the boundary conditions (16), in terms of poloidal and toroidal fields, the energy relation (39) becomes:

$$\frac{1}{2}\frac{d}{dt}\Big\{(1+c)\Big[\mid\chi_{xz}\mid^2 + \mid\chi_{yz}\mid^2 + \mid\Delta_1\chi\mid^2\Big] + \mid\psi_y\mid^2 + \mid\psi_x\mid^2$$

$$b\mid\theta\mid^2\Big\} = -\mathcal{R}(1 + c + \frac{b}{P_r})(\theta, \Delta_1\chi) + 2c\Omega\big\{(\chi_{xz}, \psi_x) + (\chi_{yz}, \psi_y)\big\}$$

$$-\big\{(1+c)\Big[\mid\nabla\chi_{xz}\mid^2 + \mid\nabla\chi_{yz}\mid^2 + \mid\nabla\Delta_1\chi\mid^2\Big] + \mid\nabla\psi_x\mid^2 + \mid\nabla\psi_y\mid^2 +$$

$$+\frac{b}{P_r}\mid\nabla\theta\mid^2\big\}, \tag{42}$$

where $|\cdot|^2$ stands for the $L^2(V)$ norm.

Let us introduce the function

$$E_{\mathcal{L}}(t) = \frac{1}{2}\Big\{(1+c)\Big[\mid\chi_{xz}\mid^2 + \mid\chi_{yz}\mid^2 + \mid\Delta_1\chi\mid^2\Big] + \mid\psi_y\mid^2 + \mid\psi_x\mid^2 + b\mid\theta\mid^2\Big\}. \tag{43}$$

The inequalities

$$b > 0, \qquad 1 + c > 0, \tag{44}$$

are sufficient to ensure that $E_{\mathcal{L}}(t)$ is definite positive.

The definition (43), with (42) yields

$$\frac{d}{dt}E_{\mathcal{L}} = -\mathcal{R}(1 + c + \frac{b}{P_r})(\theta, \Delta_1\chi) + 2c\Omega\big\{(\chi_{xz}, \psi_x) + (\chi_{yz}, \psi_y)\big\} \tag{45}$$

$$-\big\{(1+c)\Big[\mid\nabla\chi_{xz}\mid^2 + \mid\nabla\chi_{yz}\mid^2 + \mid\nabla\Delta_1\chi\mid^2\Big] + \mid\nabla\psi_x\mid^2 + \mid\nabla\psi_y\mid^2$$

$$+\frac{b}{P_r} \mid \nabla\theta \mid^2 \}.$$

We determine a condition ensuring that

$$\frac{d}{dt}E_{\mathcal{L}}\leq 0, \quad \forall t\geq 0. \tag{46}$$

Let us define

$$\mathcal{I} \equiv -(1+c+\frac{b}{P_r})(\theta,\Delta_1\chi)+\frac{2c\Omega}{\mathcal{R}}\big\{(\chi_{xz},\psi_x)+(\chi_{yz},\psi_y)\big\} \tag{47}$$

$$\mathcal{E} \equiv \Big\{(1+c)\Big[\mid \nabla\chi_{xz} \mid^2 + \mid \nabla\chi_{yz} \mid^2 + \mid \nabla\Delta_1\chi \mid^2 \Big] + \mid \nabla\psi_x \mid^2 + \mid \nabla\psi_y \mid^2 \Big]$$

$$+\frac{b}{P_r} \mid \nabla\theta \mid^2 \}. \tag{48}$$

The equation (45) becomes:

$$\frac{d}{dt}E_{\mathcal{L}} = \mathcal{R}\mathcal{I} - \mathcal{E} = -\mathcal{E}\Big(1-\mathcal{R}\frac{\mathcal{I}}{\mathcal{E}}\Big). \tag{49}$$

As the functions $E_{\mathcal{L}}$ and $\mathcal{E}(t)$ are not definite positive $\forall b, c \in \mathbf{R}$, we consider, separately, the cases , $\Big(E_{\mathcal{L}} > 0 \wedge (\mathcal{E}(t) > 0 \vee \mathcal{E}(t) < 0)\Big)$ and $\Big(E_{\mathcal{L}} < 0 \wedge (\mathcal{E}(t) > 0 \vee \mathcal{E}(t) < 0)\Big)$.

$E_{\mathcal{L}} > 0$ $\mathcal{E} > 0$

If

$$\mathcal{R} < \sqrt{Ra_*}, \tag{50}$$

where

$$\frac{1}{\sqrt{R_{a*}}} = \max\frac{\mathcal{I}}{\mathcal{E}}, \tag{51}$$

in the class of admissible functions satisfying the boundary conditions (16), from (49), (50) and (51) we deduce

$$\frac{d}{dt}E_{\mathcal{L}} \leq -\Big(1-\frac{\mathcal{R}}{\sqrt{R_{a*}}}\Big)\mathcal{E}. \tag{52}$$

Hence, in this case, if (50) is satisfied, the functional $E_{\mathcal{L}}$ is a decreasing function of t. The inequality (46) represents a stability uniqueness criterion [3], [4].

$E_{\mathcal{L}} > 0$ $\mathcal{E} < 0$

From (49) it follows that if the inequality

$$\sqrt{Ra_*} < \mathcal{R}, \tag{53}$$

is satisfied, with

$$\frac{1}{\sqrt{R_{a*}}} = \min \frac{\mathcal{I}}{\mathcal{E}}, \tag{54}$$

in the class of admissible functions satisfying the boundary conditions (16), then

$$\frac{d}{dt} E_{\mathcal{L}} \leq 0, \quad \forall t \geq 0.$$

In the case $E_{\mathcal{L}} < 0$ we proceed as follows.
Let us define$E^*_{\mathcal{L}} = -E_{\mathcal{L}}$, from (49) we have

$$\frac{d}{dt} E^*_{\mathcal{L}} = \mathcal{E}\Big(1 - \mathcal{R}\frac{\mathcal{I}}{\mathcal{E}}\Big). \tag{55}$$

In the case $\mathcal{E} < 0$, the inequality (50) implies

$$\frac{d}{dt} E^*_{\mathcal{L}} \leq 0, \quad \forall t \geq 0,$$

In the case $\mathcal{E} > 0$, the inequality (53) implies

$$\frac{d}{dt} E *_{\mathcal{L}} \leq 0, \quad \forall t \geq 0.$$

Because, as we shall see later, the inequality (53) contradits the results of the linear instability theory, we can deduce that, if the maximum problem (51) admits a solution, the inequality (50)represents s stability uniqueness criterion [3], [4].

5 The maximum problem and the stability bound

We will study the variational problem (51) and later determine the parameters b, c in terms of the physical quantities, such that $\sqrt{R_{a*}}$ will be maximal.

The Euler Lagrange equations associated with the maximum problem (51) are:

$$-(1 + c + \frac{b}{P_r})\Delta_1\theta + 2c\frac{\Omega}{\mathcal{R}}\Delta_1\psi_z + \frac{2}{\sqrt{R_{a*}}}(1+c)\Delta\Delta\Delta_1\chi = 0,$$

$$-(1 + c + \frac{b}{P_r})\Delta_1\chi + \frac{b}{P_r}\frac{2}{\sqrt{R_{a*}}}\Delta\theta = 0, \tag{56}$$

$$2c\frac{\Omega}{\mathcal{R}}\Delta_1\chi_z + \frac{2}{\sqrt{R_{a*}}}\Delta\Delta_1\psi = 0.$$

Taking into account (14), (15), the system of Euler equations equivalently read

$$\begin{aligned} -(1+c+\frac{b}{P_r})\Delta_1\theta + 2c\frac{\Omega}{\mathcal{R}}\zeta_z - \frac{2}{\sqrt{R_{a*}}}(1+c)\Delta\Delta w = 0, \\ (1+c+\frac{b}{P_r})w + \frac{b}{P_r}\frac{2}{\sqrt{R_{a*}}}\Delta\theta = 0, \\ -2c\frac{\Omega}{\mathcal{R}}w_z + \frac{2}{\sqrt{R_{a*}}}\Delta\zeta = 0. \end{aligned} \tag{57}$$

We shall suppose that the principle of exchange of stabilities holds, i.e. we assume that the instability occurs as a stationary convection.
In the class of normal mode perturbations

$$\begin{aligned} w(\vec{x}) = W(z)exp[i(k_1x_1 + k_2x_2)] \\ \zeta(\vec{x}) = Z(z)exp[i(k_1x_1 + k_2x_2)] \\ \theta(\vec{x}) = \Theta(z)exp[i(k_1x_1 + k_2x_2)], \end{aligned} \tag{58}$$

the equations (57) become

$$\begin{aligned} k^2(1+c+\frac{b}{P_r})\Theta + 2c\frac{\Omega}{\mathcal{R}}DZ - \frac{2}{\sqrt{R_{a*}}}(1+c)(D^2-k^2)^2W = 0, \\ (1+c+\frac{b}{P_r})W + \frac{b}{P_r}\frac{2}{\sqrt{R_{a*}}}(D^2-k^2)\Theta = 0, \\ -2c\frac{\Omega}{\mathcal{R}}DW + \frac{2}{\sqrt{R_{a*}}}(D^2-k^2)Z = 0, \end{aligned} \tag{59}$$

where $k^2 = k_1^2 + k_2^2$ is the wave number.

To (59) we add the following boundary conditions:

$$W = D^2W = \Theta = D^2\Theta = DZ = 0. \tag{60}$$

Owing to (14), and (16), we assume [5]

$$W(z) = \sum_{n=1}^{\infty} W_n \sin(n\pi z), \tag{61}$$

From (57)and (61) we have

$$R_{a*}(\mathcal{R}^2, k^2, n^2\pi^2, b, c) = \frac{4(1+c)\mathcal{R}^2(n^2\pi^2+k^2)^3}{4\Omega^2 n^2\pi^2 c^2 + k^2\mathcal{R}^2\frac{P_r}{b}(1+c+\frac{b}{P_r})^2}. \tag{62}$$

Differentiating (62) with respect to the parameters b and c we obtain

$$\frac{\partial}{\partial b}R_{a*} = 0 \Leftrightarrow \frac{b}{P_r} = 1+c, \quad \frac{\partial}{\partial c}R_{a*} = 0 \Leftrightarrow c = -2. \tag{63}$$

Substituting (63) in (62) we obtain R_{a*} as a function of $\mathcal{R}^2$

$$R_{a*}(\mathcal{R}^2, k^2, n^2\pi^2, b, c) = \frac{\mathcal{R}^2(n^2\pi^2+k^2)^3}{-4\Omega^2 n^2\pi^2 + k^2\mathcal{R}^2}, \tag{64}$$

defined on the subset $-4\Omega^2 n^2\pi^2 + k^2\mathcal{R}^2 > 0$.
The critical Rayleigh function of the linear instability theory, that is

$$\mathcal{R}^2_{cr} = \frac{(n^2\pi^2+k^2)^3 + 4\Omega^2 n^2\pi^2}{k^2}, \tag{65}$$

belongs to the subset where the denominator of (64) is definite positive.

Evaluating (64) for $\mathcal{R}^2 = \mathcal{R}^2_{cr}$, we obtain

$$\mathcal{R}^2_{cr}(k^2, n^2\pi^2) = R_{a*}(k^2, n^2\pi^2). \tag{66}$$

Obviously, the inequality (53), calculated for $R_{a*}(k^2, n^2\pi^2) = \mathcal{R}^2_{cr}(k^2, n^2\pi^2)$, implies $\mathcal{R}^2 > \mathcal{R}^2_{cr}$, that should be a sufficient condition of linear stability theory too (the contribution of all nonlinear terms vanish), in opposition to the well known results of linear stability theory.
Hence we proved the following theorem

Theorem 5.1 *If the principle of exchange of stabilities holds, the zero solution of (6)-(9), corresponding to the basic conduction state is nonlinearly globally stable if*

$$\mathcal{R}^2 < R_{a*},$$

where $\mathcal{R}^2_{cr} = R_{a*}(\mathcal{R}^2_{cr}, k^2, n^2)$, *attains its minimum where* $n = 1$. *Whence the linear and non linear stability bounds, obtained for* $n = 1$

$$\mathcal{R}^2_{cr}(k^2) = R_{a*}(\mathcal{R}^2_{cr}, k^2).$$

coincide.

6 Conclusions

We studied the nonlinear stability of the motionless state for a Newtonian fluid in a rotating horizontal layer, subject to an adverse temperature gradient, that is the classical Bénard problem with rotation.

In Section 3 we derived the perturbation evolution equations in terms of toroidal and poloidal fields.

In this way, we can *integrate the solenoidality equation, reduce the number of scalar fields applying no more differential operator to the perturbation evolution equations, and, first of all, obtain some perturbation evolution equations in suitable subspaces of* $L^2(V)$. This allows us to obtain an energy relation for the Lyapunov function in which *all the nonlinear terms disappear and the skewsymmetric rotation term is preserved.*

In Section 4 we studied the nonlinear Lyapunov stability introducing some functionals definite positive. We determine a sufficient condition for global stability satisfied on a subset of the parameter's space given by the solution of the variational problem arising from the energy inequality.

After solving, in the class of normal modes, the Euler-Lagrange equations associated with the maximum problem, , we maximize the stability domain with respect to the parameters introduced in the Lyapunov functional and we deduce *if the principle of exchange of stabilities holds, the equality between the linear and nonlinear critical parameters for the global stability.*

We observe that, anyhow, in this paper we applied an idea similar to [23] [24] [25] [27] [28], where, studying the nonlinear stability of a binary mixture in a plane layer *we incorporated the Joseph's idea of parameters differentiation directly into the evolution equations obtaining equations with better symmetries, which incorporate the given equations.* In this way, in [23] the velocity term in the temperature equation contributed to the symmetric part of the obtained equations. *Otherwise, if the initial evolution equations were used this contribution was null and, correspondingly, the stability criterion, weaker.*

In this paper, similarly, the rotation term disappears if the initial evolution equations were used.

The given problem governing the perturbation evolution was changed in order to obtain an optimum energy relation. The initial equations were replaced by some others equivalent to the initial ones.

All this drastically changed the linear part of the initial equations and allows us a much more advantageous symmetrization and an equivalent formulation of the stability problem, in which the Euler system associated to the maximum problem of the nonlinear stability is nothing else but the one that governs the linear instability, whence a linearization principle in the sense of the coincidence of both linear and nonlinear stability bounds.

References

[1] S.Chandrasekhar,*Hydrodynamic and Hydromagnetic stability*, Oxford, Clarendon Press, 1968.

[2] D.D. Joseph, *Global Stability of the Conduction-Diffusion Solution,* Arch. Rat. Mech. Anal., **36**, 4 (1970), 285-292.

[3] D.D. Joseph, *Stability of fluid motions* I-II, Springer, Berlin, 1976.

[4] A. Georgescu, *Hydrodynamic stability theory*, Kluwer, Dordrecht, 1985.

[5] B. Straughan, *The Energy Method, Stability and Nonlinear Convection*, 2nd. ed. Ser. in Appl. Math. Sci., **91**, Springer, New York, 2003.

[6] G. P. Galdi, M. Padula, *A new approach to energy theory in the stability of fluid motion*, Arch. Rat. Mech. Anal., **110**, 3, (1990), 187-286.

[7] G. P. Galdi, B. Straughan, *A nonlinear analysis of the stabilizing effect of rotation in the Bénard problem.*, Proc. Royal Soc. of London, Ser. A, **402** (1823), (1985), 257-283.

[8] S. Rionero, *Sulla stabilità magnetoidrodinamica in media con vari tipi di condizioni al contorno*, Ricerche di Mat., **17**, 4 (1968), 64-78.

[9] S. Rionero, G. Mulone, *On the nonlinear stability of a thermodiffusive fluid mixture in a mixed problem*, J. Math. Anal. Appl., **124**, (1987), 165-188.

[10] S. Rionero, G. Mulone, *A nonlinear stability analysis of the magnetic Bé nard problem through the Liapunov direct method,*, Arch.Rat. Mech. Anal., **103**, 4 (1988), 347-368.

[11] G. Mulone, S. Rionero , *The rotating Bénard problem: new stability results for any Prandtl and Taylor numbers*, Continuum Mech. Thermody., **9**, (1997), 347-363.

[12] G. Mulone , *Stabilizing effects in dynamical systems: linear and nonlinear stability conditions*, Far East J. Appl. Math., **15**, 2, (2004), 117-134.

[13] L. Palese, *On the nonlinear stability of the MHD anisotropic Bénard problem*, Int. J. Engng. Sc., **43**, (2005), 1265-1282.

[14] A. Georgescu, I. Oprea, *Bifurcation theory from the applications point of view*, Monografii Matematice. Universitatea de vest din Timişoara, **51**, 1994.

[15] G. Prodi, *Teoremi di tipo locale per il sistema di Navier-Stokes e stabilità delle soluzioni stazionarie*, Rend. Sem. Mat. Padova, **32**, (1962), 374-397.

[16] V. I. Yudovich, *On the stability of stationary flow of viscous incompressible fluids*, Dokl. Akad. Nauk. SSSR, **161**, 5 (1965), 1037-1040. (Russian) (translated as Soviet Physics Dokl., **10**, 4 (1965), 293-295.)

[17] D. H. Sattinger, *The mathematical problem of hydrodynamic stability*, J. Math. Mech., **19**, 9 (1970), 197-217.

[18] Yudovich, V., I., On the stability of forced oscillations of the fluids, Dokl. Akad. Nauk. S S S R, **195**, 2 (1970), 292-296. (Russian)

[19] Yudovich, V., I., On the stability of self oscillations of the fluid, Dokl. Akad. Nauk. S S S R, **195**, 3 (1970), 574-576. (Russian)

[20] Yudovich, V. I., The linearization method in hydrodynamical stability theory, Trans. of Math. Monographs, **74**, American Mathematical Society, Providence, R I, 1989.

[21] Joseph, D. D., On the stability of the Boussinesq equations, Arch. Rat. Mech. Anal., **20** (1965), 59-71.

[22] Joseph, D. D., Non linear stability of the Boussinesq equations by the method of energy, Arch. Rat. Mech. Anal., **22** (1966), 163-184.

[23] A. Georgescu, L. Palese, *Extension of a Joseph's criterion to the non linear stability of mechanical equilibria in the presence of thermodiffusive conductivity,* Theor. and Comp. Fluid Mech., **8**, 1996, 403-413.

[24] A. Georgescu, L. Palese, A. Redaelli, *On a New Method inHydrodynamic Stability Theory,* Math. Sc. Res. Hot Line, **4**, 7 (2000), 1-16.

[25] A. Georgescu, L. Palese, A. Redaelli, *The Complete Form for the Joseph Extended Criterion,* Annali Univ. Ferrara , Sez. VII, Sc. Mat. **48**, (2001), 9-22.

[26] A. Georgescu, L. Palese, *Stability Criteria for Fluid Flows,* Advances in Math. for Appl. Sc., **81**, World Scient. Singapore, 2010.

[27] A. Georgescu, L. Palese, *On the nonlinear stability of a binary mixture with chemical surface reactions,* Annals of the Academy of Romanian Scientists, Series on Math. and Appl. **3**, 1, (2011), 106-115 .

[28] A. Georgescu, L. Palese, *A linearization principle for the stability of the chemical equilibrium of a binary mixture* Romai Journal, **6**, 2 (2010), 131-138.

[29] B. J. Schmitt, W. von Wahl, *Decomposition of Solenoidal Fields into Poloidal Fields, Toroidal Fields and the Mean Flow. Applications to the Boussinesq-Equations*, Lecture Notes in Math., **1530**, Springer, Berlin (1992), 291-305.

[30] S. L. Sobolev, *Applications of Functional Analysis in Mathematical Physics,* **7**, American Mathematical Society, 1963.

[31] S. G. Miklin, *An advanced course of Mathematical Physics,* North-Holland, Amsterdam London 1970.