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EXISTENCE AND APPROXIMATION FOR A STEADY FLUID-STRUCTURE INTERACTION PROBLEM USING FICTITIOUS DOMAIN APPROACH WITH PENALIZATION*

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Abstract

In the present paper, we use a penalization of the Stokes equation in order to obtain approximate solutions in a larger domain including the domain occupied by the structure. The coefficients of the fluid problem, excepting the penalizing term, are constant and independent of the deformation of the structure, which represents an advantage of this approach. Subtracting the structure equations from the fictitious fluid equations in the structure domain and using the Green's formula, we obtain a weak formulation where the continuity of the stress at the interface does not appear explicitly. This is a second advantage of this model, because the computation of the stress at the fluid-structure interface is not easy from the theoretical point of view as well as for the numerical approximation. This problem is a free boundary problem and a fundamental difficulty is to find the free interface between the

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fluid and the structure, which is unknown and has to be identified together with the solution of the given system of equations.
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1 Introduction

The interaction of structures and fluids is the object of intense research due both to mathematical challenging problems and important practical applications. The present paper is devoted to the study of the behavior of an elastic structure immersed in a viscous incompressible fluid. We suppose that the Reynolds number is suitably small so we use Stokes equation to model the flow motion. The displacement of the structure under the flow motion will be modeled by linear elasticity equations, under the small deformations assumption. In this paper, we study the steady case.

Existence for steady interaction between an incompressible fluid and an elastic structure was proved in: [22], [13], [14], [3], [23], [10]. In these papers, the fluid equations are reformulated in a reference configuration. Consequently, the coefficients of the fluid problem are non-constant and depend on the structure deformation.

In the present paper, we use a penalization of the Stokes equation in order to obtain approximate solutions in a larger domain including the domain occupied by the structure. The coefficients of the fluid problem, excepting the penalizing term, are constant and independent of the deformation of the structure, which represents an advantage of this approach. Subtracting the structure equations from the fictitious fluid equations in the structure domain and using the Green's formula, we obtain a weak formulation where the continuity of the stress at the interface does not appear explicitly. This is a second advantage of this model, because the computation of the stress at the fluid-structure interface is not easy from the theoretical point of view as well as for the numerical approximation. We underline that this problem is a free boundary problem and a fundamental difficulty is to find the free interface between the fluid and the structure, which is unknown and has to be identified together with the solution of the given system of equations. In this respect, one of the main ingredients of our approach is similar with the method developed in [25] for the identification of domains in shape optimization problems.

The fluid-structure interaction problems can be solved numerically by the Arbitrary Lagrangian Eulerian method, where the fluid equations are written over a moving mesh which follows the structure displacement. This method was successfully employed in: [28], [8], [19], [24], [20].

The methods entering in the category of fixed domain are: fictitious domain with distributed Lagrange multiplier [12], immersed boundary method [21], an approach using Lagrangian coordinates for the fluid as well as for the structure equations [17] or Eulerian framework [7]. In the present paper, we use the fictitious domain approach with penalization, not with Lagrange multiplier.

2 A steady fluid-structure interaction problem

Let $D \subset \mathbb{R}^2$ be a bounded open domain with boundary $\partial D = \Sigma_1 \cup \Sigma_2$. Let Ω_0^S be the undeformed structure domain, and suppose that its boundary admits the decomposition $\partial\Omega_0^S = \Gamma_D \cup \Gamma_0$, where Γ_0 is a relatively open subset of the boundary. On Γ_D we impose zero displacement for the structure. We assume that $\Omega_0^S \subset D$ and $\Gamma_D \subset \Sigma_2$.

Suppose that the structure is elastic and denote by $\mathbf{u} = (u_1, u_2) : \Omega_0^S \to \mathbb{R}^2$ its displacement. A particle of the structure with initial position at the point $\mathbf{X}$ will occupy the position

$$\mathbf{x} = \varphi(\mathbf{X}) = \mathbf{X} + \mathbf{u}(\mathbf{X})$$

in the deformed domain $\Omega_u^S = \varphi\left(\Omega_0^S\right)$.

We assume that $\Omega_u^S \subset D$ and the fluid occupies $\Omega_u^F = D \setminus \overline{\Omega}_u^S$. We set $\Gamma_u = \varphi(\Gamma_0)$ and we suppose that Γ_u does not touch the container wall, i.e. $\partial D \cap \Gamma_u = \emptyset$. We recall that Γ_0 is relatively open subset. We obtain that $\overline{\Omega}_u^S \cap \overline{\Omega}_u^F = \Gamma_u$ which represents the fluid-structure interface. The boundary of the deformed structure is $\partial\Omega_u^S = \Gamma_D \cup \Gamma_u$ and the boundary of the fluid domain admits the decomposition $\partial\Omega_u^F = \Sigma_1 \cup (\Sigma_2 \setminus \Gamma_D) \cup \Gamma_u$. The fluid-structure geometrical configuration is represented in Figure 1.

We introduce some notations. Generally, the fluid equations are described using Eulerian coordinates, while for the structure equations, the Lagrangian coordinates are employed. The gradients with respect to the Eulerian coordinates $\mathbf{x} \in \Omega_u^S$ of a scalar field $q : D \to \mathbb{R}$ or a vector field

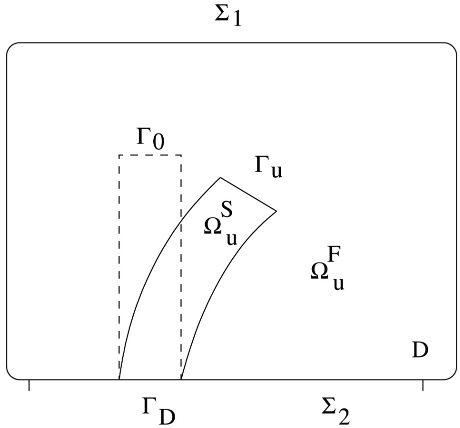

Figure 1: Geometrical configuration.

$\mathbf{w} = (w_1, w_2) : D \to \mathbb{R}^2$ are denoted by

$$\nabla q = \begin{pmatrix} \frac{\partial q}{\partial x_1} \\ \frac{\partial q}{\partial x_2} \end{pmatrix}, \quad \nabla \mathbf{w} = \begin{pmatrix} \frac{\partial w_1}{\partial x_1} & \frac{\partial w_1}{\partial x_2} \\ \frac{\partial w_2}{\partial x_1} & \frac{\partial w_2}{\partial x_2} \end{pmatrix}.$$

The scalar product of two vectors $\mathbf{v}$ and $\mathbf{w}$ of $\mathbb{R}^2$ is denoted as

$$\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{2} v_i \, w_i.$$

If $\sigma = (\sigma_{ij})_{1 \leq i,j \leq 2}$ and $\tau = (\tau_{ij})_{1 \leq i,j \leq 2}$ are two tensors, we denote

$$\sigma : \tau = \sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_{ij} \tau_{ij}.$$

The divergence operators with respect to the Eulerian coordinates of a vector field $\mathbf{w} = (w_1, w_2) : D \to \mathbb{R}^2$ and of a tensor $\sigma = (\sigma_{ij})_{1 \leq i,j \leq 2}$ are denoted by

$$\nabla \cdot \mathbf{w} = \frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2}, \quad \nabla \cdot \sigma = \begin{pmatrix} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} \\ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} \end{pmatrix}.$$

Similarly, when the derivatives are with respect to the Lagrangian coordinates $\mathbf{X} = \varphi^{-1}(\mathbf{x}) \in \Omega_0^S$, we use the notations: $\nabla_{\mathbf{X}} \mathbf{u}$, $\nabla_{\mathbf{X}} \cdot \mathbf{u}$, $\nabla_{\mathbf{X}} \cdot \sigma$.

If $\mathbf{A}$ is a square matrix, we denote by det $\mathbf{A}$, $\mathbf{A}^{-1}$, $\mathbf{A}^T$ its determinant, the inverse and the transposed matrix, respectively. We write cof $\mathbf{A} = (\det \mathbf{A}) \left(\mathbf{A}^{-1}\right)^T$ the co-factor matrix of $\mathbf{A}$. We write $\mathbf{A}^{-T} = \left(\mathbf{A}^{-1}\right)^T$.

We denote by $\mathbf{F}(\mathbf{X}) = \mathbf{I} + \nabla_{\mathbf{X}} \mathbf{u}(\mathbf{X})$ the gradient of the deformation and by $J(\mathbf{X}) = \det \mathbf{F}(\mathbf{X})$ the Jacobian determinant, where $\mathbf{I}$ is the unit matrix.

Strong formulation

The problem is to find the structure displacement $\mathbf{u} : \overline{\Omega}_0^S \to \mathbb{R}^2$, the fluid velocity $\mathbf{v} : \overline{\Omega}_u^F \to \mathbb{R}^2$ and the fluid pressure $p : \overline{\Omega}_u^F \to \mathbb{R}$ such that:

$$
\begin{aligned}
-\nabla_{\mathbf{X}} \cdot \sigma^S(\mathbf{u}) &= \mathbf{f}^S, \quad \text{in } \Omega_0^S && (1)\\
\mathbf{u} &= 0, \quad \text{on } \Gamma_D && (2)\\
-\nabla \cdot \sigma^F(\mathbf{v}, p) &= \mathbf{f}^F, \quad \text{in } \Omega_u^F && (3)\\
\nabla \cdot \mathbf{v} &= 0, \quad \text{in } \Omega_u^F && (4)\\
\mathbf{v} &= 0, \quad \text{on } \Sigma_1 && (5)\\
\mathbf{v} &= 0, \quad \text{on } \Sigma_2 \setminus \Gamma_D && (6)\\
\mathbf{v} &= 0, \quad \text{on } \Gamma_u && (7)\\
\omega\left(\sigma^F(\mathbf{v}, p)\, \mathbf{n}^F\right) \circ \varphi &= -\sigma^S(\mathbf{u})\, \mathbf{n}^S, \quad \text{on } \Gamma_0 && (8)
\end{aligned}
$$

where $\mathbf{f}^S : \Omega_0^S \to \mathbb{R}^2$ are the applied volume forces on the structure and $\mathbf{n}^S$ is the structure unit outward vector normal to $\partial\Omega_0^S$. Similarly, we define $\mathbf{f}^F : \Omega_u^F \to \mathbb{R}^2$ and $\mathbf{n}^F$ the fluid unit outward vector normal to $\partial\Omega_u^S$. We have denoted by $\sigma^S(\mathbf{u}) : \Omega_0^S \to \mathbb{R}^4$ and $\sigma^F(\mathbf{v}, p) : \Omega_u^F \to \mathbb{R}^4$ the Cauchy stress tensors of the structure and fluid, respectively. We point out that the stress tensor of the structure is defined on the undeformed structure domain Ω_0^S, while the Cauchy stress tensors of the fluid is defined in the deformed domain Ω_u^F. The constitutive relations will be precised later. We have used the notation $\omega(\mathbf{X}) = \left\| J\, \mathbf{F}^{-T} \mathbf{n}^S \right\|_{\mathbb{R}^2} = \left\| \mathrm{cof}(\mathbf{F})\, \mathbf{n}^S \right\|_{\mathbb{R}^2}$ for $\mathbf{X}$ on $\partial\Omega_0^S$, which is a kind of Jacobian determinant for the change of variable formula for integral over surface.

The equations (1), (2) concern the structure, while (3)-(6) concern the fluid. The equations (7), (8) represent the boundary conditions on the interface.

Remark 1 *The fluid and the structure domains Ω_u^F, Ω_u^S depend on the structure displacement u which is unknown. Consequently, the system (1)–(8) is a free boundary problem.*

3 Fictitious domain approach using penalization

In this section we present in an informal and intuitive manner the ideas behind our approximation approach. We introduce two more equations concerning the fluid fields, but written on the deformed structure domain:

$$-\nabla \cdot \sigma^F(\mathbf{v}, p) + \frac{1}{\epsilon}\mathcal{P}(\mathbf{v}) = \mathbf{f}^F, \quad \text{in } \Omega_u^S \tag{9}$$

$$\nabla \cdot \mathbf{v} = 0, \quad \text{in } \Omega_u^S \tag{10}$$

where $\epsilon > 0$ is a penalization parameter,

$$\mathcal{P}(\mathbf{v}) = \left(|v_1|^{\alpha-1} sgn(v_1), |v_2|^{\alpha-1} sgn(v_2)\right) \tag{11}$$

where $\mathbf{v} = (v_1, v_2)$ and $1 < \alpha < 2$ is a real number. This choice of the penalization term will be justified later.

Next, we define the characteristic functions $\chi_u^S : D \to \mathbb{R}$ and $\chi_u^F : D \to \mathbb{R}$

$$\chi_u^S(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \overline{\Omega}_u^S \\ 0, & \mathbf{x} \in D \setminus \overline{\Omega}_u^S \end{cases} \qquad \text{and} \quad \chi_u^F = 1 - \chi_u^S.$$

Combining (3) and (9), it follows that

$$-\nabla \cdot \sigma^F(\mathbf{v}, p) + \frac{1}{\epsilon}\chi_u^S \mathcal{P}(\mathbf{v}) = \mathbf{f}^F, \quad \text{in } D. \tag{12}$$

Similarly, we have from (4) and (10)

$$\nabla \cdot \mathbf{v} = 0, \quad \text{in } D. \tag{13}$$

Remark 2 *In view of the equation (9), the "fictitious" fluid velocity and pressure defined on the structure domain Ω_u^S depend on ϵ. In the following, we denote by $\mathbf{v}_\epsilon$ and p_ϵ the fluid velocity and pressure defined all over the domain D.*

Weak formulation

Let $\mathbf{w}^S : \Omega_0^S \to \mathbb{R}^2$ be such that $\mathbf{w}^S = 0$ on Γ_D. Using Green's formula, we obtain from equation (1):

$$\int_{\Omega_0^S} \sigma^S(\mathbf{u}) : \nabla_{\mathbf{X}} \mathbf{w}^S \, d\mathbf{X} = \int_{\Omega_0^S} \mathbf{f}^S \cdot \mathbf{w}^S \, d\mathbf{X} + \int_{\Gamma_0} \sigma^S(\mathbf{u}) \mathbf{n}^S \cdot \mathbf{w}^S dS. \tag{14}$$

Let us define $\widetilde{\mathbf{w}}^S : \Omega_u^S \to \mathbb{R}^2$ by $\widetilde{\mathbf{w}}^S = \mathbf{w}^S \circ \varphi^{-1}$. We get that $\widetilde{\mathbf{w}}^S = 0$ on Γ_D. Using Green's formula, we obtain from equation (9):

$$\int_{\Omega_u^S} \sigma^F(\mathbf{v}_\epsilon, p_\epsilon) : \nabla \widetilde{\mathbf{w}}^S \, d\mathbf{x} + \frac{1}{\epsilon} \int_{\Omega_u^S} \mathcal{P}(\mathbf{v}_\epsilon) \cdot \widetilde{\mathbf{w}}^S \, d\mathbf{x} = \int_{\Omega_u^S} \mathbf{f}^F \cdot \widetilde{\mathbf{w}}^S \, d\mathbf{x} - \int_{\Gamma_u} \sigma^F(\mathbf{v}_\epsilon, p_\epsilon) \mathbf{n}^F \cdot \widetilde{\mathbf{w}}^S \, ds. \quad (15)$$

The previous equation is equivalent to the variational formulation written in the undeformed domain Ω_0^S:

$$\int_{\Omega_0^S} J \left(\sigma^F(\mathbf{v}_\epsilon, p_\epsilon) \circ \varphi\right) \mathbf{F}^{-T} : \nabla_{\mathbf{X}} \mathbf{w}^S \, d\mathbf{X} + \frac{1}{\epsilon} \int_{\Omega_0^S} J\mathcal{P}(\mathbf{v}_\epsilon \circ \varphi) \cdot \mathbf{w}^S \, d\mathbf{X} = \int_{\Omega_0^S} J \left(\mathbf{f}^F \circ \varphi\right) \cdot \mathbf{w}^S \, d\mathbf{X} - \int_{\Gamma_0} \omega \left(\sigma^F(\mathbf{v}_\epsilon, p_\epsilon) \mathbf{n}^F \circ \varphi\right) \cdot \mathbf{w}^S \, dS. \quad (16)$$

Details about this kind of transformation could be found in [6], Chapter 1.2.

Subtracting (16) from (14) and taking into account the interface condition (8), we obtain that

$$\begin{aligned} &\int_{\Omega_0^S} \sigma^S(\mathbf{u}) : \nabla_{\mathbf{X}} \mathbf{w}^S \, d\mathbf{X} - \int_{\Omega_0^S} \mathbf{f}^S \cdot \mathbf{w}^S \, d\mathbf{X} \\ &= \int_{\Omega_0^S} J \left(\sigma^F(\mathbf{v}_\epsilon, p_\epsilon) \circ \varphi\right) \mathbf{F}^{-T} : \nabla_{\mathbf{X}} \mathbf{w}^S \, d\mathbf{X} \\ &+ \frac{1}{\epsilon} \int_{\Omega_0^S} J\mathcal{P}(\mathbf{v}_\epsilon \circ \varphi) \cdot \mathbf{w}^S \, d\mathbf{X} - \int_{\Omega_0^S} J \left(\mathbf{f}^F \circ \varphi\right) \cdot \mathbf{w}^S \, d\mathbf{X} \end{aligned} \quad (17)$$

for all $\mathbf{w}^S : \Omega_0^S \to \mathbb{R}^2$ such that $\mathbf{w}^S = 0$ on Γ_D.

From (12), we get

$$\int_D \sigma^F(\mathbf{v}_\epsilon, p_\epsilon) : \nabla \mathbf{w} \, d\mathbf{x} + \frac{1}{\epsilon} \int_D \chi_u^S \mathcal{P}(\mathbf{v}_\epsilon) \cdot \mathbf{w} \, d\mathbf{x} = \int_D \mathbf{f}^F \cdot \mathbf{w} \, d\mathbf{x} \quad (18)$$

for all $\mathbf{w} : D \to \mathbb{R}^2$ such that $\mathbf{w} = 0$ on ∂D.

For $q : D \to \mathbb{R}$, we obtain from (13) that

$$\int_D (\nabla \cdot \mathbf{v}_\epsilon) \, q \, d\mathbf{x} = 0. \quad (19)$$

The weak formulation is to find:

- structure displacement $\mathbf{u} : \overline{\Omega}_0^S \to \mathbb{R}^2$, $\mathbf{u} = 0$ on Γ_D,
- fluid velocity $\mathbf{v}_\epsilon : D \to \mathbb{R}^2$, $\mathbf{v}_\epsilon = 0$ on ∂D,
- fluid pressure $p_\epsilon : D \to \mathbb{R}$

such that (17), (18), (19) hold. The spaces where $\mathbf{w}$ and q belong will be introduced later on.

Remark 3 *We have to point out that the boundary conditions on the interface (7), (8) do not appear in the above weak formulation. The condition (7) will be approached by the penalization term. The condition (8) can be obtained in a weak sense from (17) and (18), if we impose some regularity to the unknowns. This will be done after the introduction of the constitutive relations. Notice that the equation (17) does not represent the structure equation, but the difference of structure and fluid equations on the structure domain. This technique employed in [4] permits us to eliminate (8).*

4 Constitutive relations

For an arbitrary $\mathbf{w} : D \to \mathbb{R}^2$, we introduce the tensor

$$\epsilon(\mathbf{w}) = \frac{1}{2}\left(\nabla\mathbf{w} + (\nabla\mathbf{w})^T\right).$$

If σ is a symmetric tensor, we have

$$\sigma : \nabla\mathbf{w} = \frac{1}{2}\sigma : \nabla\mathbf{w} + \frac{1}{2}\sigma^T : (\nabla\mathbf{w})^T = \sigma : \epsilon(\mathbf{w}).$$

Now, we present the constitutive relations of the structure and of the fluid. We assume that the structure verifies the linear elasticity equation, under the assumption of small deformations. The stress tensor of the structure written in the Lagrangian framework is

$$\lambda^S (\nabla \cdot \mathbf{u}) \mathbf{I} + 2\mu^S \epsilon(\mathbf{u})$$

where $\lambda^S, \mu^S > 0$ are the Lamé coefficients and $\mathbf{I}$ is the unit matrix.

Let us introduce the bi-linear form

$$a_S\left(\mathbf{u}, \mathbf{w}^S\right) = \int_{\Omega_0^S} \left(\lambda^S (\nabla\cdot\mathbf{u})\left(\nabla\cdot\mathbf{w}^S\right) + 2\mu^S\epsilon(\mathbf{u}) : \epsilon\left(\mathbf{w}^S\right)\right) d\mathbf{X}.$$

We assume that the fluid is Newtonian and the Cauchy stress tensor is given by

$$\sigma^F(\mathbf{v},p) = -p\,\mathbf{I} + 2\mu^F \epsilon(\mathbf{v})$$

where $\mu^F > 0$ is the viscosity of the fluid. We have

$$\sigma^F(\mathbf{v},p) : \nabla\mathbf{w} = \sigma^F(\mathbf{v},p) : \epsilon(\mathbf{w}) = 2\mu^F \epsilon(\mathbf{v}) : \epsilon(\mathbf{w}) - (\nabla\cdot\mathbf{w})\,p.$$

Introduce the notation

$$\begin{aligned} a_F(\mathbf{v},\mathbf{w}) &= \int_D 2\mu^F \epsilon(\mathbf{v}) : \epsilon(\mathbf{w})\,d\mathbf{x} \\ b_F(\mathbf{w},p) &= -\int_D (\nabla\cdot\mathbf{w})\,p\,d\mathbf{x}. \end{aligned}$$

The functional spaces will be precised later.

5 Parametrization and regularization of the characteristic function

Let $j \in W^{1,\infty}(D)$ be a parametrization of $\Omega_0^S \subset D$, i.e. :

$$\begin{aligned} j(\mathbf{x}) &> 0, \ \mathbf{x} \in \Omega_0^S, \\ j(\mathbf{x}) &< 0, \ \mathbf{x} \in D \setminus \overline{\Omega}_0^S, \\ j(\mathbf{x}) &= 0, \ \mathbf{x} \in \partial\Omega_0^S. \end{aligned}$$

The parametrization is not necessarily unique.

Let $\mathbf{u} \in \left(W^{1,\infty}(\Omega_0^S)\right)^2$ be Lipschitz with constant less than 1. Denote, as before, $\Omega_u^S = \varphi(\Omega_0^S)$, where $\varphi(\mathbf{X}) = \mathbf{X} + \mathbf{u}(\mathbf{X})$. $\mathbf{u}$ will be the displacement of the structure and it will be shown to satisfy the condition $\Omega_u^S \subset D$. Then $\varphi : \overline{\Omega}_0^S \to \overline{\Omega}_u^S$ is bijective and bilipschitzian and

$$j_u(\mathbf{y}) = \begin{cases} j(\mathbf{x}), & \mathbf{y} = \varphi(\mathbf{x}) \in \Omega_u^S \\ 0, & \mathbf{y} \in \partial\Omega_u^S \\ -dist(\mathbf{y}, \overline{\Omega}_u^S), & \mathbf{y} \notin \overline{\Omega}_u^S \end{cases}$$

is a parametrization of Ω_u^S, $j_u \in W^{1,\infty}(D)$.

If H is the Heaviside function $H : \mathbb{R} \to \{0,1\}$,

$$H(r) = \begin{cases} 1, & r \geq 0 \\ 0, & r < 0 \end{cases}$$

and $\widetilde{H}$ is a Lipschitz regularization of H (to be precised later), then $H(j_u(\cdot))$ is the characteristic function of Ω_u^S and $\widetilde{H}(j_u(\cdot))$ is its approximation, with Lipschitz properties in D. The approximation properties are to be explained later. The technique of parametrization and regularization for the unknown geometries was introduced in [18] and a thorough discussion may be found in [26]. It was employed in [25] for shape optimization problems with elliptic equations and in [15] for the steady Navier-Stokes equations.

We specify now how to choose the Lipschitz regularization $\widetilde{H}$ of the Heaviside mapping H.

If Ω_0^S is an open bounded set, for any $\epsilon > 0$ there exists $\Omega_0^\epsilon \subset\subset \Omega_0^S$, such that $\Omega_0^\epsilon \to \Omega_0^S$ in the Hausdorff-Pompeiu sense, for $\epsilon \to 0$. Since $j : D \to \mathbb{R}$ is Lipschitz continuous and $\Omega_0^\epsilon \subset\subset \Omega_0^S$, $j > 0$ in Ω_0^S, there exists $\mu_\epsilon > 0$ such that $j(\mathbf{x}) \geq \mu_\epsilon > 0$, for all $\mathbf{x} \in \Omega_0^\epsilon$. Denote $\Omega_u^\epsilon = (\mathbf{id} + \mathbf{u})\,(\Omega_0^\epsilon)$, where $\mathbf{id}$ is the identity mapping. Consequently,

$$\mu_\epsilon \leq \min_{\mathbf{y}\in\overline{\Omega}_u^\epsilon} j_u(y), \quad \forall \mathbf{u} \in \left(W^{1,\infty}(\Omega_0^S)\right)^2.$$

Then we take $\widetilde{H} = H^{\mu_\epsilon}$, the Yosida regularization of H

$$H^{\mu_\epsilon}(r) = \begin{cases} 1, & r \geq \mu_\epsilon \\ \frac{r}{\mu_\epsilon} & 0 \leq r < \mu_\epsilon \\ 0, & r < 0 \end{cases}$$

It follows that $H^{\mu_\epsilon}\left(j_u(\mathbf{x})\right) = 1$ for all $\mathbf{x} \in \Omega_u^\epsilon$.

Let us introduce the Hilbert spaces

$$\begin{aligned} W^S &= \left\{\mathbf{w}^S \in \left(H^1\left(\Omega_0^S\right)\right)^2;\ \mathbf{w}^S = 0 \text{ on } \Gamma_D\right\}, \\ W &= \left(H_0^1\left(D\right)\right)^2, \\ Q &= L_0^2\left(D\right) = \{q \in L^2\left(D\right);\ \int_D q\,dx = 0\}. \end{aligned}$$

We assume for the moment that $\mathbf{f}^F \in \left(L^2(D)\right)^2$, $\mathbf{f}^S \in \left(L^2(\Omega_0^S)\right)^2$.

For a given $\mathbf{u} \in \left(W^{1,\infty}(\Omega_0^S)\right)^2$, such that $\|\mathbf{u}\|_{1,\infty,\Omega_0^S} < 1$ and $\mathbf{u} = 0$ on Γ_D, we define:

- fluid velocity $\mathbf{v}_\epsilon \in \left(H_0^1(D)\right)^2$,
- fluid pressure $p_\epsilon \in Q$,

- structure displacement $\mathbf{u}_\epsilon \in W^S$,

as the solution of the following weakly coupled system of PDE's:

$$\begin{aligned} a_F\left(\mathbf{v}_\epsilon, \mathbf{w}\right) + b_F\left(\mathbf{w}, p_\epsilon\right) & \\ +\frac{1}{\epsilon}\int_D \widetilde{H}(j_u)\mathcal{P}\left(\mathbf{v}_\epsilon\right)\cdot\mathbf{w}\,d\mathbf{x} &= \int_D \mathbf{f}^F\cdot\mathbf{w}d\mathbf{x}, \quad \forall\mathbf{w}\in W \qquad (20) \\ b_F\left(\mathbf{v}_\epsilon, q\right) &= 0, \quad \forall q\in Q \qquad (21) \end{aligned}$$

$$\begin{aligned} a_S\left(\mathbf{u}_\epsilon, \mathbf{w}^S\right) &= \int_{\Omega_0^S} \mathbf{f}^S\cdot\mathbf{w}^S\,d\mathbf{X} \\ &+ \int_{\Omega_0^S} J\left(\sigma^F\left(\mathbf{v}_\epsilon, p_\epsilon\right)\circ\varphi\right)\mathbf{F}^{-T} : \nabla_{\mathbf{X}}\mathbf{w}^S\,d\mathbf{X} \\ &+ \frac{1}{\epsilon}\int_{\Omega_0^S} J\widetilde{H}(j_u\circ\varphi)\mathcal{P}\left(\mathbf{v}_\epsilon\circ\varphi\right)\cdot\mathbf{w}^S\,d\mathbf{X} \\ &- \int_{\Omega_0^S} J\left(\mathbf{f}^F\circ\varphi\right)\cdot\mathbf{w}^S\,d\mathbf{X},\ \forall\mathbf{w}^S\in W^S \qquad (22) \end{aligned}$$

where $\varphi(\mathbf{X}) = \mathbf{X} + \mathbf{u}(\mathbf{X})$, $\mathbf{F}(\mathbf{X}) = \mathbf{I} + \nabla_{\mathbf{X}}\mathbf{u}(\mathbf{X})$, $J(\mathbf{X}) = \det\,\mathbf{F}(\mathbf{X})$.

From (18) and (17), using the constitutive relations and the regularization of the characteristic function, we get (20) and (22), respectively.

Remark 4 *The map* $\mathbf{u}$ *appears into the coefficient* $\widetilde{H}(j_u)$ *in (20), as well as into the terms of (22) coming from the fluid equations in the right hand side. But the coefficients of* a_F, b_F, a_S *are constants.*

Define the nonlinear operator

$$T_\epsilon : \left\{\mathbf{u}\in\left(W^{1,\infty}(\Omega_0^S)\right)^2;\ \|\mathbf{u}\|_{1,\infty,\Omega_0^S} < 1,\ \mathbf{u} = 0 \text{ on } \Gamma_D\right\} \to \left(W^{1,\infty}(\Omega_0^S)\right)^2$$

by

$$T_\epsilon(\mathbf{u}) = \mathbf{u}_\epsilon.$$

We recall that we have assumed $\Omega_u^S \subset D$. In the following, we will prove that T_ϵ is well defined and that it has at least one fixed point under some additional hypotheses.

6 The approximating problem

Denote by $\|\cdot\|_{m,s,\Omega}$ the usual norm of the Sobolev space $W^{m,s}(\Omega)$. When $s = 2$, we use the well known notation $H^m(\Omega) = W^{m,2}(\Omega)$.

We denote by α' the number $\alpha/(\alpha-1)$ so that $2 < \alpha'$ and $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$.

Next, we show some estimations for the solutions of the fluid problem (20)-(21) and, respectively, of the structure problem (22). Following for example [11], the properties below hold:

$$\exists \alpha_F > 0,\ \forall \mathbf{w} \in W, \quad \alpha_F \|\mathbf{w}\|_{1,2,D}^2 \le a_F(\mathbf{w},\mathbf{w}) \tag{23}$$

$$\exists M_F > 0,\ \forall \mathbf{v},\mathbf{w} \in W, \quad |a_F(\mathbf{v},\mathbf{w})| \le M_F \|\mathbf{v}\|_{1,2,D} \|\mathbf{w}\|_{1,2,D} \tag{24}$$

$$\exists \beta > 0, \qquad \inf_{q\in Q, q\neq 0} \sup_{w\in W, w\neq 0} \frac{b_F(\mathbf{w},q)}{\|\mathbf{w}\|_{1,2,D} \|q\|_{0,2,D}} \ge \beta \tag{25}$$

$$\exists N_F > 0,\ \forall \mathbf{w} \in W,\ \forall q \in Q, \quad |b_F(\mathbf{w},q)| \le N_F \|\mathbf{w}\|_{1,2,D} \|q\|_{0,2,D} \tag{26}$$

When $\mathbf{u} \in \left(W^{1,\infty}(\Omega_0^S)\right)^2$, for every $0 < \delta < 1$, there exists $0 < \eta_\delta < 1$ such that

$$1 - \delta \le \det(\mathbf{I} + \nabla\mathbf{u}) \le 1 + \delta, \quad \text{a.e. } \mathbf{x} \in \Omega_0^S \tag{27}$$

for all $\mathbf{u}$ that satisfy $\|\mathbf{u}\|_{1,\infty,\Omega_0^S} \le \eta_\delta$.

Notice that the coefficient $\widetilde{H}(j_u)$ is Lipschitz and $0 \le \widetilde{H}(j_u(\mathbf{x})) \le 1$ for all $\mathbf{x} \in \overline{D}$. Define $\phi : L^2(D)^2 \to \mathbb{R}$ by $\phi(\mathbf{v}) = \frac{1}{\alpha}\left(|v_1|^\alpha + |v_2|^\alpha\right)$, where $1 < \alpha < 2$ and $\mathbf{v} = (v_1, v_2)$. This is a convex continuous function.

Let us define

$$V = \left\{ \mathbf{w} \in \left(H_0^1(D)\right)^2;\ \nabla\cdot\mathbf{w} = 0 \text{ on } D \right\}$$

and let V' be its dual.

Lemma 1 *The operator $\frac{1}{\epsilon}\widetilde{H}(j_u)\partial\phi(\cdot) : V \to V'$ is maximal monotone.*

Proof. Here $\partial\phi(\cdot)$ is the subdifferential of ϕ defined above in $L^2(D)^2 \times L^2(D)^2$. It exists in any point and the above operator is monotone since the pairing in $V \times V'$ extends the scalar product in $L^2(D)^2$ and the coefficient $\frac{1}{\epsilon}\widetilde{H}(j_u)$ is positive.

To prove its maximality, we use Minty's theorem, [2, p. 39]. We introduce the regularization $\phi_\delta : L^2(D)^2 \to \mathbb{R}$ of ϕ, which is Fréchet differentiable and

$\nabla\phi_\delta \subset L^2(D)^2 \times L^2(D)^2$ is Lipschitzian. Then, the operator $\frac{1}{\epsilon}\widetilde{H}(j_u)\nabla\phi_\delta$ is monotone and Lipschitzian in $L^2(D)^2 \times L^2(D)^2$. Its restriction to $V \times V'$ is monotone and continuous, consequently it is maximal monotone. By Minty's theorem, there is a unique solution $\mathbf{v}_\delta \in V$ of the equation

$$F\,\mathbf{v}_\delta + \frac{1}{\epsilon}\widetilde{H}(j_u)\nabla\phi_\delta\,(\mathbf{v}_\delta) = \mathbf{w}$$

for any $\mathbf{w} \in V'$, where $F : V \to V'$ is the duality mapping.

Taking into account that $0 = \nabla\phi_\delta\,(0)$, we get immediately that $\{\mathbf{v}_\delta;\ \delta > 0\}$ is bounded in V and W. On a subsequence, we have $\mathbf{v}_\delta \to \widehat{\mathbf{v}}$ weakly in W and strongly in $L^2(D)^2$. Since $(Id + \delta\partial\phi)^{-1}$ is nonexpansive in $L^2(D)^2$ and $(Id + \delta\partial\phi)^{-1}\,\widehat{\mathbf{v}} \to \widehat{\mathbf{v}}$ for $\delta \to 0$ in $L^2(D)^2$, we obtain that $(Id + \delta\partial\phi)^{-1}\,\mathbf{v}_\delta \to \widehat{\mathbf{v}}$ in $L^2(D)^2$.

Moreover, $\partial\phi$ is defined everywhere in $L^2(D)^2$ and it is locally bounded. Then $\partial\phi\,(Id + \delta\partial\phi)^{-1}\,(\mathbf{v}_\delta) = \nabla\phi_\delta(\mathbf{v}_\delta)$ is bounded in $L^2(D)^2$ and we may assume that $\partial\phi\,(Id + \delta\partial\phi)^{-1}\,(\mathbf{v}_\delta) \to \mathbf{z}$ weakly in $L^2(D)^2$, on a subsequence. The demiclosedness of $\partial\phi$ gives $\mathbf{z} \in \partial\phi(\widehat{\mathbf{v}})$. Passing to the limit in the approximating equation, we get

$$F\,\widehat{\mathbf{v}} + \frac{1}{\epsilon}\widetilde{H}(j_u)\partial\phi(\widehat{\mathbf{v}}) \ni \mathbf{w}.$$

This ends the proof of the Lemma, again by Minty's theorem. $\square$

Proposition 1 *There exists a unique solution of (20)-(21) such that* $\mathbf{v}_\epsilon \in \left(H_0^1(D)\right)^2$ *and* $p_\epsilon \in Q$.

Proof. The operator defined by the bilinear form $a_F(\cdot,\cdot)$ in $V \times V'$ is maximal monotone continuous and coercive. Its sum with the operator $\frac{1}{\epsilon}\widetilde{H}(j_u)\partial\phi(\cdot)$ is, consequently maximal monotone onto V' due to its coercivity. Such properties are discussed in [2, ch. II]. Then, we get the existence of a unique weak solution $\mathbf{v}_\epsilon \in V$ of

$$a_F\left(\mathbf{v}_\epsilon, \mathbf{w}\right) + \frac{1}{\epsilon}\int_D \widetilde{H}(j_u)\partial\phi\left(\mathbf{v}_\epsilon\right)\cdot\mathbf{w}\,d\mathbf{x} = \int_D \mathbf{f}^F\cdot\mathbf{w}\,d\mathbf{x},\ \forall\mathbf{w} \in V.$$

We obtain that the element of W' defined by

$$\mathbf{w} \mapsto a_F\left(\mathbf{v}_\epsilon, \mathbf{w}\right) + \frac{1}{\epsilon}\int_D \widetilde{H}(j_u)\partial\phi\left(\mathbf{v}_\epsilon\right)\cdot\mathbf{w}\,d\mathbf{x} - \int_D \mathbf{f}^F\cdot\mathbf{w}\,d\mathbf{x}$$

belongs to the polar set $V^0 = \{h \in W';\ \langle h, \mathbf{w}\rangle = 0,\ \forall \mathbf{w} \in V\}$.

Taking into account (25)-(26) and the Lemma 4.1, p. 58 from [11], there exists a unique $p_\epsilon \in Q$, such that (20) holds. From $\mathbf{v}_\epsilon \in V$, we get (21).

The choice (11) of the penalization operator is justified by the expression of $\partial\phi(\cdot)$, the subdifferential of $\phi(\mathbf{v}) = \frac{1}{\alpha}\left(|v_1|^\alpha + |v_2|^\alpha\right)$. In fact, we have

$$\frac{1}{\epsilon}\int_D \widetilde{H}(j_u)\partial\phi\left(\mathbf{v}_\epsilon\right)\cdot \mathbf{w}\, d\mathbf{x} = \frac{1}{\epsilon}\int_D \widetilde{H}(j_u)\mathcal{P}\left(\mathbf{v}_\epsilon\right)\cdot \mathbf{w}\, d\mathbf{x}.$$

□

Proposition 2 *Let D, Ω_0^S be open bounded sets of class $\mathcal{C}^2$. We assume that $\mathbf{f}^F \in \left(L^{\alpha'}(D)\right)^2$, $\mathbf{f}^S \in \left(L^{\alpha'}(\Omega_0^S)\right)^2$. Then $\mathbf{v}_\epsilon \in \left(W^{2,\alpha'}(D)\right)^2$, $p_\epsilon \in W^{1,\alpha'}(D)$ and $\mathbf{u}_\epsilon \in \left(W^{2,\alpha'}(\Omega_0^S)\right)^2$, where $\mathbf{v}_\epsilon$, p_ϵ, $\mathbf{u}_\epsilon$ are the solution of (20)-(22).*

Proof. The existence of a weak solution $\mathbf{v}_\epsilon \in \left(H_0^1(D)\right)^2$ and $p_\epsilon \in Q$ of (20)-(21) has already been discussed in Proposition 1. From $\mathbf{v}_\epsilon \in \left(H^1(D)\right)^2$, then $\mathbf{v}_\epsilon \in \left(L^\alpha(D)\right)^2$ since $\alpha < 2$. It follows that

$$\int_D \left(|\mathbf{v}_\epsilon|^{\alpha-1}\right)^{\alpha'} d\mathbf{x} = \int_D |\mathbf{v}_\epsilon|^\alpha\, d\mathbf{x} < \infty \tag{28}$$

and consequently $|\mathbf{v}_\epsilon|^{\alpha-1} \in \left(L^{\alpha'}(D)\right)^2$. Since the coefficient $\widetilde{H}(j_u)$ is Lipschitz and $0 \le \widetilde{H}(j_u(\mathbf{x})) \le 1$ for all $\mathbf{x} \in \overline{D}$, we obtain that

$$\frac{1}{\epsilon}\widetilde{H}(j_u)\mathcal{P}\left(\mathbf{v}_\epsilon\right) \in \left(L^{\alpha'}(D)\right)^2.$$

Passing the penalization term of (20) in the right-hand side, we obtain that $\mathbf{v}_\epsilon$ and p_ϵ are solution of the Stokes problem:

$$\begin{aligned} a_F\left(\mathbf{v}_\epsilon, \mathbf{w}\right) + b_F\left(\mathbf{w}, p_\epsilon\right) &= \int_D \left(\mathbf{f}^F - \frac{1}{\epsilon}\widetilde{H}(j_u)\mathcal{P}\left(\mathbf{v}_\epsilon\right)\right)\cdot \mathbf{w}\, d\mathbf{x},\ \forall \mathbf{w} \in W \\ b_F\left(\mathbf{v}_\epsilon, q\right) &= 0,\ \forall q \in Q \end{aligned}$$

and assuming $\mathbf{f}^F \in \left(L^{\alpha'}(D)\right)^2$, we get $\mathbf{v}_\epsilon \in \left(W^{2,\alpha'}(D)\right)^2$ and $p_\epsilon \in W^{1,\alpha'}(D)$ by the regularity results for the Stokes equations from [27, p. 35].

The existence and regularity of $\mathbf{u}_\epsilon$ is a consequence of standard results for elliptic systems. □

Proposition 3 *For every $\epsilon > 0$, the nonlinear operator T_ϵ has at least one fixed point in*

$$B_\delta = \{\mathbf{u} \in W^{1,\infty}(\Omega_0^S)^2; ||\mathbf{u}||_{1,\infty,\Omega_0^S} \leq \eta_\delta,\ \mathbf{u} = 0 \ on\, \Gamma_D\}$$

if f^F, f^S are "small" in their own norms.

Proof. We shall prove that $T_\epsilon(B_\delta) \subset B_\delta$ and that T_ϵ is continuous.

The regularity result from Proposition 2 can be completed by estimates expressing the bounded dependence of the solutions $\mathbf{v}_\epsilon$, p_ϵ, $\mathbf{u}_\epsilon$ on the data f^F, f^S:

$$\begin{aligned} &\|\mathbf{v}_\epsilon\|_{2,\alpha',D} + \|p_\epsilon\|_{1,\alpha',D} \leq \\ &C_1\left(\left\|\mathbf{f}^F\right\|_{0,\alpha',D} + \frac{1}{\epsilon}\left\|\widetilde{H}(j_u)\mathcal{P}(\mathbf{v}_\epsilon)\right\|_{0,\alpha',D}\right) \end{aligned} \tag{29}$$

$$\begin{aligned} &\|\mathbf{u}_\epsilon\|_{2,\alpha',\Omega_0^S} \leq \\ &C_2\left(\|\mathbf{v}_\epsilon\|_{2,\alpha',D} + \|p_\epsilon\|_{1,\alpha',D} + \left\|\mathbf{f}^S\right\|_{0,\alpha',\Omega_0^S} + \left\|\mathbf{f}^F\right\|_{0,\alpha',D}\right). \end{aligned} \tag{30}$$

Here $\epsilon >$ is fixed and the constants C_1, C_2 are independent of ϵ and $\mathbf{u} \in B_\delta$.

We have that

$$\left|\widetilde{H}(j_u)\mathcal{P}(\mathbf{v}_\epsilon)\right|^{\alpha'} = \left|\widetilde{H}(j_u)\left|\mathbf{v}_\epsilon\right|^{\alpha-1} sgn(\mathbf{v}_\epsilon)\right|^{\alpha'} \leq |\mathbf{v}_\epsilon|^{(\alpha-1)\alpha'} = |\mathbf{v}_\epsilon|^{\alpha}.$$

The mapping $\widetilde{H}(j_u)$ has support in Ω_u^S and we get

$$\int_D \left|\widetilde{H}(j_u)\mathcal{P}(\mathbf{v}_\epsilon)\right|^{\alpha'} d\mathbf{x} = \int_{\Omega_u^S} \left|\widetilde{H}(j_u)\mathcal{P}(\mathbf{v}_\epsilon)\right|^{\alpha'} d\mathbf{x} \leq \int_{\Omega_u^S} |\mathbf{v}_\epsilon|^{\alpha} d\mathbf{x}$$

and then

$$\left\|\widetilde{H}(j_u)\mathcal{P}(\mathbf{v}_\epsilon)\right\|_{0,\alpha',D} \leq \left(\int_{\Omega_u^S} |\mathbf{v}_\epsilon|^{\alpha} d\mathbf{x}\right)^{1/\alpha'} = \|\mathbf{v}_\epsilon\|_{0,\alpha,\Omega_u^S}^{\alpha/\alpha'} = \|\mathbf{v}_\epsilon\|_{0,\alpha,\Omega_u^S}^{\alpha-1}.$$

Using (29) and (30), we finally get

$$\|\mathbf{u}_\epsilon\|_{2,\alpha',\Omega_0^S} \leq C\left(\left\|\mathbf{f}^F\right\|_{0,\alpha',D} + \left\|\mathbf{f}^S\right\|_{0,\alpha',\Omega_0^S} + \frac{1}{\epsilon}\|\mathbf{v}_\epsilon\|_{0,\alpha,\Omega_u^S}^{\alpha-1}\right).$$

Now, we put $\mathbf{w} = \mathbf{v}_\epsilon \in W$ in (20), it follows that

$$a_F(\mathbf{v}_\epsilon, \mathbf{v}_\epsilon) + \frac{1}{\epsilon}\int_D \widetilde{H}(j_u)\mathcal{P}(\mathbf{v}_\epsilon)\cdot \mathbf{v}_\epsilon \, d\mathbf{x} = \int_D \mathbf{f}^F \cdot \mathbf{v}_\epsilon d\mathbf{x}. \tag{31}$$

Since $\widetilde{H}(j_u) \geq 0$ and $\mathcal{P}(\mathbf{v}_\epsilon)\cdot \mathbf{v}_\epsilon \geq 0$, we have

$$\int_D \widetilde{H}(j_u)\mathcal{P}(\mathbf{v}_\epsilon)\cdot \mathbf{v}_\epsilon \, d\mathbf{x} \geq 0.$$

Using (23) and the Cauchy inequality, we get

$$\alpha_F \|\mathbf{v}_\epsilon\|^2_{1,2,D} \leq \left\|\mathbf{f}^F\right\|_{0,2,D} \|\mathbf{v}_\epsilon\|_{0,2,D} \leq \left\|\mathbf{f}^F\right\|_{0,2,D} \|\mathbf{v}_\epsilon\|_{1,2,D}.$$

But $\alpha < 2$, we get finally

$$\|\mathbf{u}_\epsilon\|_{2,\alpha',\Omega_0^S} \leq C\left(\left\|\mathbf{f}^F\right\|_{0,\alpha',D} + \left\|\mathbf{f}^S\right\|_{0,\alpha',\Omega_0^S} + \frac{1}{\epsilon}\left(\left\|\mathbf{f}^F\right\|_{0,\alpha',D}\right)^{\alpha-1}\right).$$

For $\epsilon > 0$ fixed, if f^F, f^S are "small" in their own norms, then $T_\epsilon(B_\delta) \subset B_\delta$. Here, we also use the Sobolev embedding theorem that gives, in particular, that $\mathbf{u}_\epsilon$ is Lipschitzian in each component.

As the Sobolev theorem ensures compactness as well, we just have to show that the operator T_ϵ is continuous in B_δ.

We study first the coefficient $\widetilde{H}(\cdot) = H^{\mu_\epsilon}(j_u(\cdot))$ (ϵ is fixed here), with μ_ϵ defined in section 5.

Let $u_n \to u$ strongly in $W^{1,\infty}(\Omega_0^S)^2$, $u_n, u \in B_\delta$. We show that $\bar{\Omega}^S_{u_n} \to \bar{\Omega}^S_u$ in the Hausdorff-Pompeiu sense. By the compactness of the Hausdorff-Pompeiu metric, the limit exists on a sub-sequence (again denoted by n) and we have

$$\lim_{n\to\infty} \bar{\Omega}^S_{u_n} = \{y \in \mathbb{R}^2; \exists x_n \in \Omega_0^S,\ y = \lim_{n\to\infty}(x_n + u_n(x_n))\}.$$

We may assume $x_n \to x_0$, then $y = x_0 + \lim_{n\to\infty} u_n(x_0) + \lim_{n\to\infty}(u_n(x_n) - u_n(x_0)) = x_0 + u(x_0)$, by the uniform Lipschitz property of $u_n \in B_\delta$. Then $\lim_{n\to\infty} \bar{\Omega}^S_{u_n} \subset \bar{\Omega}^S_u$. The converse inclusion is obvious:

$$\forall y \in \Omega^S_u \Rightarrow y = z + u(z) = \lim_{n\to\infty}(z + u_n(z)),$$

for some $z \in \Omega_0^S$. As the limit is unique, the convergence is valid without taking sub-sequences. Due to the regularity properties of Ω^S_u, $\Omega^S_{u_n}$, the above convergence is equivalent with the convergence of the open sets $\Omega^S_{u_n} \to \Omega^S_u$ in the Hausdorff-Pompeiu complementary sense, [26, p. 469].

Let $\mathcal{K} \subset \Omega^S_u$ be some compact subset. Then $\mathcal{K} \subset \Omega^S_{u_n}$ for n "big enough" by the Γ-property of the Hausdorff-Pompeiu complementary convergence of open sets, [26, p. 465].

If $y \in \mathcal{K}$ is arbitrary, then there exist $x_n \in \Omega^S_{u_n}$, $x \in \Omega^S_u$; $x_n = (I+u_n)^{-1}(y)$, $x = (I+u)^{-1}y$ and we may assume $x_n \to z$ (a limit of a subsequence). Clearly, by the uniform Lipschitz property of u_n, $(I+u_n)(z) - (I+u_n)(x_n) \to 0$, therefore

$$y = (I+u_n)(x_n) = (I+u)(x) = (I+u)(z)$$

and $z = x$ as $I+u$ is one-to-one.

It follows that $x_n \to x$ so $(I+u_n)^{-1}y \to (I+u)^{-1}y \quad \forall y \in \mathcal{K}$ and $j_{u_n}(y) \to j_u(y)$, $\forall y \in \mathcal{K}$.

As $\mathcal{K}$ may be "extended" to Ω^S_u, the definition of $j_u(\cdot)$ shows that $j_{u_n} \to j_u$ a.e. in D. Consequently $H^{\mu_\epsilon}(j_{u_n}) \to H^{\mu_\epsilon}(j_u)$ a.e. in D and the boundedness of $\widetilde{H}$, gives $H^{\mu_\epsilon}(j_{u_n}) \to H^{\mu_\epsilon}(j_u)$ strongly in $L^r(D)$, $\forall r \geq 1$.

As $u_n, u \in B_\delta$, the corresponding solutions of (20), (21) and (22), denoted shortly v_n, v, p_n, p and $T_\epsilon u_n$, $T_\epsilon u$ (ϵ is fixed) are bounded in their spaces, uniformly in n. One can take weak convergent sub-sequences, denoted again by n. In particular the penalization operator satisfies $\mathcal{P}(v_n) \to \mathcal{P}(v)$ uniformly and it is possible to pass to the limit in the penalization integral due to the above convergence property of $\widetilde{H}$. This allows to pass to the limit in (20)-(22) and to show that v, p are indeed the limits of v_n, p_n and $T_\epsilon u_n \to T_\epsilon u$ in B_δ. The Schauder fixed point theorem achieves the proof. $\square$

Remark 5 *The solution of $T_\epsilon u = u$ is, in general, not unique due to the nonlinear character of T_ϵ. The above argumentation may be compared with the approach in [13], although the penalized equations, the geometric domains and the functional spaces are different.*

Proposition 4 *We have $\{\mathbf{v}_\epsilon\}$ bounded in $\left(H^1(D)\right)^2$ and*

$$\frac{1}{\epsilon}\int_D \widetilde{H}(j_{u_\epsilon})\,|\mathbf{v}_\epsilon|^\alpha\, d\mathbf{x} \leq \frac{1}{2\alpha_F}\left\|\mathbf{f}^F\right\|^2_{0,2,D} \tag{32}$$

for all $\epsilon > 0$. Moreover, $\{\mathbf{u}_\epsilon\}$ is bounded in $\left(W^{1,\infty}(\Omega^S_0)\right)^2$.

Proof. From (31), using (23), the inequalities Cauchy and $ab \le \frac{1}{2\lambda}a^2 + \frac{\lambda}{2}b^2$, we obtain

$$\begin{aligned}&\alpha_F \left\|\mathbf{v}_\epsilon\right\|_{1,2,D}^2 + \frac{1}{\epsilon}\int_D \widetilde{H}(j_u)\mathcal{P}\left(\mathbf{v}_\epsilon\right)\cdot \mathbf{v}_\epsilon \, d\mathbf{x} \\ &\le \left\|\mathbf{f}^F\right\|_{0,2,D}\left\|\mathbf{v}_\epsilon\right\|_{0,2,D} \\ &\le \frac{1}{2\alpha_F}\left\|\mathbf{f}^F\right\|_{0,2,D}^2 + \frac{\alpha_F}{2}\left\|\mathbf{v}_\epsilon\right\|_{1,2,D}^2\end{aligned}$$

Relation (32) is a consequence of the definition of $\mathcal{P}$.

The boundedness of $\{\mathbf{u}_\epsilon\}$ is given by $\mathbf{u}_\epsilon \in B_\delta$, according to the previous proposition. □

Remark 6 *One can infer weak convergence properties from Proposition 4. The experiments from the next section show that our method has good numerical convergence and stability properties, too. From a theoretical point of view, it is necessary to clarify as well the hypothesis that the data f^F, f^S have to be "small", which may induce same undesired dependence on $\epsilon > 0$.*

Proposition 5 *If $u_\epsilon \to u^*$ weakly in $W^{2,\alpha'}(\Omega_0^S)$, $v_\epsilon \to v^*$ weakly in $W^{2,\alpha'}(D)$, $p_\epsilon \to p^*$ weakly in $W^{1,\alpha'}(D)$, then v^* and p^* satisfy (3), (4) in $\Omega_{u^*}^F$, (7) on Γ_{u^*}, (5) and (6). The mapping u^* satisfies (1) and (2).*

Proof. We have that $\Omega_{u_\epsilon}^S \to \Omega_{u^*}^S$ in the complementary Hausdorff-Pompeiu metric, as in the proof of Proposition 3. And similarly $\Omega_{u_\epsilon}^F \to \Omega_{u^*}^F$ in the same topology.

Notice that *supp* $\tilde{H}(j_{u_\epsilon}(\cdot)) =$ *supp* $H^{\mu_\epsilon}(j_{u_\epsilon}(\cdot))$ is contained in $\bar{\Omega}_{u_\epsilon}^S$ by the construction of $\tilde{H}$ and j_{u_ϵ}. Let w and q be some test functions in W, respectively Q, with their support in $\Omega_{u^*}^F$. By the Γ-property, for ϵ "small", *supp* $w \subset \Omega_{u_\epsilon}^F$, *supp* $q \subset \Omega_{u_\epsilon}^F$. Then, in particular, the penalization integral in (20) vanishes due to *supp* $\tilde{H}(j_{u_\epsilon}(\cdot)) \subseteq \bar{\Omega}_{u_\epsilon}^S$. One can pass to the limit $\epsilon \to 0$, in (20), (21) and obtain (3), (4) in a weak form, satisfied by v^* and p^*.

For the boundary condition (7), we use the estimate (32) and the relation $\tilde{H}(j_{u_\epsilon}) = 1$ on $\Omega_{u_\epsilon}^\epsilon$. We get

$$\frac{1}{\epsilon}\int_{\Omega_{u_\epsilon}^\epsilon} \left|\mathbf{v}_\epsilon\right|^\alpha \, d\mathbf{x} \le \frac{1}{2\alpha_F}\left\|\mathbf{f}^F\right\|_{0,2,D}^2 .$$

If $\mathcal{K} \subset \Omega^S_{u^*}$ is some compact, then $\mathcal{K} \subset \Omega^S_{u_\epsilon}$ for ϵ "small". From the above inequality, we get that $||v_\epsilon||_{0,\alpha,\mathcal{K}} \to 0$, for $\epsilon \to 0$. Then, $v^* = 0$ a.e. on $\mathcal{K}$. Since $\mathcal{K}$ was arbitrary, we infer that $v^* = 0$ a.e. in $\Omega^S_{u^*}$.

The domain $\Omega^S_{u^*}$ is Lipschitzian and the trace theorem gives (7), i.e. $v^* = 0$ on Γ_{u^*}. By construction, v_ϵ verifies (5), (6), then these boundary conditions hold for v^* also.

Notice that $\tilde{H}(j_{u_\epsilon}) = 1$ on $\Omega^\epsilon_{u_\epsilon}$ by the definition of $\tilde{H}$. Take $K \subset \Omega^S_0$ to be any compactly embedded subdomain. Then $K \subset \Omega^\epsilon_0$ for ϵ sufficiently small and $\tilde{H}(j_{u_\epsilon}(x)) = 1$ for any $x \in (I + u_\epsilon)(K)$.

Let $\mathbf{w}^S \in \left(H^1(\Omega^S_0)\right)^2$ have compact support K in Ω^S_0. Since $\mathbf{u} \in \left(W^{1,\infty}(\Omega^S_0)\right)^2$ is such that $\|\mathbf{u}\|_{1,\infty,\Omega^S_0} \leq \eta_\delta$, then $\widetilde{\mathbf{w}}^S = \mathbf{w}^S \circ \varphi^{-1}$ belongs to $\left(H^1(\Omega^S_u)\right)^2$ with compact support in Ω^S_u. We define $\widetilde{\mathbf{w}} \in \left(H^1(D)\right)^2$ the extension by 0 of $\widetilde{\mathbf{w}}^S$.

We have the identities

$$\begin{aligned}
& a_F\left(\mathbf{v}_\epsilon, \widetilde{\mathbf{w}}\right) + b_F\left(\widetilde{\mathbf{w}}, p_\epsilon\right) = \int_D \sigma^F\left(\mathbf{v}_\epsilon, p_\epsilon\right) : \nabla \widetilde{\mathbf{w}}\, d\mathbf{x} \\
&= \int_D \left(-\nabla \cdot \sigma^F\left(\mathbf{v}_\epsilon, p_\epsilon\right)\right) \cdot \widetilde{\mathbf{w}}\, d\mathbf{x} = \int_{\Omega^S_u} \left(-\nabla \cdot \sigma^F\left(\mathbf{v}_\epsilon, p_\epsilon\right)\right) \cdot \widetilde{\mathbf{w}}^S d\mathbf{x} \\
&= \int_{\Omega^S_0} \left(-\nabla_{\mathbf{X}} \cdot \left[J\left(\sigma^F\left(\mathbf{v}_\epsilon, p_\epsilon\right) \circ \varphi\right) \mathbf{F}^{-T}\right]\right) \cdot \mathbf{w}^S d\mathbf{X} \\
&= \int_{\Omega^S_0} \left[J\left(\sigma^F\left(\mathbf{v}_\epsilon, p_\epsilon\right) \circ \varphi\right) \mathbf{F}^{-T}\right] : \nabla_{\mathbf{X}} \mathbf{w}^S d\mathbf{X}.
\end{aligned}$$

Also, we have

$$\int_D \mathbf{f}^F \cdot \widetilde{\mathbf{w}}\, d\mathbf{x} = \int_{\Omega^S_u} \mathbf{f}^F \cdot \widetilde{\mathbf{w}}^S\, d\mathbf{x} = \int_{\Omega^S_0} J\left(\mathbf{f}^F \circ \varphi\right) \cdot \mathbf{w}^S\, d\mathbf{X}$$

and

$$\int_D \widetilde{H}(j_u)\mathcal{P}\left(\mathbf{v}_\epsilon\right) \cdot \widetilde{\mathbf{w}}\, d\mathbf{x} = \int_{\Omega^S_u} \mathcal{P}\left(\mathbf{v}_\epsilon\right) \cdot \widetilde{\mathbf{w}}^S\, d\mathbf{x} = \int_{\Omega^S_0} J\mathcal{P}\left(\mathbf{v}_\epsilon \circ \varphi\right) \cdot \mathbf{w}^S\, d\mathbf{X}.$$

Puting the test function $\widetilde{\mathbf{w}}$ in (20) and $\mathbf{w}^S$ in (22) and adding the two relations, the above argument yields the conclusion (ϵ small):

$$a_S(\mathbf{u}_\epsilon, \mathbf{w}^S) = \int_{\Omega^S_0} \mathbf{f}^S \cdot \mathbf{w}^S d\mathbf{X},\ \forall \mathbf{w}^S \in \left(H^1(K)\right)^2,\ \forall K \subset \Omega^S_0.$$

Since $\mathbf{u}_\epsilon \in W^{2,\alpha'}(\Omega_0^S)$ and

$$a_S(\mathbf{u}_\epsilon, \mathbf{w}^S) = -\int_{\Omega_0^S} \left(\nabla_{\mathbf{X}} \cdot \sigma^S(\mathbf{u}_\epsilon)\right) \cdot \mathbf{w}^S d\mathbf{X},\ \forall \mathbf{w}^S \in \left(H^1(K)\right)^2,\ \forall K \subset \Omega_0^S$$

we get

$$-\int_{\Omega_0^S} \left(\nabla_{\mathbf{X}} \cdot \sigma^S(\mathbf{u}_\epsilon)\right) \cdot \mathbf{w}^S d\mathbf{X} = \int_{\Omega_0^S} \mathbf{f}^S \cdot \mathbf{w}^S d\mathbf{X},\ \forall \mathbf{w}^S \in \left(H^1(K)\right)^2,\ \forall K \subset \Omega_0^S.$$

By passing to the limit in the above relation, we obtain (1) in the sense of $L^{\alpha'}(\Omega_0^S)$. By construction $\mathbf{u}_\epsilon \in W^S$, so (2) holds for $\mathbf{u}^*$ also. $\square$

Let us discuss now the condition (8). Let $\mathbf{u} \in \left(W^{1,\infty}(\Omega_0^S)\right)^2$ be such that $\|\mathbf{u}\|_{1,\infty,\Omega_0^S} < 1$ and $\mathbf{u} = 0$ on Γ_D, and let $\mathbf{v}_\epsilon \in W^{2,\alpha'}(D)$, $p_\epsilon \in W^{1,\alpha'}(D)$, $\mathbf{u}_\epsilon \in W^S$ be a solution of (20)-(22) with $\alpha' > 2$.

We have proved at the end of the proof of Proposition 5 that

$$-\nabla_{\mathbf{X}} \cdot \sigma^S(\mathbf{u}_\epsilon) = \mathbf{f}^S,\ \text{in } L^{\alpha'}(\Omega_0^S).$$

It follows that

$$a_S(\mathbf{u}_\epsilon, \mathbf{w}^S) = \int_{\Omega_0^S} \mathbf{f}^S \cdot \mathbf{w}^S d\mathbf{X} + \int_{\Gamma_0} \sigma^S(\mathbf{u}_\epsilon)\, \mathbf{n}^S \cdot \mathbf{w}^S\, dS \tag{33}$$

for all $\mathbf{w}^S \in W^S$.

For $\mathbf{w} \in W$, we have

$$\begin{aligned} a_F(\mathbf{v}_\epsilon, \mathbf{w}) + b_F(\mathbf{w}, p_\epsilon) &= \int_D \sigma^F(\mathbf{v}_\epsilon, p_\epsilon) : \nabla \mathbf{w}\, d\mathbf{x} \\ &= \int_D \left(-\nabla \cdot \sigma^F(\mathbf{v}_\epsilon, p_\epsilon)\right) \cdot \mathbf{w}\, d\mathbf{x} \end{aligned}$$

and taking into account the equality (20), we get

$$-\nabla \cdot \sigma^F(\mathbf{v}_\epsilon, p_\epsilon) + \frac{1}{\epsilon} \widetilde{H}(j_u) \mathcal{P}(\mathbf{v}_\epsilon) = \mathbf{f}^F,\ \text{in } L^{\alpha'}(D).$$

The same equality holds also in $L^{\alpha'}(\Omega_u^S)$. Let us define for each $\mathbf{w}^S \in W^S$ the function $\widetilde{\mathbf{w}}^S = \mathbf{w}^S \circ \varphi^{-1}$ which belongs to $\left(H^1(\Omega_u^S)\right)^2$. Using the Green formula, we obtain

$$\int_{\Omega_u^S} \sigma^F\left(\mathbf{v}_\epsilon, p_\epsilon\right) \cdot \widetilde{\mathbf{w}}^S d\mathbf{x} + \frac{1}{\epsilon}\int_{\Omega_u^S} \widetilde{H}(j_u)\mathcal{P}\left(\mathbf{v}_\epsilon\right) \cdot \widetilde{\mathbf{w}}^S d\mathbf{x}$$
$$= \int_{\Omega_u^S} \mathbf{f}^F \cdot \widetilde{\mathbf{w}}^S d\mathbf{x} - \int_{\Gamma_u} \sigma^F\left(\mathbf{v}_\epsilon, p_\epsilon\right)\mathbf{n}^F \cdot \widetilde{\mathbf{w}}^S ds.$$

We rewrite the previous equation in the undeformed domain Ω_0^S:

$$\begin{aligned}
&\int_{\Omega_0^S} J\left(\sigma^F\left(\mathbf{v}_\epsilon, p_\epsilon\right)\circ\varphi\right)\mathbf{F}^{-T} : \nabla_{\mathbf{X}}\mathbf{w}^S d\mathbf{X} \\
&+\frac{1}{\epsilon}\int_{\Omega_0^S} J\widetilde{H}(j_u\circ\varphi)\mathcal{P}\left(\mathbf{v}_\epsilon\circ\varphi\right)\cdot\mathbf{w}^S d\mathbf{X} \\
&= \int_{\Omega_0^S} J\left(\mathbf{f}^F\circ\varphi\right)\cdot\mathbf{w}^S d\mathbf{X} - \int_{\Gamma_0}\omega\left(\sigma^F\left(\mathbf{v}_\epsilon, p_\epsilon\right)\mathbf{n}^F\circ\varphi\right)\cdot\mathbf{w}^S dS. \quad (34)
\end{aligned}$$

From (33), (34) and (22), we get

$$-\int_{\Gamma_0}\omega\left(\sigma^F\left(\mathbf{v}_\epsilon, p_\epsilon\right)\mathbf{n}^F\circ\varphi\right)\cdot\mathbf{w}^S dS = \int_{\Gamma_0}\sigma^S\left(\mathbf{u}_\epsilon\right)\mathbf{n}^S\cdot\mathbf{w}^S dS$$

for all $\mathbf{w}^S \in W^S$. This shows that the approximating solutions $\mathbf{v}_\epsilon, p_\epsilon, \mathbf{u}_\epsilon$ satisfy (8) for any ϵ.

7 Partitioned procedures based on the fixed point iterations

We propose a partitioned procedures algorithm in order to approach the solution of the penalized fluid-structure interaction problem.

The penalized term $\mathcal{P}\left(\mathbf{v}\right) = \Big(\left|v_1\right|^{\alpha-1} sgn\left(v_1\right), \left|v_2\right|^{\alpha-1} sgn\left(v_2\right)\Big)$, where $1 < \alpha < 2$ is non-linear in $\mathbf{v}$, so the fluid problem (20)-(21) is non-linear. But, if α is close to 2, we can approach $\mathcal{P}\left(\mathbf{v}\right)$ by $\mathbf{v}$. We can also replace $\widetilde{H}(j_u)$ by the characteristic function χ_u^S in (20).

Under the assumption of small displacements for the structure, we can approach the Jacobian determinant J by 1 and the gradient of the deformation $\mathbf{F}$ by the identity matrix $\mathbf{I}$. Then, the right-hand side in (37) becomes simpler than in (22).

Algorithm 1

Step 1. Given the initial displacement of the structure $\mathbf{u}_\epsilon^0 \in W^S$, compute the characteristic function $\chi^S_{u_\epsilon^0}$, put $k := 0$

Step 2. Find the velocity $\mathbf{v}_\epsilon^k \in \left(H_0^1(D)\right)^2$, and the pressure $p_\epsilon^k \in Q$ by solving the fluid problem

$$a_F\left(\mathbf{v}_\epsilon^k, \mathbf{w}\right) + b_F\left(\mathbf{w}, p_\epsilon^k\right) + \frac{1}{\epsilon}\int_D \chi^S_{u_\epsilon^k}\mathbf{v}_\epsilon^k \cdot \mathbf{w}\, d\mathbf{x} = \int_D \mathbf{f}^F \cdot \mathbf{w}\, d\mathbf{x}, \forall \mathbf{w} \in W \quad (35)$$

$$b_F\left(\mathbf{v}_\epsilon^k, q\right) = 0, \forall q \in Q \quad (36)$$

Step 3. Find the new displacement of the structure $\mathbf{u}_\epsilon^{k+1} \in W^S$ by solving

$$\begin{aligned} a_S\left(\mathbf{u}_\epsilon^{k+1}, \mathbf{w}^S\right) &= \int_{\Omega_0^S} \left(\mathbf{f}^S - \mathbf{f}^F\right) \cdot \mathbf{w}^S\, d\mathbf{x} + \int_{\Omega_0^S} 2\mu^F \epsilon\left(\mathbf{v}_\epsilon^k\right) : \epsilon\left(\mathbf{w}^S\right)\, d\mathbf{x} \\ &- \int_{\Omega_0^S} \left(\nabla \cdot \mathbf{w}^S\right) p_\epsilon^k\, d\mathbf{x} + \frac{1}{\epsilon}\int_{\Omega_0^S} \left(\mathbf{v}_\epsilon^k \circ \varphi_\epsilon^k\right) \cdot \mathbf{w}^S\, d\mathbf{x} \quad \forall \mathbf{w}^S \in W^S \quad (37) \end{aligned}$$

where $\varphi_\epsilon^k(\mathbf{X}) = \mathbf{X} + \mathbf{u}_\epsilon^k(\mathbf{X})$.
Step 4. Stopping test: if $\left\|\mathbf{u}_\epsilon^k - \mathbf{u}_\epsilon^{k+1}\right\|_{0,\Omega_0^S} \leq tol$, then **Stop**
Step 5. Compute the characteristic function $\chi^S_{u_\epsilon^{k+1}}$, put $k := k + 1$ and **Go to Step 2.**

Remark 7 *The fluid problem (35)-(36) is linear and, for a given* $\mathbf{u}_\epsilon^k$, *it has a unique solution. Also, the structure problem (37) is linear and, for given* $\mathbf{v}_\epsilon^k$ *and* p_ϵ^k, *it has a unique solution. At the **Step 5**, an interpolation between the deformed structure mesh and the fixed fluid mesh is necessary. We have to point out that the map of the **Step 5***

$$\mathbf{u}_\epsilon^{k+1} \in W^S \mapsto \chi^S_{u_\epsilon^{k+1}} \in L^\infty(D)$$

is non linear.

8 Numerical results. Deformation of a tall building under the action of wind

We have performed numerical simulations using a 2D model adapted from [5] (see Figure 2).

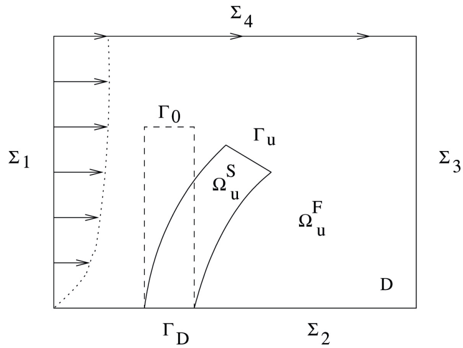

Figure 2: Geometrical configuration for the numerical results

The dimensions of a rectangular tall building are: height $H = 180\ m$, length $L = 30\ m$. The computational domain of the fluid D is a rectangle of height $H_1 = 3H$ and length $L_1 = L + 4H$, its left bottom corner is at $(0, 0)$. This does not satisfy the regularity assumptions on the geometry of Proposition 2. It is possible to "smooth" the corners, but this does not affect essentially our computations. We shall also allow nonhomogeneous Dirichlet data in the numerical experiments.

The distance between the left side of the fluid and the left side of the structure is H (see Figure 3). We denote by Σ_1, Σ_3 the left and the right vertical boundaries and by Σ_2, Σ_4 the bottom and the top boundaries, respectively.

The mechanical proprieties of the building assumed to be an elastic structure are: mass density $\rho^S = 160\ Kg/m^3$, Young modulus $E^S = 2.3 \times 10^8\ N/m^2$, Poisson's ratio $\nu^S = 0.25$, the applied volume forces on the structure $\mathbf{f}^S : \Omega_0^S \to \mathbb{R}^2$, $\mathbf{f}^S = (0, -9.81\rho^S)\ N/m^3$.

The fluid is the air with: mass density $\rho^F = 1.25\ Kg/m^3$, dynamic viscosity $\mu^F = 7.03 \times 10^{-2}\ N \cdot s/m^2$, the applied volume forces on the fluid $\mathbf{f}^S : D \to \mathbb{R}^2$, $\mathbf{f}^F = (0, -9.81\rho^F)\ N/m^3$. The inflow velocity profile is

$$\mathbf{g}(x_1, x_2) = 100 \left(\frac{x_2}{H}\right)^{0.19}\ m/s.$$

The considered boundary conditions for the fluid are more natural from the point of view of applications and differ slightly compared with the previous sections. We impose: $\mathbf{v}_\epsilon = \mathbf{g}$ on $\Sigma_1 \cup \Sigma_4$, $\mathbf{v}_\epsilon = 0$ on Σ_2 and $\sigma^F(\mathbf{v}, p)\,\mathbf{n}^F = 0$

on Σ_3. In this case, the space of fluid pressure is $Q = L^2(D)$.

The numerical tests have been produced using the software *FreeFem++* [16].

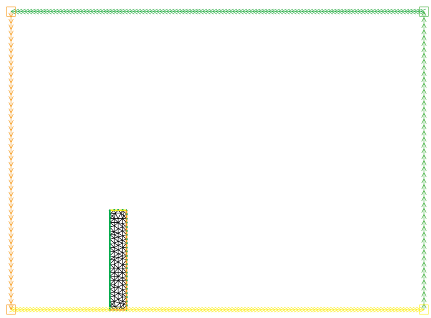

Figure 3: The mesh of the structure domain and the boundary of the fluid domain.

We use a fixed mesh for the fluid domain of 17054 triangles and 8703 vertices. The mesh of structure domain has 192 triangles and 125 vertices. For the approximation of the fluid velocity and pressure we have employed the triangular finite elements $\mathbb{P}_1$+*bubble* and $\mathbb{P}_1$ respectively. The finite element $\mathbb{P}_1$ was used in order to solve the structure problem. The characteristic function was approached by $\mathbb{P}_0$ finite element.

Remark 8 *The fluid and structure meshes are not compatible, for example, a vertex on the structure boundary is not necessary a vertex on the fluid mesh.*

We have performed the simulation using the **Algorithm 1** described in the previous section. For the stopping criterion at the **Step 4**, we have used the tolerance $tol = 0.01$.

First, we have used the initial displacement $\mathbf{u}^0 = (0, 0)$ at the **Step 1**. The stopping criterion holds after 2 iterations $\left\|\mathbf{u}_\epsilon^1 - \mathbf{u}_\epsilon^2\right\|_{0,\Omega_0^S} \leq tol$.

The maximal structural displacement is 0.12 m. We observe in Figure 4 that the deformation is due to the fluid flow and under the action of gravity.

The penalization parameter is $\epsilon = 10^{-3}$. We remark that in the deformed structure domain, the fluid velocity is almost zero (see Figure 5), more precisely

$$\|\mathbf{v}_\epsilon\|_{0,\Omega^S_{u_\epsilon}} = \sqrt{\int_D \chi^S_{u_\epsilon} \mathbf{v}_\epsilon \cdot \mathbf{v}_\epsilon \, d\mathbf{x}} = 0.009443$$

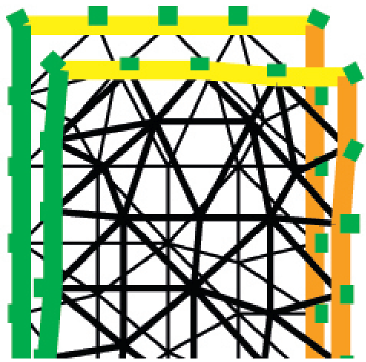

Figure 4: The initial and the deformed structure mesh (detail of the top of the building). We have used a magnification factor 50 for visibility.

We can observe in the Table 1 that $\|\mathbf{v}_\epsilon\|_{0,\Omega^S_{u_\epsilon}}$ decreasing when ϵ goes to 0, but $\|\mathbf{v}_\epsilon\|_{0,\Omega^S_{u_\epsilon}} / \epsilon$ is bounded.

ϵ	$\|\mathbf{v}_\epsilon\|_{0,\Omega^S_{u_\epsilon}}$	$\|\mathbf{v}_\epsilon\|_{0,\Omega^S_{u_\epsilon}} / \epsilon$
0.00100	0.009443	9.44397
0.00050	0.004723	9.44766
0.00010	0.000949	9.49400
0.00005	0.000607	12.1533

Table 1: The fluid velocity in the fictitious domain.

We have also tested the initial displacement $\mathbf{u}^0 = (0.0003 \cdot x_2^2, 0)$ at the **Step 1**. We obtain the same solution, but after 4 iterations.

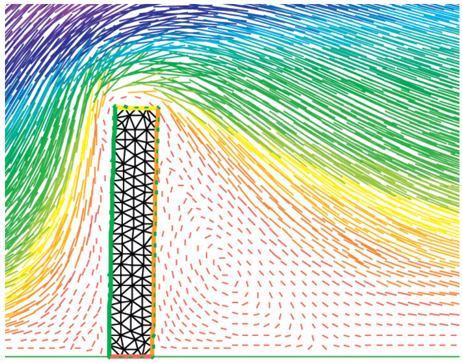

Figure 5: The fluid velocity around the final position of the structure (detail).

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